



Available at
<http://pvamu.edu/aam>
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied
Mathematics:
An International Journal
(AAM)

Vol. 05, Issue 2 (December 2010), pp. 282 – 291
(Previously, Vol. 05, Issue 10, pp. 1379 – 1388)

Concomitants of Upper Record Statistics for Bivariate Pseudo–Weibull Distribution

Muhammad Ahsanullah
Department of Management Sciences
Rider University
Lawrenceville, New Jersey, U.S.A
ahsan@rider.edu

Saman Shahbaz, Muhammad Qaiser Shahbaz and Muhammad Mohsin
Department of Mathematics
COMSATS Institute of Information Technology
Lahore, Pakistan
samans@ciitlahore.edu.pk; qshahbaz@gmail.com; mohsinshahid@yahoo.com

Received: October 27, 2009; Accepted: September 9, 2010

Abstract

In this paper the Bivariate Pseudo–Weibull distribution has been defined as a compound distribution of two random variables to model the failure rate of component reliability. The distribution of r -th concomitant and joint distribution of r -th and s -th concomitant of record statistics of the resulting distribution have been derived. Single and product moments alongside the correlation coefficient have also been obtained. Recurrence relation for the single moments has also been obtained for the resulting distributions.

Keywords: Concomitants; Record values; Pseudo–Weibull distribution

MSC (2000) No.: 60E05, 62E15

1. Introduction

The reliability of a component in multicomponent system plays very important role for efficient working of the system. Distributions having longer right tail can be used to nicely model the failure time of components. Several probability distributions have been in use to model this sort of scenario. Some popular probability distributions include Exponential, Weibull and Log-normal. Different versions of Bivariate Exponential and Bivariate Weibull distribution have also been available in literature which can be used for modeling of two component system. Filus and Filus (2001) introduced pseudo distributions as alternate models for modeling component reliability. Chandler (1952) introduced records as sequence of random variables such that random variable at k -th place is larger (smaller) than variable at $(k-1)^{th}$ place. He called random variables as upper (lower) records in a random sample of size n from some probability distribution $f(x)$. After the introduction of the field, number of people jumped into this area of statistics. Shorrock (1973) has thoroughly discussed about record values and record time in a sequence of random variables. Ahsanullah (1979) has characterized the exponential distribution by using the record values. Ahsanullah (1991) has also derived the distributional properties of records by using the Lomax distribution. Balakrishnan et al. (1995) have obtained the recurrence relations for moments of record values for Gumbel distribution. Some moment properties of the records have been given by Ahsanullah (1992). Sufficient material is now available in the field of records that can be found in Arnold et al. (1998). Some more characterizations of the probability distributions via records have been obtained by Balakrishnan and Balasubramanian (1995). Ahsanullah (1995) and Nevzorov (2001) have given a comprehensive review of the mathematical foundation of the records.

Ahsanullah (1995) has given the distribution of k -th upper record, $X_{U(k)}$, as:

$$f_{k:n}(x_k) = \frac{1}{\Gamma(k)} f(x_k) [R(x_k)]^{k-1}, \quad (1)$$

where $R(x) = -\ln\{1 - F(x)\}$ and $F(x)$ is distribution function of X .

The joint distribution of k -th and m -th upper records, $X_{U(k)}$ and $X_{U(m)}$, can be obtained by using the following expression given by Ahsanullah (1995):

$$f_{k,m:n}(x_k, x_m) = \frac{1}{\Gamma(k)\Gamma(m-k-1)} r(x_k) f(x_m) [R(x_k)]^{k-1} [R(x_m) - R(x_k)]^{m-k-1}, \quad (2)$$

with

$$r(x) = R'(x).$$

The concomitants in case of record statistics have not been extensively studied as compared with the concomitants of order statistics. This branch is relatively new one in the field of ordered random variables. The distribution of concomitant of record values can be obtained by using the following expression given in Ahsanullah (1995):

$$f(y_k) = \int_{-\infty}^{\infty} f(y|x) f_{k:n}(x_k) dx. \quad (3)$$

where $f_{k:n}(x_k)$ is the distribution of k -th upper record given in (1).

Ahsanullah (1995) has also given joint distribution of concomitants of k -th and m -th record values as:

$$f(y_k, y_m) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_m} f(y_k|x_k) f(y_m|x_m) f_{k,m:n}(x_k, x_m) dx_k dx_m, \quad (4)$$

where $f_{k,m:n}(x_k, x_m)$ is given in (2).

In this paper we have obtained the distribution of the concomitants of upper record statistics for Bivariate Pseudo-Weibull distribution. The organization of the paper is as follows: The Bivariate Pseudo-Weibull distribution is defined in section 2, distribution of r -th concomitant for Bivariate Pseudo-Weibull distribution with properties has been studied in section 3, joint distribution of two concomitants is given in section 4 and the recurrence relation for single moments is presented in section 5 of the paper.

2. Bivariate Pseudo-Weibull Distribution

The pseudo distributions are relatively new addition to probability distributions. Fillis and Fillis (2001 and 2006) have introduced the Pseudo-Normal and Pseudo-Gamma distributions as linear combinations of the normal and gamma random variables. We define the Bivariate Pseudo-Weibull distribution following the lines of Shahbaz and Ahmad (2009) as under:

Let random variable X has a Weibull distribution with parameter β and γ_1 . The density function of X is:

$$f(x; \beta, \gamma_1) = \beta \gamma_1 x^{\gamma_1 - 1} \exp(-\beta x^{\gamma_1}), \beta > 0, \gamma_1 > 0, x > 0. \quad (5)$$

Further, let another random variable Y has the Weibull distribution with parameter $\phi(x)$ and γ_2 , where $\phi(x)$ is some function of random variable X . The density function of Y is:

$$f\{y; \phi(x), \gamma_2 | x\} = \phi(x) \gamma_2 y^{\gamma_2 - 1} \exp\{-\phi(x) y^{\gamma_2}\}; \phi(x) > 0, \gamma_2 > 0, y > 0. \quad (6)$$

The Bivariate Pseudo-Weibull distribution is defined as the compound distribution of (5) and (6). The density function of the Bivariate Pseudo-Weibull distribution is given as:

$$f(x, y) = \beta \phi(x) \gamma_1 \gamma_2 x^{\gamma_1 - 1} y^{\gamma_2 - 1} \exp\left[-\{\beta x^{\gamma_1} + \phi(x) y^{\gamma_2}\}\right]; \quad (7)$$

$$\gamma_1 > 0, \gamma_2 > 0, \beta > 0, \phi(x) > 0, x > 0, y > 0.$$

The density (7) can be used to generate several distributions depending upon various choices of function $\phi(x)$. Shahbaz and Ahmad (2009) studied distribution (7) by using $\phi(x) = 1$ for which (7) is simply equal to product to two marginal densities. We have used $\phi(x) = x^{\gamma_1}$ in (7) to obtain following Bivariate Pseudo–Weibull distribution:

$$f(x, y) = \beta \gamma_1 \gamma_2 x^{2\gamma_1 - 1} y^{\gamma_2 - 1} \exp\left[-x^{\gamma_1} (\beta + y^{\gamma_2})\right]; \beta > 0, \gamma_1 > 0, \gamma_2 > 0, x > 0, y > 0. \quad (8)$$

In following section, distribution of concomitant of record statistics has been derived for (8).

3. Distribution of k -th Concomitant and Its Properties

The Bivariate Pseudo–Weibull distribution has been given in (7) and (8). In this section the distribution of k -th concomitants of record statistics for Bivariate Pseudo–Weibull distribution, given in (8), has been obtained.

The distribution of the k -th concomitants of record statistics can be obtained by using (3). The distribution given in (3) requires the distribution of k -th record statistics for random variable X , given below:

For the random variable X we have $R(x) = -\ln\{1 - F(x)\} = \beta x^{\gamma_1}$ and so the distribution of k -th record statistics for random variable X is:

$$f_{k:n}(x_k) = \frac{\gamma_1}{\Gamma(k)} \beta^k x^{k\gamma_1 - 1} \exp(-\beta x^{\gamma_1}); \beta > 0, \gamma_1 > 0, x > 0. \quad (9)$$

The conditional distribution, $f(y|x)$, from (6) is:

$$f(y|x) = \gamma_2 x^{\gamma_1} y^{\gamma_2 - 1} \exp(-x^{\gamma_1} y^{\gamma_2}), \gamma_1 > 0, \gamma_2 > 0, x > 0, y > 0; \quad (10)$$

Using (9) and (10) in (3), the distribution of k -th concomitant of record statistic for Bivariate Pseudo–Weibull distribution is:

$$h(y) = \frac{\gamma_1 \gamma_2 \beta^k}{\Gamma(k)} y^{\gamma_2 - 1} \int_0^\infty x^{k\gamma_1 + \gamma_1 - 1} \exp\left\{-x^{\gamma_1} (\beta + y^{\gamma_2})\right\} dx; \quad (11)$$

Integrating (11), the probability distribution of k -th concomitant of record statistics is given as:

$$h(y) = \frac{k \gamma_2 \beta^k y^{\gamma_2 - 1}}{(\beta + y^{\gamma_2})^{k+1}}, \quad k, \beta, \gamma_2, y > 0. \quad (12)$$

The distribution function for (12) is:

$$F(y) = 1 - \frac{\beta^k}{(\beta + y^{\gamma_2})^k}, \quad k, \beta, \gamma_2 > 0, y > 0. \quad (13)$$

By using (13), the p -th percentile of the distribution is:

$$\xi_p = \left[\left\{ \beta^k / (1-p) \right\}^{1/k} - \beta \right]^{1/\gamma_2}. \quad (14)$$

The median can be easily obtained by using $p = 0.5$ in (14). The mode can be obtained from (12) as:

$$y = \left\{ \frac{\beta(\gamma_2 - 1)}{k\gamma_2 + 1} \right\}^{1/\gamma_2}. \quad (15)$$

Also by using (12) and (13), the hazard rate function of the distribution is:

$$r(y) = \frac{k\gamma_2 y^{\gamma_2 - 1}}{\beta + y^{\gamma_2}}, \quad k, \beta, \gamma_2, y > 0. \quad (16)$$

The r -th moment of the distribution given in (12) is obtained as:

$$\mu'_r = E(y^r) = \int_0^\infty y^r h(y) dy = \int_0^\infty y^r \frac{k\gamma_2 \beta^k y^{\gamma_2 - 1}}{(\beta + y^{\gamma_2})^{k+1}} dy. \quad (17)$$

Simplifying (17), the r -th moment of the concomitant is:

$$\mu'_r = E(y^r) = \frac{\beta^r \Gamma(k - r/\gamma_2) \Gamma(r/\gamma_2 + 1)}{\Gamma(k)}, \quad r < k\gamma_2. \quad (18)$$

The mean and the variance of the concomitants of record statistics for Pseudo-Weibull distribution can be obtained by using (18). Table 1 and Table 2 shows numerical values of mean and the variance of the concomitants of record statistics for Pseudo-Weibull distribution for $k = 5$ and for different values of β and γ_2 .

Table 1. Mean of the concomitants of record statistics for Pseudo–Weibull distribution for $k = 5$.

β	γ_2					
	0.5	0.6	0.7	0.8	0.9	1.0
0.5	0.08333	0.08708	0.09491	0.10440	0.11459	0.12500
0.6	0.10000	0.10450	0.11389	0.12528	0.13751	0.15000
0.7	0.11667	0.12192	0.13287	0.14616	0.16043	0.17500
0.8	0.13333	0.13933	0.15185	0.16704	0.18335	0.20000
0.9	0.15000	0.15675	0.17083	0.18792	0.20627	0.22500
1.0	0.16667	0.17416	0.18981	0.20880	0.22919	0.25000

Table 2. Variance of the concomitants of record statistics for Pseudo–Weibull distribution for $k = 5$.

β	γ_2					
	0.5	0.6	0.7	0.8	0.9	1.0
0.5	0.24306	0.07950	0.04700	0.03512	0.02936	0.02604
0.6	0.35000	0.11448	0.06768	0.05057	0.04227	0.03750
0.7	0.47639	0.15582	0.09212	0.06883	0.05754	0.05104
0.8	0.62222	0.20352	0.12032	0.08991	0.07515	0.06667
0.9	0.78750	0.25758	0.15227	0.11379	0.09512	0.08438
1.0	0.97222	0.31800	0.18799	0.14048	0.11743	0.10417

4. Joint Distribution of the Concomitants and Moments

In this section we have derived the joint distribution of concomitants of record statistics for Pseudo–Weibull distribution given in (8). The expression (4) can be used to obtain the joint distribution of two concomitant of record statistics. From (4) we can see that the distribution given in (4) requires the joint distribution of two record statistics given in (2). The joint distribution (2) for random variables $X_1 = X_{U(k)}$ and $X_2 = X_{U(m)}$ for parent Weibull distribution is given as:

$$f(x_1, x_2) = \frac{1}{\Gamma(k)\Gamma(m-k)} (\beta x_1^{\gamma_1})^{k-1} (\beta x_2^{\gamma_1} - \beta x_1^{\gamma_1})^{m-k-1} \beta \gamma_1 x_1^{\gamma_1-1} \beta \gamma_1 x_2^{\gamma_1-1} \exp(-\beta x_2^{\gamma_1}).$$

After simplification, the joint distribution of two record statistics for Weibull distribution is:

$$f(x_1, x_2) = \frac{\beta^m \gamma_1^2}{\Gamma(k)\Gamma(m-k)} x_1^{k\gamma_1-1} x_2^{\gamma_1-1} (x_2^{\gamma_1} - x_1^{\gamma_1})^{m-k-1} \exp(-\beta x_2^{\gamma_1}). \tag{19}$$

Using (19) and conditional distribution of Y given X in (4), the joint distribution of k -th and m -th concomitant is obtained as under:

$$g(y_1, y_2) = \frac{\gamma_1^2 \gamma_2^2 \beta^m}{\Gamma(k) \Gamma(m-k)} \int_0^\infty \int_0^{x_2} \left[y_1^{\gamma_1-1} y_2^{\gamma_2-1} x_1^{k\gamma_1+\gamma_1-1} x_2^{2\gamma_1-1} (x_2^{\gamma_1} - x_1^{\gamma_1})^{m-k-1} \right. \\ \left. \exp(-x_1^{\gamma_1} y_1^{\gamma_2}) \exp\{-x_2^{\gamma_1} (\beta + y_2^{\gamma_2})\} \right] dx_1 dx_2.$$

$$g(y_1, y_2) = \frac{\gamma_1^2 \gamma_2^2 \beta^m}{\Gamma(k) \Gamma(m-k)} y_1^{\gamma_1-1} y_2^{\gamma_2-1} \int_0^\infty x_2^{2\gamma_1-1} \exp\{-x_2^{\gamma_1} (\beta + y_2^{\gamma_2})\} \\ \left[\int_0^{x_2} x_1^{k\gamma_1+\gamma_1-1} (x_2^{\gamma_1} - x_1^{\gamma_1})^{m-k-1} \exp(-x_1^{\gamma_1} y_1^{\gamma_2}) dx_1 \right] dx_2.$$

or

$$g(y_1, y_2) = \frac{\gamma_1^2 \gamma_2^2 \beta^m}{\Gamma(k) \Gamma(m-k)} y_1^{\gamma_1-1} y_2^{\gamma_2-1} \int_0^\infty x_2^{2\gamma_1-1} \exp\{-x_2^{\gamma_1} (\beta + y_2^{\gamma_2})\} I(x_2) dx_2, \quad (20)$$

where

$$I(x_2) = \int_0^{x_2} x_1^{k\gamma_1+\gamma_1-1} (x_2^{\gamma_1} - x_1^{\gamma_1})^{m-k-1} \exp(-x_1^{\gamma_1} y_1^{\gamma_2}) dx_1. \quad (21)$$

Simplifying (21), we have:

$$I(x_2) = \frac{\Gamma(k+1) \Gamma(m-k)}{\gamma_1 \Gamma(m+1)} (x_2^{\gamma_1})^m {}_1F_1(k+1, m+1, -x_2^{\gamma_1} y_1^{\gamma_2}), \quad (22)$$

where ${}_1F_1(a, b; x)$ is Kummer confluent Hypergeometric function, defined as:

$${}_1F_1(a, b; x) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{j!} x^j.$$

Using (22) in (20) and simplifying we get:

$$g(y_1, y_2) = k \gamma_2^2 \beta^m y_1^{\gamma_2-1} y_2^{\gamma_2-1} (\beta + y_2^{\gamma_2})^{k-m-1} (y_1^{\gamma_2} + y_2^{\gamma_2} + \beta)^{-(k+2)} \\ \left[(m+1)(y_1^{\gamma_2} + y_2^{\gamma_2} + \beta) - (k+1)y_1^{\gamma_2} \right], \quad (23)$$

where $Y_1 = Y_k$ and $Y_2 = Y_m$.

The product moments for distribution (23) are obtained below:

The q -th and r -th moments of two concomitants is defined as:

$$\mu'_{q,r} = E(y_1^q, y_2^r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_1^q y_2^r g(y_1, y_2) dy_1 dy_2; \tag{24}$$

Using (23) in (24) we have:

$$\begin{aligned} \mu'_{q,r} = E(y_1^q, y_2^r) = & \int_0^{\infty} \int_0^{\infty} \left[y_1^q y_2^r k \gamma_2^2 \beta^m y_1^{\gamma_2-1} y_2^{\gamma_2-1} (\beta + y_2^{\gamma_2})^{k-m-1} \right. \\ & \left. (y_1^{\gamma_2} + y_2^{\gamma_2} + \beta)^{-(k+2)} \{ (m+1)(y_1^{\gamma_2} + y_2^{\gamma_2} + \beta) - (k+1)y_1^{\gamma_2} \} \right] dy_1 dy_2. \end{aligned}$$

Simplifying above equation, the product moments are:

$$\begin{aligned} \mu'_{q,r} = & \beta^{m+(q+r-m\gamma_2)/\gamma_2} \left\{ (m+1)(k - q/\gamma_2 - 1) \left(\frac{m\gamma_2 - q - r}{\gamma_2} - 1 \right) - \beta(k+1)(q/\gamma_2 + 1) \right\} \\ & \left\{ \frac{\Gamma(k - q/\gamma_2 - 1)\Gamma(q/\gamma_2 + 1)\Gamma(r/\gamma_2 + 1)\Gamma\{(m\gamma_2 - q - r)/\gamma_2 - 1\}}{\Gamma(k)\Gamma(m - q/\gamma_2 + 1)} \right\}. \end{aligned} \tag{25}$$

The covariance and correlation can be obtained by using (25). Table 3 shows some numerical values of covariance for $k = 5, m = 6$ and for different values of β and γ_2 .

Table 3: Covariance of the concomitants of record statistics for Pseudo-Weibull distribution for $k = 5, m = 6$.

β	γ_2					
	0.5	0.6	0.7	0.8	0.9	1.0
0.5	0.00056	0.00170	0.00188	0.00187	0.00169	0.00134
0.6	-0.00026	0.00261	0.00289	0.00276	0.00242	0.00188
0.7	-0.00310	0.00352	0.00404	0.00379	0.00324	0.00249
0.8	-0.00977	0.00418	0.00525	0.00490	0.00414	0.00316
0.9	-0.02286	0.00421	0.00641	0.00605	0.00508	0.00386
1.0	-0.04583	0.00317	0.00738	0.00717	0.00603	0.00458

5. Recurrence Relation for Single Moments

It is easy to see from (12) and (13) that

$$\bar{H}_k(y) = 1 - H_k(y) = h_k(y) \frac{\beta + y^{\gamma_2}}{k\gamma_2 y^{\gamma_2-1}}. \tag{26}$$

The relation in (26) will be used to establish recurrence relations for moments of the concomitant of the k -th record statistic for Pseudo- Weibull distribution.

Theorem: For $r < \gamma_2 k$ and r integer

$$\mu_r = \frac{r\beta}{k\gamma_2 - r} \mu_{r-\gamma_2}.$$

Proof:

$$\begin{aligned} \mu_r &= E(Y^r) = \int_0^\infty y^r h_k(y) dy. \\ &= \int_0^\infty r y^{r-1} \bar{H}_k(y) dy. \end{aligned} \quad (27)$$

Substituting (26) in (27) and after simplification, we get

$$= \frac{r\beta}{k\gamma_2} \mu_{r-\gamma_2} + \frac{r}{k\gamma_2} \mu_r.$$

Thus,

$$\mu_r = \frac{r\beta}{k\gamma_2 - r} \mu_{r-\gamma_2};$$

as asserted.

REFERENCES

- Ahsanullah, M. (1979). Characterization of exponential distribution by record values, *Sankhya*, Vol. **41**, B, pp 116–121.
- Ahsanullah, M. (1991). Record values of Lomax distribution, *Statisti. Nederlandica*, Vol. 41, No. 1, pp 21–29.
- Ahsanullah, M. (1992). Record values of independent and identically distributed continuous random variables, *Pak. J. Statist.* Vol. 8, No. 2, pp 9–34.
- Ahsanullah, M. (1995). *Record Statistics*, Nova Science Publishers, USA.
- Arnold, B. C., Balakrishnan, N. and Nagaraja, H. N. (1992). *A First Course in Order Statistics*, John Wiley, New York.
- Balakrishnan, N. and Balasubramanian, K. (1995). A characterization of geometric distribution based on record values, *J. Appl. Statist. Science*, Vol. 2, No. 1, pp 73–87.

- Balakrishnan, N., Ahsanullah, M. and Chan, P. S. (1995). Relations for single and product moments of record values from Gumbel distribution, *Stat. and Prob. Lett.*, Vol. 15, No. 3, pp 223–227.
- Candler, K. N. (1952). The distribution and frequency of record values, *J. Roy. Statist. Soc.*, Vol.14, B, pp 220–228.
- Filus, J.K. and Filus, L.Z. (2001). On some Bivariate Pseudo normal densities. *Pak. J. Statist.* Vol. 17, No. 1, pp 1-19.
- Filus, J.K. and Filus, L.Z. (2006). On some new classes of Multivariate Probability Distributions. *Pak. J. Statist.* Vol. 22, No. 1, pp 21–42.
- Nevzorov, V. B., (2001). *Record: Mathematical Theory*, Translations of Mathematical Monographs, Vol. 194, American Mathematical Society.
- Shahbaz, S. and Ahmad (2009). Concomitants of order statistics for Bivariate Pseudo–Weibull distribution, *World App. Sci. J.*, Vol. 6, No. 10, pp 1409–1412.
- Shorrock, R. W. (1973). Record values and inter–record times, *J. Appl. Prob.*, Vol. 10, No. 3, pp 543–555.