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Analytical solution for nonlinear Gas Dynamic equation by Homotopy Analysis Method

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Abstract

In this paper, the Homotopy Analysis Method (HAM) is used to implement the homogeneous gas dynamic equation. The analytical solution of this equation is calculated in form of a series with easily computable components.

Keywords: Nonlinear Gas Dynamic Equation; Homotopy Analysis Method; Partial Differential Equation

MSC (2000) No.: 47J30; 35G25

1. Introduction

Recently various iterative methods are employed for the numerical and analytical solution of partial differential equation. In this paper, the Homotopy analysis method (1992) is applied to solve a kind of partial differential equations.

In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely Homotopy Analysis Method (HAM), [Liao (1992, 1995, 2002, 2003, 2005)]. This method has been successfully applied to solve many types of nonlinear problems see Ayub (2004a, 2004b), Jafari (2009) and Liao (2004c, 2005a).

The HAM offers certain advantages over routine numerical methods. Numerical methods use discretization which gives rise to rounding off errors causing loss of accuracy, and requires large computer memory and time. This computational method yields analytical solutions and has certain advantages over standard numerical methods. The HAM method is better since it does not involve discretization of the variables and hence is free from rounding off errors and does not require large computer memory or time.

The paper has been organized as follows. In Section 2 the Homotopy Analysis Method is described. In Section 3 HAM is applied for nonlinear homogeneous gas dynamics equation. Discussion and conclusions are presented in Section 4.

2. Basic idea of HAM

Consider the following differential equation

$$N[u(r,t)] = 0, \tag{1}$$

where N is a nonlinear operator, r and t are independent variables, $u(r, t)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao (2003) constructs the so called zero order deformation equation

$$(1-p)L[\varphi(r,t;p)-u_0(r,t)] = phH(r,t)N[\varphi(r,t;p)], \tag{2}$$

where $p \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is a nonzero auxiliary parameter, $H(r,t) \neq 0$ is non zero auxiliary function, L is an auxiliary linear operator, $u_0(r,t)$ is an initial guess of $u(r,t)$, $u(r,t:p)$ is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when p = 0 and p =1, it holds

$$\varphi(r,t;0) = u_0(r,t) \ , \qquad \varphi(x,t;1) = u(r,t), \tag{3}$$

respectively. Thus, as p increases from 0 to 1, the solution $\phi(x,t;p)$ varies from the initial guesses $u_0(r,t)$ to the solution $u(r,t)$. Expanding $\phi(x,t;p)$ in Taylor series with respect to p, we have

$$\phi(r,t;p) = u_0(r,t) + \sum_{m=1}^{+\infty} u_m(r,t)\, p^m, \tag{4}$$

where

$$u_m(r,t) = \frac{1}{m!}\frac{\partial^m \phi(r,t;p)}{\partial p^m}\Big|_{p=0}. \tag{5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (4) converges at $p = 1$, then we have

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{+\infty} u_m(r,t). \tag{6}$$

Define the vector

$$\overrightarrow{u_n} = \left\{u_0\,,\, u_1, ..., u_n\right\}.$$

Differentiating equation (2) m times with respect to the embedding parameter p and then setting p =0 and finally dividing them by $m!$, we obtain the m^{th} order deformation equation

$$L[u_m - \chi_m u_{m-1}] = hH(r,t)\Re_m(\overrightarrow{u_{m-1}}), \tag{7}$$

where

$$\Re_m(\overrightarrow{u_{m-1}}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(r,t;p)]}{\partial p^{m-1}},$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{9}$$

Applying L^{-1} on both side of equation (7), we get

$$u_m(r,t) = \chi_m u_{m-1}(r,t) + h L^{-1}[H(r,t)\Re_m(\overrightarrow{x_{m-1}})]. \tag{10}$$

In this way, it is easily to obtain u_m for $m \ge 1$, at M^{th} order, we have

$$u(x,t) = \sum_{m=0}^{M} u_m(x,t). \tag{11}$$

When $M \to +\infty$, we get an accurate approximation of the original equation (1). For the convergence of the above method we refer the reader to Liao (2003). If equation (1) admits unique solution, then this method will produce the unique solution. If equation (1) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

3. Applying HAM

In this section, we apply this method for solving the nonlinear gas dynamic equation.

Example

Consider the homogeneous differential [Evans (2002)]

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}-u(1-u)=0; \quad 0\le x\le 1, \qquad t>0, \tag{12}$$

with initial conditions $u(x,0)=ae^{-x}$.

To solve the equation (12) by means of homotopy analysis method, according to the initial conditions denoted in equation (12), it is natural to choose

$$u_0(x,t)=ae^{-x}. \tag{13}$$

We choose the linear operator

$$L[\phi(x,t;p)]=\frac{\partial\phi(x,t;p)}{\partial t}, \tag{14}$$

with the property $L[c]=0$. Where c is constant. We now define a nonlinear operator as

$$N[\phi(x,t;p)]=\phi_t(x,t;p)+\phi(x,t;p)\,\phi_x(x,t;p)-\phi(x,t;p)(1-\phi(x,t;p)). \tag{15}$$

Using above definition, with assumption $H(x,t)=1$. We construct the *zeroth* order deformation equation

$$(1-p)L[\phi(r,t;p)-u_0(r,t)]=p\hbar H(r,t)N[\phi(r,t;p)],$$

obviously, when $p=0$ and $p=1$,

$$\phi(x,t;0)=u_0(x,t), \quad \phi(x,y;1)=u(x,t), \tag{17}$$

Thus, we obtain the mth order deformation equations

$$L[u_m-\chi_m u_{m-1}]=h\Re_m(\vec{u}_{m-1}), \tag{18}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}}{\partial t} + \frac{1}{2}\frac{\partial \sum_{i=0}^{m-1} u_i u_{m-1-i}}{\partial x} + \sum_{i=0}^{m-1} u_i u_{m-1-i} - u_{m-1} .$$

Now, the solution of the m^{th} order deformation equation (18)

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + h L^{-1}[\Re_m(\vec{x}_{m-1})]. \tag{19}$$

Finally, we have

$$u(r,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t).$$

From equations (13) and (19) and subject to initial condition

$$u_m(x,0) = 0 \; , \quad m \ge 1,$$

we obtain

$$\begin{aligned}
&u_0(x,t) = ae^{-x}, \\
&u_1(x,t) = -ae^{-x}ht, \\
&u_2(x,t) = \frac{1}{2}ae^{-x}t^2h^2 - ae^{-x}th^2 - ae^{-x}th, \\
&\vdots
\end{aligned}$$

Hence,

$$\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + \ldots \\
&= ae^{-x}(1 - 4th - 6th^2 + 3t^2h^2 - 4th^3 + 4t^2h^3 - \frac{2t^3h^3}{3} - th^4 + \frac{3t^2h^4}{3} - \frac{t^3h^4}{2} + \frac{t^4h^4}{24} + \ldots).
\end{aligned}$$

When $h = -1$ we have

$$u(x,t) = ae^{-x}(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \cdots) = ae^{-x}\sum_{i=0}^{+\infty}\frac{t^i}{i!} = ae^{t-x},$$

which is the exact solution of equation (12).

4. Conclusion

In this paper, the Homotopy Analysis Method has been applied to study the nonlinear gas dynamic equation. The explicit series solutions gas dynamics equation are obtained, which are the same as those results given by Adomian decomposition method for $h = -1$. This accords with the conclusion that the homotopy analysis method logically contains the Adomian decomposition method in other words the ADM is only a special case of the HAM [Liao (2004c, 2005)]. It is worth pointing out that this method presents a rapid convergence for the solutions. In conclusion, HAM provides accurate numerical solution for nonlinear problems in comparison with other methods. It also does not require large computer memory and discretization of the variables *t* and *x*. The results show that HAM is powerful mathematical tool for solving nonlinear partial differential equations. Mathematica has been used for computations in this paper.

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