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Differential Transform Method for Nonlinear Parabolic-hyperbolic Partial Differential Equations

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Abstract

In the present paper an analytic solution of non-linear parabolic-hyperbolic equations is deduced with the help of the powerful differential transform method (DTM). To illustrate the capability and efficiency of the method four examples for different cases of the equation are solved. The method can easily be applied to many problems and is capable of reducing the size of computational work.

Keywords: Differential transform method; Parabolic-hyperbolic equations.

MSC 2000: 35Mxx

1. Introduction

Since the beginning of 1986, Zhou and Pukhov have developed a so-called differential transformation method (DTM) for electrical circuits problems. The DTM is a technique that uses Taylor series for the solution of differential equations in the form of a polynomial. It is different from the high-order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally tedious for high order equations. The Differential transform method leads to an iterative procedure for obtaining an analytic series solutions of functional equations. In recent years researchers have applied the method to various linear and nonlinear problems such as two point boundary value problems by Chen and Liu(1998), partial differential equations by Jang et al. (2001), differential-algebraic equations by Ayaz (2004), integro-differential equations by Arikoglu and Ozkol (2005), fractional differential equations by Arikoglu and Ozkol (2007), the KdV and mKdV equations by Kangalgil and Ayaz (2009), the Schrödinger equations by Ravi Kanth and Aruna (2009), Analytic solution for Telegraph equation by Biazar and Eslami (2010), Systems of Volterra Integral Equations of the First Kind by Biazar and Eslami (2010), and Approximate analytical solution for the fractional modified KdV by Kurulay and Bayram (2010).

In recent years, special equations of the composite type have received attention in many papers. In this paper, we consider the Cauchy problem for the nonlinear parabolic-hyperbolic equation of the following type

$$\left(\frac{\partial}{\partial t}-\Delta\right)\left(\frac{\partial^2}{\partial t^2}-\Delta\right)u=F(u),$$

with initial conditions

$$\frac{\partial^k u}{\partial t^k}(0,X)=\varphi_k(X),\quad X=(x_1,x_2,\ldots,x_i),\quad k=0,1,2,$$

where the nonlinear term is represented by $F(u)$, and Δ is the Laplace operator in R^n.

2. Basic Idea of Differential Transform Method

The basic definitions and fundamental operations of the two-dimensional differential transform are defined as follows; see Chen and Ho (1999). The differential transform function of a function say $u(x,y)$ is in the following form

$$U(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,y)}{\partial x^k\partial y^h}\right]_{(x_0,y_0)}, \tag{1}$$

where $u(x,y)$ is the original function and $U(k,h)$ is the transformed function.

The differential inverse transform of $U(k,h)$ is defined as

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)(x-x_0)^k(y-y_0)^h, \tag{2}$$

in a real application, and when (x_0,y_0) are taken as $(0,0)$, then the function $u(x,y)$ is expressed by a finite series and Eq. (2) can be written as

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,y)}{\partial x^k\partial y^h}\right]x^k y^h. \tag{3}$$

Equation (3) implies that the concept of two-dimensional differential transform is derived from two-dimensional Taylor series expansion. In this study we use the lower case letters to represent the original functions and upper case letters to stand for the transformed functions (T-functions). From the definitions of Equations (1) and (2), it is readily proved that the transformed functions comply with the following basic mathematical operations.

Similarity an m-dimensional differential transform of $u(x_1,x_2,\ldots,x_m)$ is defined

$$U(k_1,k_2,\ldots,k_m)=\frac{1}{k_1!k_2!\ldots k_m!}\left[\frac{\partial^{k_1+k_2+\ldots+k_m}u(x_1,x_2,\ldots,x_m)}{\partial x_1^{k_1}\partial x_2^{k_2}\ldots\partial x_m^{k_m}}\right]_{(0,0,\ldots,0)}, \tag{4}$$

where $u(x_1,x_2,\ldots,x_m)$ is the original and $U(k_1,k_2,\ldots,k_m)$ is the transformed function. The differential inverse transform of $u(x_1,x_2,\ldots,x_m)$ is defined as follows

$$u(x_1,x_2,\ldots,x_m)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_m=0}^{\infty}U(k_1,k_2,\ldots,k_m)x_1^{k_1}x_2^{k_2}\ldots x_m^{k_m}, \tag{5}$$

and from Equations (4) and (5) can be concluded

$$u(x_1,x_2,\ldots,x_m)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_m=0}^{\infty}\frac{1}{k_1!k_2!\ldots k_m!}\left[\frac{\partial^{k_1+k_2+\ldots+k_m}u(x_1,x_2,\ldots,x_m)}{\partial x_1^{k_1}\partial x_2^{k_2}\ldots\partial x_m^{k_m}}\right]_{(0,0,\ldots,0)}x_1^{k_1}x_2^{k_2}\ldots x_m^{k_m}. \tag{6}$$

From Equations (2) – (6), one can easily prove that the transformed functions comply with the following basic mathematical operations.

1. If $u(x_1, x_2, \ldots x_m) = f_1(x_1, x_2, \ldots x_m) \pm f_2(x_1, x_2, \ldots x_m)$, then

$$U(k_1, k_2, \ldots k_m) = F_1(k_1, k_2, \ldots k_m) \pm F_2(k_1, k_2, \ldots k_m).$$

2. If $u(x_1, x_2, \ldots x_m) = \lambda g(x_1, x_2, \ldots x_m)$, then

$U(k_1, k_2, \ldots k_m) = \lambda G(k_1, k_2, \ldots k_m)$, where, λ is a constant.

3. If $u(x_1, x_2, \ldots x_m) = \dfrac{\partial g\left(x_1, x_2, \ldots x_m\right)}{\partial x_i}$, then

$$U(k, h) = (k_i + 1)G(k_1, \ldots k_i + 1, \ldots k_m), \quad 1 \le i \le m.$$

4. If $u(x, y) = \dfrac{\partial^{r+s} g\left(x_1, x_2, \ldots x_m\right)}{\partial x_i^{\,r} \partial x_j^{\,s}}, \quad 1 \le i \ne j \le m$ then

$$U(k_1, k_2, \ldots k_m) = (k_i + 1)(k_i + 2) + \ldots + (k_i + r)(k_j + 1)(k_j + 2) + \ldots (k_j + s)G(k_1, \ldots k_i + r, \ldots, k_j + s, \ldots k_m).$$

5. If $u(x_1, x_2, \ldots x_m) = x_1^{h_1} x_2^{h_2} \ldots x_m^{h_m}$ then

$U(k_1, k_2, \ldots k_m) = \delta(k_1 - h_1)\delta(k_2 - h_2) \ldots \delta(k_m - h_m)$, where $\delta(k_i - h_i) = \begin{cases} 1, & k_i = h_i, \\ o & otherwise. \end{cases}$

6. If $u(x_1, x_2) = f_1(x_1, x_2) f_2(x_1, x_2)$, then

$$U(k, h) = \sum_{r=0}^{k} \sum_{s=0}^{h} F_1(r, h - s) F_2(k - r, s).$$

7. If $u(x_1, x_2, x_3) = f_1(x_1, x_2.x_3) f_2(x_1, x_2, x_3)$, then

$$U(k_1, k_2, k_3) = \sum_{r=0}^{k_1} \sum_{s=0}^{k_2} \sum_{p=0}^{k_3} F_1(r, k_2 - s, k_3 - p) F_2(k_1 - r, s, p).$$

3. Examples

To illustrate the capability, reliability and simplicity of the method, four examples for different cases of the equation will be discussed here.

Example 1. Consider the following equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u = 2\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2}\right)^2 - 144u, \tag{7}$$

subject to the following initial conditions

$$
\begin{aligned}
&u(0,x) = -x^4, \\
&\frac{\partial u}{\partial t}(0,x) = 0, \\
&\frac{\partial^2 u}{\partial t^2}(0,x) = 0,
\end{aligned}
\tag{8}
$$

with the exact solution

$$u(t,x) = -x^4 + 4t^3.$$

Taking the differential transform of (7), leads to

$$
\begin{aligned}
&(k+1)(k+2)(k+3)U(k+3,h) = (k+1)(h+1)(h+2)U(k+1,h+2) \\
&+(k+1)(k+2)(h+1)(h+2)U(k+2,h+2) - (h+1)(h+2)(h+3)(h+4)U(k,h+4) \\
&+2\sum_{r=0}^{k}\sum_{s=0}^{h}(r+1)(k-r+1)(k-r+2)U(r+1,h-s)U(k-r+2,s) \\
&-\sum_{r=0}^{k}\sum_{s=0}^{h}(h-s+1)(h-s+2)(s+1)(s+2)U(r,h-s+2)U(k-r,s+2) - 144U(k,h).
\end{aligned}
\tag{9}
$$

From the initial conditions given by Equations (8), we have

$$
\begin{aligned}
&U(0,h) = \begin{cases} -1, & h = 4, \\ 0, & h = 0,1,2,3,5,\ldots \end{cases} \\
&U(1,h) = 0, \quad h = 0,1,2,\ldots \\
&U(2,h) = 0, \quad h = 0,1,2,\ldots.
\end{aligned}
\tag{10}
$$

Substituting Equation (10) into Equation (9) and by recursive method, the results are listed as follows

$$U(k,h) = \begin{cases} -1, & \textit{for } k = 0 \textit{ and } h = 4, \\ 4, & \textit{for } k = 3 \textit{ and } h = 0, \\ 0, & \textit{otherwise.} \end{cases}$$

We obtained the series solution as

$$u(t,x) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)t^k x^h = -x^4 + 4t^3,$$

which is an exact solution of the problem given in Equations (7)-(8).

Example 2. Consider the following equation

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)u=(\frac{\partial^2 u}{\partial t^2})^2-(\frac{\partial^2 u}{\partial x^2})^2-2u^2, \tag{11}$$

subject to the initial conditions

$$\begin{aligned} &u(0,x)=e^x,\\ &\frac{\partial u}{\partial t}(0,x)=e^x,\\ &\frac{\partial^2 u}{\partial t^2}(0,x)=e^x, \end{aligned} \tag{12}$$

with the exact solution

$$u(t,x)=e^{x+t}.$$

One can readily find the differential transform of (11), as follows.

$$\begin{aligned} &(k+1)(k+2)(k+3)U(k+3,h)=(k+1)(h+1)(h+2)U(k+1,h+2)\\ &+(k+1)(k+2)(h+1)(h+2)U(k+2,h+2)-(h+1)(h+2)(h+3)(h+4)U(k,h+4)\\ &+\sum_{r=0}^{k}\sum_{s=0}^{h}(r+1)(r+2)(k-r+1)(k-r+2)U(r+2,h-s)U(k-r+2,s)\\ &+\sum_{r=0}^{k}\sum_{s=0}^{h}(h-s+1)(h-s+2)(s+1)(s+2)U(r,h-s+2)U(k-r,s+2)-2\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)U(k-r,s). \end{aligned} \tag{13}$$

The transformed version of Equation (12) is

$$\begin{aligned} &U(0,h)=\frac{1}{h!}, \quad h=0,1,2,\ldots\\ &U(1,h)=\frac{1}{h!}, \quad h=0,1,2,\ldots\\ &U(2,h)=\frac{1}{h!}, \quad h=0,1,2,\ldots. \end{aligned} \tag{14}$$

Substituting (14) in (13), all spectra can be found as

$$U(k,h)=\frac{1}{k!h!}.$$

Substituting $U(k,h)$ into Equation (3), we obtain

$$u(t,x)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}t^k x^h=(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots)(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots),$$

which is the expansion of the function e^{x+t} and is an exact solution of the problem given in Equations (11)-(12).

Example 3. Consider the following equation

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)u=u\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial x}, \tag{15}$$

with initial conditions,

$$\begin{aligned}&u(0,x)=\cos x,\\&\frac{\partial u}{\partial t}(0,x)=-\sin x,\\&\frac{\partial^2 u}{\partial t^2}(0,x)=-\cos x.\end{aligned} \tag{16}$$

Taking the differential transform leads to

$$\begin{aligned}&(k+1)(k+2)(k+3)U(k+3,h)=\\&(k+1)(h+1)(h+2)U(k+1,h+2)+(k+1)(k+2)(h+1)(h+2)U(k+2,h+2)\\&-(h+1)(h+2)(h+3)(h+4)U(k,h+4)+\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,h-s)U(k-r+1,s)\\&+\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)(k-r+2)(h-s+1)U(k-r+2,s)U(r,h-s+1).\end{aligned} \tag{17}$$

From the initial conditions (17) we can write

$$\begin{aligned}&U(0,h)=\begin{cases}\dfrac{(-1)^{\frac{h}{2}}}{h!}, & for\ h\ is\ even,\\ 0, & for\ h\ is\ odd,\end{cases}\\&U(1,h)=\begin{cases}0, & for\ h\ is\ even,\\ -\dfrac{(-1)^{\frac{h-1}{2}}}{h!}, & for\ h\ is\ odd,\end{cases}\\&U(2,h)=\begin{cases}\dfrac{(-1)^{\frac{h}{2}+1}}{h!}, & for\ h\ is\ even,\\ 0, & for\ h\ is\ odd.\end{cases}\end{aligned} \tag{18}$$

Substituting (18) in (17), all spectra can be found as

$$U(k,h)=\begin{cases}\dfrac{(-1)^{\frac{k+h}{2}}}{k!h!}, & \textit{for } k \textit{ is even and } h \textit{ is even},\\ -\dfrac{(-1)^{\frac{k+h-2}{2}}}{k!h!}, & \textit{for } k \textit{ is odd and } h \textit{ is odd},\\ 0, & \textit{otherwise.}\end{cases}$$

Substituting $U(k,h)$ into Equation (3), we have series solution for u and follow closed form solution

$$\begin{aligned}u(t,x)&=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)t^k x^h=\sum_{k=0,2,\ldots}^{\infty}\sum_{h=0,2,\ldots}^{\infty}\frac{(-1)^{\frac{k+h}{2}}}{k!h!}t^k x^h-\sum_{k=1,3,\ldots}^{\infty}\sum_{h=1,3,\ldots}^{\infty}\frac{(-1)^{\frac{k+h-2}{2}}}{k!h!}t^k x^h\\&=k\sum_{k=0,2,\ldots}^{\infty}\frac{1}{k!}(-1)^{\frac{k}{2}}t^k\sum_{h=0,2,\ldots}^{\infty}\frac{1}{h!}(-1)^{\frac{h}{2}}x^h-\sum_{k=1,3,\ldots}^{\infty}\frac{1}{k!}(-1)^{\frac{k-1}{2}}t^k\sum_{h=1,3,\ldots}^{\infty}\frac{1}{h!}(-1)^{\frac{h-1}{2}}t^h\\&=\cos t\cos x-\sin t\sin x,\end{aligned}$$

which is an exact solution.

Example 4. Consider the following equation

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}\right)\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}\right)u=(\frac{\partial u}{\partial t})^2-4u^2, \tag{19}$$

with initial conditions,

$$\begin{aligned}&u(0,x_1,x_2)=\sinh(x_1+x_2),\\&\frac{\partial u}{\partial t}(0,x_1,x_2)=2\sinh(x_1+x_2),\\&\frac{\partial^2 u}{\partial t^2}(0,x_1,x_2)=4\sinh(x_1+x_2).\end{aligned} \tag{20}$$

and exact solution

$$u(t,x_1,x_2)=\sinh(x_1+x_2)e^t.$$

Taking the differential transform of (19), leads to

$$
\begin{aligned}
&(k+1)(k+2)(k+3)U(k+3,h,m)=(k+1)(h+1)(h+2)U(k+1,h+2,m)\\
&+(k+1)(m+1)(m+2)U(k+1,h,m+2)+(k+1)(k+2)(h+1)(h+2)U(k+2,h+2,m)\\
&-(h+1)(h+2)(h+3)(h+4)U(k,h+4,m)-2(h+1)(h+2)(m+1)(m+2)U(k,h+2,m+2)\\
&+(k+1)(k+2)(m+1)(m+2)U(k+2,h,m+2)-(m+1)(m+2)(m+3)(m+4)U(k,h,m+4)\\
&-\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(m-p+1)(m-p+2)(p+1)(p+2)U(r,h-s,m-p+2)U(k-r,s,p+2)\\
&-4\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}U(r,h-s,m-p)U(k-r,s,p).
\end{aligned}
\tag{21}
$$

From the initial conditions given by Equation (20)

$$
U(0,h,m)=\begin{cases}\dfrac{1}{h!m!}, & \textit{for } h \textit{ is odd and } m \textit{ is even},\\ \dfrac{1}{h!m!}, & \textit{for } h \textit{ is even and } m \textit{ is odd},\\ 0 & \textit{otherwise},\end{cases}
$$

$$
U(1,h,m)=\begin{cases}\dfrac{2}{h!m!}, & \textit{for } h \textit{ is odd and } m \textit{ is even},\\ \dfrac{2}{h!m!}, & \textit{for } h \textit{ is even and } m \textit{ is odd},\\ 0, & \textit{otherwise},\end{cases}
$$

$$
U(2,h,m)=\begin{cases}\dfrac{4}{h!m!}, & \textit{for } h \textit{ is odd and } m \textit{ is even},\\ \dfrac{4}{h!m!}, & \textit{for } h \textit{ is even and } m \textit{ is odd},\\ 0, & \textit{otherwise}.\end{cases}
\tag{22}
$$

Substituting (22) in (21), all spectra can be found as follows

$$
U(k,h,m)=\begin{cases}\dfrac{2^k}{k!h!m!}, & \textit{for } k=0,1,2,\ldots, \textit{ h is odd and } m \textit{ is even},\\ \dfrac{2^k}{k!h!m!}, & \textit{for } k=0,1,2,\ldots, \textit{ h is even and } m \textit{ is odd},\\ 0, & \textit{otherwise}.\end{cases}
$$

We obtain a series solution for u and rearranging this solution yield the closed form solution as follows

$$u(t,x_1,x_2)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}U(k,h)t^k x_1^h x_2^m=\sum_{k=0,1,\ldots}^{\infty}\sum_{h=1,3,\ldots}^{\infty}\sum_{m=0,2,\ldots}^{\infty}\frac{2^k}{k!h!m!}t^k x_1^h x_2^m+\sum_{k=0,1,\ldots}^{\infty}\sum_{h=0,2,\ldots}^{\infty}\sum_{m=1,3,\ldots}^{\infty}\frac{2^k}{k!h!m!}t^k x_1^h x_2^m$$

$$=\sum_{k=0,1,\ldots}^{\infty}\frac{2^k}{k!}t^k\sum_{h=1,3,\ldots}^{\infty}\frac{1}{h!}x_1^h\sum_{m=0,2,\ldots}^{\infty}\frac{1}{m!}x_2^m+\sum_{k=0,1,\ldots}^{\infty}\frac{2^k}{k!}t^k\sum_{h=0,2,\ldots}^{\infty}\frac{1}{h!}x_1^h\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m!}x_2^m$$

$$=\sinh(x_1+x_2)e^{2t},$$

which is an exact solution.

4. Conclusion

Application of the DTM to parabolic-hyperbolic partial differential equations has been presented successfully. The results show that the differential transform method is a powerful and efficient technique for finding analytic solutions for parabolic-hyperbolic partial differential equations. The results obtained reinforce the claim of high efficiency of the DTM. The Maple Package was used to calculate the series obtained by differential transform method.

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