



On Some Fractional Integral Operators Involving Generalized Gauss Hypergeometric Functions

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Abstract

The object of this paper is to give a generalization of Gauss hypergeometric function, and to investigate its basic properties. Further, we define some fractional integral operators and their inverses in terms of the Mellin transform. Several well known integral operators, including Saigo operators can be derived from the results established here.

Keywords: Fractional operators; Hypergeometric functions; Mellin transform

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1. Generalized Gauss Hypergeometric Function

The important role played by special functions, particularly the hypergeometric function in solving numerous problems of Mathematical Physics, Engineering and Applied Mathematics is well known [see, for instance, Kilbas et al. (2004), Kiryakova (1994), Mathai et al. (1973),

Samko et al. (1993) and Virchenko et al. (2001)]. This fact has inspired the investigation of several generalizations of hypergeometric functions. With this objective in mind, we introduce a generalization of the Gauss's hypergeometric function in the following form:

$${}_r\tilde{F}(a, b; c; z) = {}_r\tilde{F} \equiv \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} {}_1\Phi_1^{\tau, \beta} \left(\alpha; \gamma; -\frac{r}{t(1-t)} \right) dt, \tag{1}$$

where $\operatorname{Re} c > \operatorname{Re} b > 0$, $r > 0$; $r = 0$, $|z| < 1$; $\operatorname{Re} \gamma > \operatorname{Re} \alpha > 0$, $\tau, \beta \in \mathfrak{R}$ $\tau > 0$, $\tau - \beta < 1$, $B(\dots)$ is the beta function [Abramowitz et al. (1972)].

The function ${}_1\Phi_1^{\tau, \beta}(a; c; z)$ denotes the generalized confluent hypergeometric (introduced by Virchenko (2006)) as follows:

$${}_1\Phi_1^{\tau, \beta}(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} {}_1\Psi_1 \left[\begin{matrix} (c; \tau) \\ (c; \beta) \end{matrix} \middle| zt^\tau \right] dt, \tag{2}$$

where ${}_1\Psi_1(z)$ is the particular case of the so-called generalized Wright hypergeometric function ${}_p\Psi_q(z)$ (introduced by Wright (1935)) defined for complex a_i, b_j and real A_i, B_j ($i = \overline{1, p}$, $j = \overline{1, q}$) by the series

$${}_p\Psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} \middle| z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(a_i + A_i n)}{\prod_{j=1}^q \Gamma(b_j + B_j n)} \cdot \frac{z^n}{n!}. \tag{3}$$

In a series of papers, Wright (1935, 1940) also investigates its asymptotic behavior for the large values of argument z under the condition

$$\sum_{j=1}^q B_j - \sum_{i=1}^p A_i > -1. \tag{4}$$

Certain basic properties of ${}_p\Psi_q(z)$ were also studied in Kilbas et al. (2002). In particular, it was proved that ${}_p\Psi_q(z)$ is an entire function of z , under condition (4).

Further, for $\beta = \tau$, generalized Gauss hypergeometric function given by (1) has the following next form

$${}_r\tilde{F}(a, b; c; z) = {}_r\tilde{F} \equiv \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} {}_1\Phi_1^\tau \left(\alpha; \gamma; -\frac{r}{t(1-t)} \right) dt. \quad (5)$$

In the next sections, we prove certain basic properties (Theorems 1-6, below) for the generalized Gauss's hypergeometric function ${}_r\tilde{F}(z)$ and we also define some fractional integral operators and their inverses. Corresponding results for the Gauss hypergeometric function and integral operators of Kalla & Saxena (1969) can be easily derived from general results established here.

2. Basic Properties of ${}_r\tilde{F}(z)$

As indicated above, this section consists of six theorems which provide some properties of this function. The first theorem deals with a series representation, while the second gives us some differential properties.

Theorem 1. *The series representation of ${}_r\tilde{F}(z)$.* Under the valid conditions of existence of the function ${}_r\tilde{F}(z)$ (see (1)), the following formula holds true,

$${}_r\tilde{F}(a, b; c; z) = \frac{1}{B(b, c-b)} \sum_{n=0}^{\infty} (a)_n {}_{\tau, \beta}B_\alpha^\gamma(b+n, c-b; r) \frac{z^n}{n!}, \quad (6)$$

where $(a)_n$ is the Pochhammer symbol and ${}_{\tau, \beta}B_\alpha^\gamma(\dots)$ is the (τ, β) -generalized beta-function defined by [see Virchenko (2006)]:

$${}_{\tau, \beta}B_\alpha^\gamma(x, y; r, \delta, \omega) = \int_0^1 t^{x-1} (1-t)^{y-1} {}_1\Phi_1^{\tau, \beta} \left(\alpha; \gamma; -\frac{r}{t^\delta (1-t)^\omega} \right) dt, \quad (7)$$

here $\operatorname{Re} x > 0$, $\operatorname{Re} y > 0$, $\tau > 0$, $\beta > 0$; $\delta > 0$, $\omega > 0$ and ${}_1\Phi_1^{\tau, \beta}$ is given by (2).

Theorem 2. *The differential relations for ${}_r\tilde{F}(z)$.* With due regard to convergence, the function ${}_r\tilde{F}(a, b; c; z)$ satisfies the following differential formulas:

$$\frac{d^n {}_r\tilde{F}(a, b; c; z)}{dz^n} = \frac{(a)_n (b)_n}{(c)_n} {}_r\tilde{F}(a+n, b+n; c+n; z), \quad (8)$$

$$\frac{d^n}{dz^n} (z^{a+n-1} \tilde{F}(a, b; c; z)) = (a)_n z^{a-1} {}_r\tilde{F}(a+n, b; c; z), \quad (9)$$

$$z \frac{d}{dz} {}_r\tilde{F}(a, b; c; z) = a \left({}_r\tilde{F}(a+1, b; c; z) - {}_r\tilde{F}(a, b; c; z) \right). \quad (10)$$

Some recurrence formulae, addition and multiplication properties are given in the following theorems:

Theorem 3. The functional relations for ${}_r\tilde{F}(z)$. Under the conditions of existence of the function ${}_r\tilde{F}(z)$ the next functional relations are valid:

$$b {}_r\tilde{F}(a, b+1; c; z) + (c-b-1) {}_r\tilde{F}(a, b; c; z) = (c-1) {}_r\tilde{F}(a, b; c-1; z), \quad (11)$$

$$c {}_r\tilde{F}(a, b; c; z) - (c-b) {}_r\tilde{F}(a, b; c+1; z) = b {}_r\tilde{F}(a, b+1; c+1; z). \quad (12)$$

Theorem 4. The addition and multiplication theorem. If the conditions of existence of the function ${}_r\tilde{F}(z)$ are valid, then the following functional relations hold

$${}_r\tilde{F}(a, b; c; x+y) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{y^n}{n!} {}_r\tilde{F}(a+n, b+n; c+n; x), \quad (13)$$

$${}_r\tilde{F}(a, b; c; xy) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n (1-y)^n}{n!} {}_r\tilde{F}(a+n, b+n; c+n; x). \quad (14)$$

The following two theorems deals with integral representations and transformation formulae for ${}_r\tilde{F}(z)$.

Theorem 5. Integral representations of ${}_r\tilde{F}(z)$. For the valid conditions of existence of the function ${}_r\tilde{F}(z)$, we have the following integral representations of ${}_r\tilde{F}(a, b; c; z)$:

$${}_r\tilde{F}(a, b; c; z) = \frac{2^{c-b}}{B(b, c-b)(1+z)^a} \int_1^{\infty} \frac{(t-1)^{b-1} (t+1)^{a-c}}{\left(1 - \frac{z-1}{z+1}t\right)^a} {}_1\Phi_1^{\tau, \beta} \left(\alpha; \gamma; -\frac{r(t+1)^2}{2(t-1)} \right) dt, \quad (15)$$

$${}_r\tilde{F}(a, b; c; z) = \frac{2}{B(b, c-b)} \int_0^{\frac{\pi}{2}} \frac{(\sin \omega)^{2b-1} (\cos \omega)^{2c-2b-1}}{(1-z \sin^2 \omega)^a} {}_1\Phi_1^{\tau, \beta} \left(\alpha; \gamma; -\frac{4r}{\sin^2 2\omega} \right) d\omega, \quad (16)$$

$${}_r\tilde{F}(a, b; c; z) = \frac{2^{b-a}}{\mathbf{B}(b, c-b)} \int_0^\infty \frac{(sh\theta)^{2a-2c+1} \cdot (ch\theta-1)^{2c-a-b-1}}{\left(\frac{1}{2}-z+\frac{1}{2}ch\theta\right)^a} {}_1\Phi_1^{\tau, \beta}\left(\alpha; \gamma; -\frac{r(ch\theta-1)^{-3}}{2sh^{-4}\theta}\right) d\theta, \quad (17)$$

$${}_r\tilde{F}(a, b; c; z) = \frac{2^{c-b}}{\mathbf{B}(b, c-b)} \int_0^\infty \frac{(sh\theta)^{2b-1} (ch\theta+1)^{a-c-b+1}}{(1+z+ch\theta(1-z))^a} {}_1\Phi_1^{\tau, \beta}\left(\alpha; \gamma; -\frac{r(ch\theta+1)^3}{2sh^2\theta}\right) d\theta. \quad (18)$$

$sh\theta$ and $ch\theta$ stand for usual hyperbolic sine and cosine, respectively.

Theorem 6. Transformations for ${}_r\tilde{F}(z)$. Under the conditions of existence of the functions ${}_r\tilde{F}(a, b; c; z)$ the following generalized relations of Kummer' types are valid:

$${}_r\tilde{F}(a, b; a+b+1-c; 1-z) = z^{-a} {}_r\tilde{F}(a, a-c+1; a+b-c+1; 1-z^{-1}), \quad (19)$$

$${}_r\tilde{F}(a+1-c, b+1-c; 2-c; z) = (1-z)^{c-a-1} {}_r\tilde{F}\left(a+1-c, 1-b; 2-c; \frac{z}{z-1}\right), \quad (20)$$

$${}_r\tilde{F}(c-a, c-b; c+1-a-b; 1-z) = z^{a-c} {}_r\tilde{F}(c-a, 1-a; c+1-a-b; 1-z^{-1}), \quad (21)$$

$${}_r\tilde{F}(a, b; c; z) = (1-z)^{-a} {}_r\tilde{F}\left(a, c-b; c; \frac{z}{z-1}\right). \quad (22)$$

Proofs.

We observe that the Theorems 1-3 can be easily derived, using definitions (1) & (7), and simple transformations. To prove the results (13) and (14), we use formula (8) of Theorem 2, and then Taylor theorem about analytical functions [Kilbas et al. (2004)].

Proof of the Theorem 5:

Using main integral representation (1) of ${}_r\tilde{F}(a, b; c; z)$ and substituting $t = \frac{y-1}{y+1}$, $t = \sin^2 \omega$, $t = \frac{2}{ch\theta+1}$, $t = \frac{ch\theta-1}{ch\theta+1}$, we get respectively formulae (15), (16), (17) and (18).

Proof of the Theorem 6:

Simple transformations lead to results (19) to (22). For example, let us prove (19) and (22). We make the following transformations:

$$\begin{aligned} {}_r\tilde{F}(a, b; a+b+1-c; 1-z) &= \frac{1}{\mathbf{B}(b, a-c+1)} \int_0^1 t^{b-1} (1-t)^{a-c} (1-(1-z)t)^{-a} {}_1\Phi_1^{\tau, \beta} \left(\alpha; \gamma; -\frac{r}{t(1-t)} \right) dt \\ &= \frac{1}{\mathbf{B}(b, a-c+1)} \int_0^1 (1-y)^{b-1} y^{a-c} (1-(1-z)(1-y))^{-a} {}_1\Phi_1^{\tau, \beta} \left(\alpha; \gamma; -\frac{r}{(1-y)y} \right) dy \\ &= z^{-a} {}_r\tilde{F}(a, a-c+1; a+b+1-c; 1-z^{-1}). \end{aligned}$$

Similarly, we have

$$\begin{aligned} {}_r\tilde{F}(a, b; c; z) &= \frac{1}{\mathbf{B}(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} {}_1\Phi_1^{\tau, \beta} \left(\alpha; \gamma; -\frac{r}{t(1-t)} \right) dt \\ &= \frac{1}{\mathbf{B}(b, c-b)} \int_0^1 (1-y)^{b-1} y^{c-b-1} (1-z(1-y))^{-a} {}_1\Phi_1^{\tau, \beta} \left(\alpha; \gamma; -\frac{r}{(1-y)y} \right) dy \\ &= (1-z)^{-a} {}_r\tilde{F} \left(a, c-b; c; \frac{z}{z-1} \right). \end{aligned}$$

This proves the result (22).

3. The Integral Operators with ${}_r\tilde{F}(z)$.

Riemann-Liouville operators of fractional integration and differentiation are basic in the theory of fractional calculus. Several authors, like Love, Saxena, Kalla & Saxena (1969), Saigo, McBride etc. [see, Kiryakova (1994)] have defined different forms of hypergeometric operators of fractional integration. Kalla (1969) has defined integral operators involving Fox H-function. In this part of the paper, we define some fractional integral operators involving the function ${}_r\tilde{F}(z)$ given by equation (5) and their inverses in terms of the Mellin transform. The corresponding results for hypergeometric operators mentioned above can be derived from general formulas established here.

Let us introduce the following four integral operators:

$$\left({}_r\tilde{I}_x^{a,b,c} f \right)(x) = \frac{(x+d)^{-a}}{\Gamma(a+c)} \int_0^x (x-t)^{a+c-1} {}_r\tilde{F} \left(a, -b-c; a-c; \frac{x-t}{x+d} \right) f(t) dt, \quad (23)$$

where $a > 0, c > 0$; b and d are real, the function $f(x)$ is continuous on (σ, ∞) ;

$$\left({}_r \tilde{B}_{c,d,\lambda}^{a,b} f\right)(x) = x^\lambda \int_1^\infty (xt)^{b+d-1} {}_r \tilde{F}(a,b;c;-xt) f(t) dt, \quad (0 < a < 1, c > 0); \quad (24)$$

$$\left({}_r \tilde{I}_x^{a,b,c,d} f\right)(x) = \frac{x^{-a-b-2d}}{\Gamma(a)} \int_0^x t^d (x-t)^{a-1} {}_r \tilde{F}\left(a+b+d, -a-c; -a; 1-\frac{t}{x}\right) f(t) dt, \\ (a > 0, d > -1; b, c \text{ are real}); \quad (25)$$

$$\left({}_r \tilde{\Psi}_{\mu,\nu,s} f\right)(x) = \frac{x^{\frac{\mu}{2}}}{\Gamma(1+\mu)} \int_0^\infty t^{\frac{1}{2}\mu-1} (x+t)^{-\mu} {}_r \tilde{F}\left(-\nu, \nu-s+1; \mu-2s+1; \frac{1}{2}-\frac{x}{2t}\right) f(t) dt. \quad (26)$$

It is interesting to observe that for $r=0$, the operator given by (25) yields to the Saigo's fractional integral operator [see Kiryakova (1994)].

The Mellin and Laplace transforms are used to obtain the inverse operators. The operators defined by (23) to (26), give the following respective inverse operators

$$\left({}_r \tilde{I}_x^{a,b,c} f\right)^{-1} = K_1^{-1} \tilde{I}_x^{-a,b+a+c,-c} M \left\{ {}_r \tilde{I}_x^{a,b,c} f, c \right\}, \quad (27)$$

where $M\{\cdot\}$ denotes the Mellin integral transform and

$$K_1 = \frac{\Gamma(\gamma)\Gamma(c)\Gamma(\alpha-c\tau)\Gamma(a+b+c)\Gamma(-b)\Gamma(a-c)}{\Gamma(-b-c)\Gamma(a+b)\Gamma(\alpha)\Gamma(\gamma-c\tau)\Gamma(a+c)}. \quad (28)$$

The inverse operator for (24) is as follows:

$$\left({}_r \tilde{B}_{c,d,\lambda}^{a,b} f\right)^{-1} = x^{1-b-d} K_2^{-1} D^{1-a} (1+x) \lim_{n \rightarrow \infty} L_n h(x), \quad (29)$$

where

$$h(x) = (1-x)(r(x) - r(1)),$$

$$r(x) = C^{-1} x^{\frac{2}{p}-a} U \left\{ t^{1-b-d} \frac{d}{dt} \left\{ t^{\frac{2}{p}} I_{c-b+d-\lambda, 1-\frac{2}{p}-\lambda}^{1-c-d+b} M \left\{ {}_r \tilde{B}_{c,d,\lambda}^{a,b} f, d \right\} (t) \right\} \right\} (x),$$

$$Uf(x) = x^{-\frac{2}{p}} f\left(\frac{1}{x}\right),$$

$$L_n f(x) = \frac{(-x)^{n-1}}{n!(n-2)!} \frac{d^{2n-1}}{dx^{2n-1}} \{x^n f(x)\}, \quad (n = 2, 3, 4, \dots),$$

$$K_2 = \frac{\Gamma(\gamma)\Gamma(d)\Gamma(\alpha-d\tau)\Gamma(c-b+d)\Gamma(b+d)\Gamma(a)}{B(b,c-b)\Gamma(\alpha)\Gamma(\gamma-d\tau)\Gamma(c+2d)}.$$

Now we consider the inverse operator for (25),

$$\left({}_r \tilde{I}_x^{a,b,c,d} f \right)^{-1} = K_3^{-1} \left(I_x^{a,b,c,d} \right)^{-1} M \left\{ \left({}_r \tilde{I}_x^{a,b,c,d} f \right) (r), a \right\}, \tag{30}$$

where

$$\left(I_x^{a,b,c,d} f \right) (x) = \frac{x^{-a-b-2d}}{\Gamma(a)} \int_0^x t^d (x-t)^{a-1} {}_2F_1 \left(a+b+d, -c; a; 1-\frac{t}{x} \right) f(t) dt, \tag{31}$$

$$K_3 = \frac{\Gamma(\gamma)\Gamma(\alpha-a\tau)\Gamma(a+c)\Gamma(-c)\Gamma(-a)}{\Gamma(\gamma)\Gamma(\gamma-a\tau)\Gamma(-a-c)\Gamma(c)\Gamma(a)}.$$

Finally, the inverse operator for (26) is derived as follows:

$$\left({}_r \tilde{\Psi}_{\mu,\nu,s} f \right)^{-1} = K_4^{-1} \left(T_{\frac{1}{2}^{\mu+\nu+1}}^{-1} V^{-1} W_{\frac{1}{4}^{\mu+\frac{1}{2}\nu+\frac{1}{4},-\nu-1}} V T_{\frac{1}{2}^{\mu+1}}^{-1} M \left\{ \left({}_r \tilde{\Psi}_{\mu,\nu,s} f \right) (r), s \right\} \right) (x), \tag{32}$$

where

$$T_\alpha^{-1} f(x) = x^{1-\alpha} L^{-1} \left(t^{1-\alpha} f(t), t \rightarrow x \right), \tag{33}$$

L^{-1} is the inverse Laplace' transform, $W_{\eta,\alpha}^{-1}f = W_{\eta+\alpha,-\alpha}f$ and

$W_{\eta,\alpha}f$ is the Erdelyi-Kober fractional integral operator, [see Kiryakova (1994), pp.65 - 66]

$$Vf(x) = 2^{-\frac{1}{2}}x^{-\frac{1}{2}}f\left(x^{\frac{1}{2}}\right), \quad V^{-1}f(x) = 2^{\frac{1}{2}}x^{\frac{1}{2}}f(x^2),$$

$$K_4 = \frac{\Gamma(\gamma)\Gamma(s)\Gamma(\alpha-s\tau)\Gamma(\nu+1)\cdot 2^{-\nu-1}}{\Gamma(\alpha)\Gamma(\gamma-s\tau)\Gamma(\mu+1)B(\nu-s+1,\mu-\nu-s)\Gamma(\mu+\nu+1)}.$$

The above inversion formulae can be derived easily. The representation of integral operator ${}_r\tilde{\Psi}_{\mu,\nu,s}$ in the following form can be used to prove the result (32).

$$\left({}_r\tilde{\Psi}_{\mu,\nu,s}f\right)(x) = K_5M^{-1}\left\{T_{\frac{\mu}{2}}I_{\frac{\mu}{2}-\nu-1,\nu+1}T_{\frac{\mu}{2}+\nu+1}f,r\right\}, \quad (34)$$

where

$$T_\alpha f(x) = \int_0^\infty (xt)^{\alpha-1} e^{-xt} f(t) dt,$$

$$I_{\eta,\alpha}f(x) = \frac{2}{\Gamma(\alpha)} x^\eta \int_x^\infty (t^2 - x^2)^{\alpha-1} t^{-\eta-2\alpha+1} f(t) dt, \quad (35)$$

and M^{-1} is the inverse Mellin transform.

Several special cases can be derived from these general results. Inversion formula for Saigo's operator can be derived from (31).

4. Discussions

We have given a generalization of Gauss' hypergeometric function and derived some of its properties. This function is used to define new operators of fractional integration. Their inverse operators are obtained by an appeal to the Laplace and Mellin transforms. Further, the generalized Gauss's hypergeometric function and its properties established in this paper are of general forms and may be used to define new density functions in probability theory. These operators of fractional integration can be used to solve certain fractional integrodifferential equations.

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REFERENCES

- Abramowitz, M., I. and Stegun, I.A. (1972). Handbook of Mathematical Functions with, *Formulas, Graphs and Mathematical Tables*. Dover, New York.
- Kalla, S.L. and Saxena, R.K. (1969). Integral operators involving hypergeometric functions. *Mathematische Zeitschrift*. Vol. 108, 231—234.
- Kalla, S.L. (1969), Integral operators involving Fox's H-function. *Acta Mexicana de Ciencias y Tecnologia*, Vol. 3, 117-122.
- Kilbas, A.A. and Saigo, M. (2004). *H-Transforms*, Chapman and Hall/CRC Press, Boca Raton, FL.
- Kilbas, A.A., Saigo, M and Trujillo, J.J. (2002). On the generalized Wright function, *Fractional Calculus and Applied Analysis*. Vol.5, 437-460.
- Kiryakova, V. (1994). *Generalized Fractional Calculus and Applications*, Wiley & Sons. Inc., New York.
- Mathai, A. M., and Saxena, R.K. (1973). *Generalized Hypergeometric Functions With Applications in Statistics and Physical Sciences*, Springer-Verlag, Berlin.
- Samko, S.G., Kilbas, A.A. and Marichev, O.I. (1993). *Fractional Integrals and Derivatives-Theory and Applications*, Gordon and Breach Science Publishers, New York.
- Virchenko, N. Kalla, S.L. and Al Zamel, A. (2001). Some results on a generalized hypergeometric function, *Integral Transforms and Special Functions*. Vol. 12, 89-100.
- Virchenko, N. (2006). On the generalized confluent hypergeometric function and its applications. *Fractional Calculus Applied Analysis*. Vol.9, 101-108.
- Wright, E.M. (1935). The asymptotic expansion of the generalized hypergeometric function. *Journal of the London Mathematical Society*. Vol.10, 286-293.
- Wright, E.M. (1940). The asymptotic expansion of the generalized hypergeometric function. *Journal of the London Mathematical Society* (2) .Vol. 46, 389-408.