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On Submoduloids of a Moduloid on Nexus

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Abstract

In this paper, the submoduloid of a moduloid on nexus that is generated by a subset, cyclic submoduloid and bounded sets are defined and the properties of structures on it are investigated. Also, the fractions of a moduloid on nexus are defined and shown to be isomorphic with a moduloid on nexus.

Keywords: $\mathbb{N}^{\infty}$-submoduloid; cyclic; $\mathbb{N}^{\infty}$-submoduloid; bounded sets; fractions of an $\mathbb{N}^{\infty}$-moduloid

MSC 2010 No.: 16Y60, 16U20

1. Introduction

The roots of the concepts of 'formex' (plural: formices) and 'plenix' (plural: plenices) go back to the nineteen seventies. At the time, an extensive research programme was led by H. Nooshin in the Spatial Structures Research Centre of the University of Surrey, with the aim of finding convenient ways of generating data for analysis and design of Spatial Structures consisting of many thousands of elements. The geometry of such structural systems often involves many types of symmetries that, when using, can simplify the generation of information. However, in addition to the geometric information, it is necessary to produce information about the properties of material(s) of the elements, positions and particulars of the supports and the nature and magnitudes of the external loads. Also, the information about the external loads should include the details of dead weights, snow loads, wind effects, earthquake forces, temperature changes and so on.

The starting point was the introduction of the concepts of ’formex algebra’ (Nooshin (1975); Nooshin (1984)). These concepts are used for the algebraic representation and processing of all types of geometric forms and, in particular, structural configurations (Nooshin-Disney (2000); Nooshin-Disney (2001); Nooshin-Disney (2002)). Subsequently, a software called ‘Formian’ was introduced, which provides a convenient medium for the use of formex algebra (Nooshin et al. (1993)). However, the concepts of formex algebra are general and can be used in many different fields.

Several years later, in order to be able to conveniently handle the vast amount of varied data that defines a Spatial Structure, a sophisticated form of database was evolved which called a ’plenix' (Haristchain (1980); Hee (1985)). A plenix is capable of containing any type of information either in explicit constant form or in a generic form, that is, as a parametric formulation. The term ’plenix’ comes from the Latin word ’plenus’ which is meaningfull. This choice was a reflection of intention of a plenix for being capable of representing the `full spectrum' of mathematical objects. However, these pioneering works were mainly concerned with plenices as data structures. The generic nature of a plenix as a database places the concept in a class of its own with capabilities that are far beyond any normal database. Fundamentally, a plenix is a mathematical object consisting an arrangement of a mathematical object. A plenix is like a tree structure in which every branch is a mathematical object. For instance, Figure 1 shows a graphic representation of a plenix, consisting a sequence of elements such that each of them consists asequence of elements and so on. The graphical representation of plenix Q in Figure 1 is referred to as the dendrogram of Q. The following construction is another way to represent plenix Q.

$$Q = <<<5,\ 3,\ FALSE>,\ 0>,\ 1,\ <<>,\ 8>, [-2, 1, 7]> .$$

In this plenix, the first and the third elements themselves are plenices. A dendrogram of Q is shown in Figure 1.

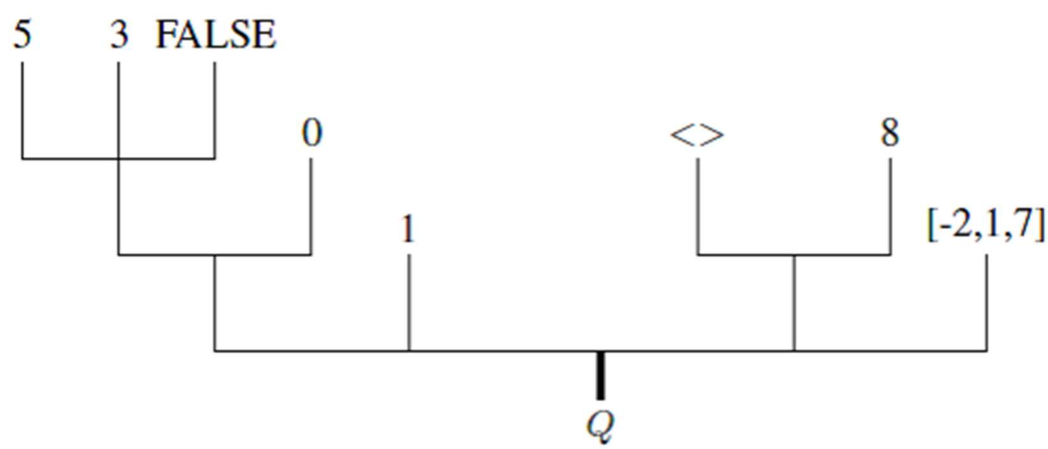

Figure 1. Dendrogram of Q

In the early two thousands, the basic idea of a plenix was further developed as a mathematical object by Bolourian (Bolourian (2009)) and Bolourian and (Bolourian-Nooshin (2004)). The aim of the research was to create an algebra based on plenices in order to define meaningful relations, operations and functions for plenices and to investigate the properties of the resulting algebra. This work turned the concept of a plenix into a proper mathematical system with the potential for applications in many branches of human knowledge.

Every panel of a plenix may be associated with a sequence of positive integers that indicates the position of a panel in the plenix. This sequence of positive integers is referred to as the address of the panel. For instance, consider the plenix Q, Figure 1, the panels addresses are given in the Table 1.

An address (i, j, k) refers to the k^{th} principal panel of the j^{th} principal panel of the i^{th} principal panel of the plenix. For example, the address of 8 is $(3, 2)$, indicating the 2^{th} principal panel of the $3rd$ principal panel of the plenix.

Table 1. Addresses of plenix of Q

Panel	Address
<< 5; 3; FALSE >; 0 >	(1)
1	(2)
<<>; 8 >	(3)
< [-2; 1; 7] >	(4)
< 5; 3; FALSE >	(1, 1)
0	(1, 2)
<>	(3, 1)
8	(3, 2)
5	(1, 1, 1)
3	(1, 1, 2)
FALSE	(1, 1, 3)

The set of addresses for all panels of a plenix is called the 'address set' of that plenix. For instance, the set

$$A=\{(1), (2), (3), (4), (1, 1), (1, 2), (3, 1), (3, 2), (1, 1, 1), (1, 1, 2), (1, 1, 3)\},$$

is the address set of plenix Q. Therefore, the address set represents the constitution of a plenix. The constitution of a plenix plays an important role in the theory of plenices. As a result, one of the interesting domains for research in plenix theory is the constitution of a plenix, irrespective of the values of its primion panel. The mathematical object that represents the constitution of a plenix is called a 'nexus'. The notion of a nexus was introduced in (Bolourian, 2009) and a nexus was defined axiomatically by using the concept of the address set. Also, an interesting fact about a relationship between plenix and nexus was shown and it is about the concept of a plenix which was defined via the concept of a nexus.

The idea of 'nexus algebra' is another important mathematical structure that has come out of Bolourian's work (Bolourian (2009)). The concept of the nexus, as an abstract algebraic structure, is certainly worthy of attention. Bolourian (Bolourian (2009)) investigated the properties of nexuses from the view point of pure mathematics. Many familiar concepts from an abstract algebra such as substructures, cyclic substructures, generators of an algebra, homomorphism of an algebra, direct product and direct sum of an algebra, metric space, prime and maximal substructures, decomposition theorem and so on, were studied deeply within the context of nexus algebra. (Afkhami et al. (2011); Afkhami et al. (2012); Bolourian (2009); Estaji-Estaji(2015); Estaji et al. (2015); Hedayati-Asadi (2014); Saeidi-Hasankhani (2011)). This means that nexus algebra has great potential as an algebraic structure.

In (Bolourian et al. (2019)) the structure of a moduloid was defined on a nexus and some basic results were investigated. The concept of a submoduloid was introduced and some interesting facts about the submoduloid such that every cyclic subnexus of the nexus N is a submoduloid of N, were proven. Furthermore, a homomorphism between two moduloids was defined and some of these results were also investigated.

In this paper, the $\mathbb{N}^{\infty}$-submoduloid of an $\mathbb{N}^{\infty}$-moduloid generated by a subset X of an $\mathbb{N}^{\infty}$-moduloid and cyclic $\mathbb{N}^{\infty}$-submoduloid are defined and the properties of the structures are investigated. Also, the notions of bounded sets and fractions of an $\mathbb{N}^{\infty}$-moduloid are defined and some of these results are investigated. For instance, every bounded set of $\mathbb{N}^{\infty}$-moduloid N is an $\mathbb{N}^{\infty}$-submoduloid of N (Theorem 3.15) and, we can count the number of addresses in B_a of an $\mathbb{N}^{\infty}$-moduloid (Theorem 3.22) and the intersection $B_a \cap B_b$ of two sets B_a and B_b is the bounded set $B_{a\times b}$ (Theorem 3.23).

The paper is organized as follows. In Section 2, we first state the basic definitions and elementary needed properties. Moreover, examples are also provided. In section 3, we state and prove the fundamental theorems related to $\mathbb{N}^{\infty}$-submoduloids. The corresponding results for $\mathbb{N}^{\infty}$-submoduloids are given in Sections 3 and 4.

2. Preliminaries

In this section, we review some existing definitions and results for the sake of completeness and reference.

Definition 2.1.

A groupoid is a set that is closed under a binary operation. A semigroup G is a groupoid with a binary operation o, which satisfies the associative property,

$$(aob)oc = ao(boc),$$

for all $a,\ b,\ c \in G$.

A monoid G is a semigroup containing an identity element. A semiring is a set R with two operations $+$ and o, such that $(R,+)$ is a commutative monoid and (R,o) is a semigroup. The operation o is distributive with respect to $+$, that is,

$$ao(b+c) = aob + aoc,$$
$$(b+c)oa = boa + coa,$$

for all $a,\ b,\ c \in R$. Also, for any $a \in R,\ ao0 = 0oa = 0$, where 0 is the identity element of the monoid $(R,+)$.

Definition 2.2.

A moduloid N over the semiring R consists of a commutative groupoid $(M,+)$ with the identity element and the scalar multiplication $\cdot : R \times M \to M$, which maps $(r,a) \to ra.$ Also, for all r and s in R, and a in M, the following equations are valid,

$$(i)\ (r + s)a = ra + sa,$$

$$(ii)\ r(a + b) = ra + rb,$$

$$(iii)\ (rs)a = r(sa),$$

$$(iv)\ 0a = r0 = 0.$$

Definition 2.3.

(i) Let $\mathbb{N}^*$ be the set of non-negative integers. Then an address is a sequence whose elements belong to $\mathbb{N}^*$. Also, $a_k = 0$ implies that $a_i = 0$, for all $i \geq k$. The sequence of zero is called the empty address and is denoted by $()$. In other words, every non-empty address is of the form $(a_1,\ a_2,\ ...,\ a_n,\ 0,\ 0,\ ...)$, where a_i and n belong to $\mathbb{N}$. Hereafter, this address will be denoted by $\left(a_1,\ a_2,\ ...,\ a_n\right)$.

(ii) A nexus N is a non-empty set of address with the following properties:

$$(a_1,\ a_2,\ ...,\ a_{n-1},\ a_n) \in N \Rightarrow (a_1,\ a_2, ...,\ a_{n-1},\ t) \in N,\ \forall\ 0 \leq t \leq a_n, \tag{1}$$

and for an infinite nexus

$$\{a_i\}_{i=1}^{\infty} \in N,\ a_i \in \mathbb{N} \Rightarrow \forall n \in \mathbb{N},\ \forall 0 \le t \le a_n,\ (a_1,\ a_2,\ ...,\ a_n - t) \in N. \tag{2}$$

Note that condition (2) does imply condition (1).

Definition 2.4.

Let N be a nexus. A subset S of N is called a subnexus of N provided that S itself is a nexus.

Definition 2.5.

Let N be a nexus and $\varnothing \ne A \subseteq N$. Then the smallest subnexus of N containing A is called the subnexus of N generated by A and is denoted by $< A >$. If $A = \{a_1,\ a_2,\ ...,\ a_n\}$, then instead of $< A >$ one can write $< a_1,\ a_2,\ ...,\ a_n >$. If A has only one element a, then the subnexus $< a >$ is called a cyclic subnexus of N. It is clear that $()$ and N is trivial subnexuses of the nexus N.

Theorem 2.6

Let N be a nexus and $\phi \ne A \subseteq N$, and a be an address in N. Then,

(i) $< A >= \{b \in N : \exists a \in A, b \le a\}$.

(ii) $< a >= \{b \in N : b \le a\}$.

Theorem 2.7.

Let N be the set of addresses. Then, N is a nexus if and only if $\phi \ne N$ and for every $a \in N, b \le a$ implies that $b \in N$.

Definition 2.8.

Let N be a nexus and $a \in N$. The level of a is said to be:

(i) n, if $a = (a_1,\ a_2,\ ...,\ a_n)$, for some $a_n \in N$,

(ii) ∞, if a is an infinite sequence of N,

(iii) 0, if $a = ()$.

The level of a is denoted by $l(a)$.

Definition 2.9.

Let $a=\{a_i\}$ and $b=\{b_i\}$, $i \in \mathbb{N}$, be two addresses. Then, $a \leq b$ if $l(a)=0$ or if one of the following cases is satisfied:

(i) If $l(a)=1$, that is $a=(a_1)$, for some $a_1 \in \mathbb{N}$ and $a_1 \leq b_1$.

(ii) If $1<l(a)<\infty$, then $l(a) \leq l(b)$ and $a_{l(a)} \leq b_{l(b)}$ and for any $1 \leq i < l(a), a_i = b_i$.

(iii) If $l(a)=\infty$, then $a=b$.

Definition 2.10.

Let N be a nexus and let $a=(a_1,\ a_2,\ ...,\ a_k)$ be an address of N. The set

$$\{(a_1,\ a_2,\ ...,\ a_k,\ a_{k+1},\ ...,\ a_n) \in N : a_{k+i} \in \mathbb{N} \quad for\ i=1,\ 2,\ ...,\ n-k\},$$

is called the 'panel' of a and is denoted by q_a. In other words, if $a=(a_1,\ a_2,\ ...,\ a_k)$, then every address b of N is an address in q_a provided that the first k terms of b are the same as the corresponding terms of a. Note that, the 'panel' of a does not include a. Also, q_0 includes all the addresses of N except for the empty address itself.

Definition 2.11.

Let N be a nexus and let a be an address of N. The set $\{b \in N : a \leq b\}$ is called the quasi panel of a and is denoted by Q_a.

Example 2.12.

Consider the nexus

$$N=\{(),\ (1),\ (2),\ (1,\ 1),\ (1,\ 2),\ (2,\ 1),\ (2,\ 2),\ (2,\ 3),\ (2,\ 2,\ 1),\ (2,\ 2,\ 2),\ (2,\ 3,\ 1),\ (2,\ 3,\ 2)\}.$$

Now, consider the address $a=(2,\ 2)$, of N. Then, $q_a=\{(2,\ 2,\ 1),\ (2,\ 2,\ 2)\}$ and

$$Q_a=\{(2,\ 2),\ (2,\ 2,\ 1),\ (2,\ 2,\ 2),\ (2,\ 3),\ (2,\ 3,\ 1),\ (2,\ 3,\ 2)\}.$$

Definition 2.13.

Let N be a nexus and let $a=(a_1,\ a_2,\ ...)$ and $b=(b_1,\ b_2,\ ...)$ be two addresses of N. Now, the operation $+$ is defined on N as follows: Suppose that there exists a k such that

$$(a_1 \vee b_1,\ a_2 \vee b_2,\ \ldots,\ a_k \vee b_k) \in N,$$

and

$$(a_1 \vee b_1,\ a_2 \vee b_2,\ \ldots,\ a_{k+1} \vee b_{k+1}) \notin N,$$

then

$$a + b := (a_1 \vee b_1, a_2 \vee b_2,\ \ldots,\ a_k \vee b_k).$$

In this case, one may write $index_N(a+b) = k$. On the other hand, if there is no such k, then

$$a + b := (a_1 \vee b_1, a_2 \vee b_2, \ldots),$$

and we write $index_N(a+b) = \infty$. Note that always $(a_1 \vee b_1) \in N$.

Example 2.14.

Consider the nexus

$$N = \{(),\ (1),\ (2),\ (1,\ 1),\ (1,\ 2),\ (2,\ 1),\ (2,\ 2),\ (2,\ 3), \\ (1,\ 2,\ 1),\ (1,\ 2,\ 2),\ (1,\ 2,\ 3),\ (1,\ 2,\ 4),\ (2,\ 3,\ 1),\ (2,\ 3,\ 2)\}.$$

Suppose that $a = (1,\ 2,\ 4)$ and $b = (2,\ 3,\ 1)$. Then,

$$a + b = (1,\ 2,\ 4) + (2,\ 3,\ 1) = (1 \vee 2,\ 2 \vee 3,\ 4 \vee 1) = (2,\ 3,\ 4).$$

As one can see, $(2,3,4) \notin N$, so by definition of the summation of two addresses of a nexus N, the last component, that is 4, must be eliminated. Since, $(2,\ 3) \in N$ then one may consider $(2,\ 3)$ as the summation of the addresses a and b. In this case, $index_N(a,b) = 2$. As another example, suppose that $a = (1)$ and $b = (2,\ 3,\ 1)$. Since $l(a) = 1$ and $l(b) = 3$, one may write $a = (1,\ 0,\ 0)$. Therefore, $a + b = (1,\ 0,\ 0) + (2,\ 3,\ 1) = (1 \vee 2,\ 0 \vee 3,\ 0 \vee 1) = (2,\ 3,\ 1) = b$.

Definition 2.15.

Let $\mathbb{N}^\infty = \mathbb{N} \cup \{0, \infty\}$, N be a nexus and the scalar multiplication

$$o : \mathbb{N}^\infty \times N \to N$$

is defined on N as follows:

$$row = \begin{cases} (a_1,\ a_2,\ ...,\ a_r), & if\ l(w) > r,\ r > 0, \\ (a_1,\ a_2,\ ...,\ a_n), & if\ l(w) \le r,\ r > 0, \\ 0, & if\ r = 0, \\ w, & if\ r = \infty, \end{cases}$$

for all, $r \in \mathbb{N}^\infty$ and $w = (a_1,\ a_2,\ ...,\ a_n) \in N$. In other words, $row = \{a_i\}_{i=1}^{l(w)\wedge r}$. From now on, operation o is called the dot product.

Theorem 2.16.

Let N be a nexus. Then $(N,+,0)$ is a moduloid over $(\mathbb{N}^\infty,\vee,\wedge,0)$ together with scalar multiplication o. For simplicity, N is called an $\mathbb{N}^\infty$-moduloid.

Definition 2.17.

Let N be an $\mathbb{N}^\infty$-moduloid, S be a non-empty subset of N and $0 \in S$. Then S is called a submoduloid of N, if $(S,+,0)$ is a moduloid over $(\mathbb{N}^\infty,\vee,\wedge,0)$. The set of all $\mathbb{N}^\infty$-submoduloid of N is denoted by $SUB_M(N)$.

Theorem 2.18.

Let S be a non-empty subset of a nexus N. Then,

(i) $S \in SUB_M(N)$ if and only if

(a) $roa \in S, \forall r \in \mathbb{N}^\infty, \forall a \in S$,

(b) $a + b \in S, \forall a, b \in S$.

(ii) If N is a unitary moduloid over $(\mathbb{N}^\infty,\vee,\wedge,0)$ and $S \in SUB_M(N)$, then S is a unitary moduloid over $(\mathbb{N}^\infty,\vee,\wedge,0)$.

Lemma 2.19.

(i) Suppose that N is a nexus, and a and b are two addresses in N. If $a \le b$, then $a + b = b$.

(ii) In a cyclic nexus N, since every two addresses are comparable, then the summation of two addresses is equal to the greater summand.

Theorem 2.20.

Let N be an $\mathbb{N}^{\infty}$-moduloid and $n \in \mathbb{N}$. Consider the subset

$$L(n) = \{a \in N : l(a) \leq n\} = noN,$$

of N. Then $L(n)$ is an $\mathbb{N}^{\infty}$-moduloid of N ($L(n)$ is called n-cut).

3. Some $\mathbb{N}^{\infty}$-submoduloids

Definition 3.1.

Let N be an $\mathbb{N}^{\infty}$-moduloid and $X \subseteq N$. Then the smallest $\mathbb{N}^{\infty}$-submoduloid of N containing X is called the $\mathbb{N}^{\infty}$-submoduliod of N generated by X and denoted by $< X >_m$. If $X = \{a\}$, then the $\mathbb{N}^{\infty}$-submoduloid of N generated by a is called the cyclic $\mathbb{N}^{\infty}$-submoduloid and denoted by $< a >_m$.

Theorem 3.2.

Let N be an $\mathbb{N}^{\infty}$-moduloid and $a = (a_1, a_2, ..., a_n)$ be an address in N. If K is a cyclic $\mathbb{N}^{\infty}$-submoduloid generated by a, that is, $K =< a >_m$, then

$$K = \{(), (a_1), (a_1, a_2), ..., (a_1, a_2, ..., a_n)\}.$$

Furthermore, the number of addresses of K is one more than the level of a, that is, $| K |= l(a)+1$
.

Proof:

Suppose that,

$$A = \{(), (a_1), (a_1, a_2), ..., (a_1, a_2, ..., a_n)\}.$$

First, one must show that A is an $\mathbb{N}^{\infty}$-submoduloid. Since A is a chain, by Lemma 2.19, the summation of two addresses in A is equal to the largest summand. Therefore, A is closed under the $+$ operation. By the structure of elements of A, this set is closed under dot product. Therefore, by Theorem 2.18, A is an $\mathbb{N}^{\infty}$-submoduloid of N and contains a. Now, one must show that $K = A$. Since K contains a and also K is an $\mathbb{N}^{\infty}$-submoduloid of N, then K is closed under dot product for all $r \in \mathbb{N}^{\infty}$. Therefore K contains

$$(), (a_1), (a_1, a_2), ..., (a_1, a_2, ..., a_n).$$

Thus, $A \subseteq K$. By Definition 3.1, K is the smallest $\mathbb{N}^{\infty}$-submoduloid of N containing a. So, $A = K$.

Example 3.3.

Consider the nexus

$$N = \{(),\ (1),\ (2),\ (3),\ (2,\ 1),\ (2,\ 2),\ (2,\ 3),\ (3,\ 1),\ (3,\ 2),\ (2,\ 3,\ 1),\ (2,\ 3,\ 2),\ (2,\ 3,\ 3),\ (2,\ 3,\ 2,\ 1),\ (2,\ 3,\ 3,\ 2)\},$$

and the address $a = (2,\ 3,\ 2,\ 2)$ in N. The dendrogram of N is shown in Figure 2. The subnexus generated by a is

$$< a >= \{(),\ (1),\ (2),\ (2,\ 1),\ (2,\ 2),\ (2,\ 3),\ (2,\ 3,\ 1),\ (2,\ 3,\ 2),\ (2,\ 3,\ 2,\ 1),\ (2,\ 3,\ 2,\ 2)\},$$

and the submoduloid generated by a is

$$< a >_m = \{(),\ (2),\ (2,\ 3),\ (2,\ 3,\ 2),\ (2,\ 3,\ 2,\ 2)\}.$$

The dendrogram of $< a >$ is shown in Figure 3. The addresses of submoduloid generated by a, that is, $< a >_m$ are shown by thick lines. As one can see, $< a >_m \subseteq < a >$ the number of addresses of $< a >_m$ is one more than the level of a, that is,

$$|< a >_m| = l(a) + 1 = 4 + 1 = 5.$$

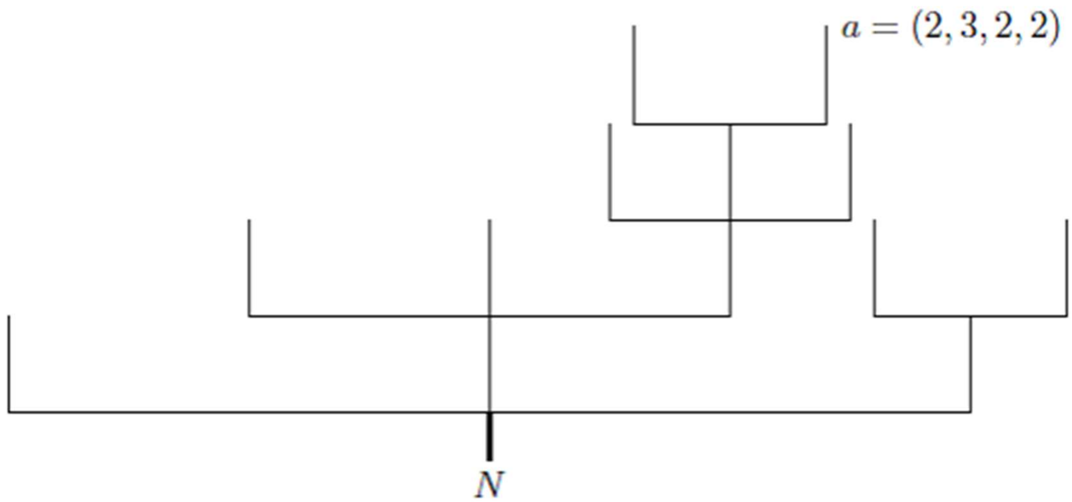

Figure 2. Dendrograms of N

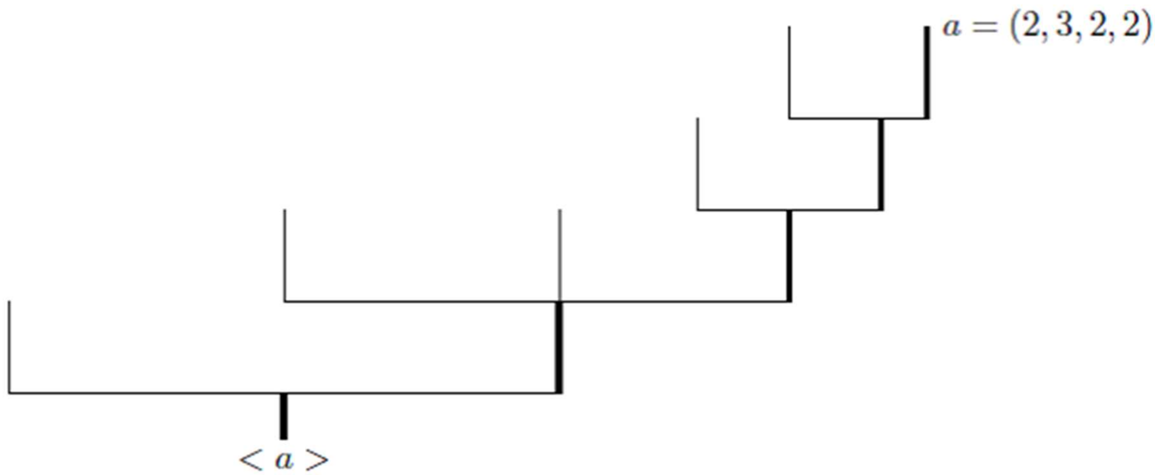

Figure 3. Dendrogram of $< a >$. The addresses of $< a >_m$ are shown by thick lines

Lemma 3.4.

Let N be an $\mathbb{N}^\infty$-moduloid and a be an address in N.

(i) If x and y are two addresses in N such that both of them are greater than or equal to a, then $a \leq x + y$.

(ii) Q_a and q_a are closed under $+$ operation.

Proof:

(i) Suppose that $a = (a_1, a_2, ..., a_n)$. Since x and y are greater than or equal to a, so they are of the forms

$$(a_1, a_2, ..., a_n, b_1, b_2, ..., b_m),$$

or

$$(a_1, a_2, ..., a_n + k, c_1, c_2, ..., c_t).$$

In both cases,

$$x + y = (a_1, a_2, ..., a_n + k, b_1 \vee c_1, b_2 \vee c_2, ..., b_j \vee c_j),$$

where $index_N(x + y) = j$ and k is a non-negative integer. Therefore, $a \leq x + y$.

(ii) By definition of Q_a, all addresses in Q_a are greater than or equal to a, so, by part (i), $x + y$ is greater than or equal to a. Therefore, $x + y \in Q_a$. Now, suppose that $x, y \in q_a$. Therefore,

$$x = (a_1, a_2, ..., a_n, b_1, b_2, ..., b_m), \qquad y = (a_1, a_2, ..., a_n, c_1, c_2, ..., c_t).$$

In this case

$$x+y=(a_1,\ a_2,\ \dots,\ a_n,\ b_1\vee c_1,\ b_2\vee c_2,\ \dots,\ b_j\vee c_j),$$

where $index_N(x+y)=j$. Thus, $x+y\in q_a$.

Definition 3.5.

Let N be a nexus, and let $a=(a_1,\ a_2,\ \dots)$ and $b=(b_1,\ b_2,\ \dots)$ be two addresses of N.

(iii) Defined the operation '$\times$' on N as follows: Suppose that there exists a k such that

$$(a_1\wedge b_1,\ a_2\wedge b_2,\ \dots,\ a_k\wedge b_k)\in N,$$

and

$$(a_1\wedge b_1,\ a_2\wedge b_2,\ \dots,\ a_{k+1}\wedge b_{k+1})\notin N,$$

then

$$a\times b:=(a_1\wedge b_1,\ a_2\wedge b_2,\ \dots,\ a_k\wedge b_k).$$

In this case, one may write $index_N(a\times b)=k$. On the other hand, if there is no such k, then

$$a\times b:=(a_1\wedge b_1,\ a_2\wedge b_2,\ \dots),$$

and we write $index_N(a\times b)=\infty$.

Note that, always $(a_1\wedge b_1)\in N$, and $a\times b=b\times a$, that is, the product of a and b is commutative and $a\times 0=0$.

(ii) Let N be an $\mathbb{N}^\infty$-moduloid and let A and B be two subsets of N. Then,

$$A\times B=\{a\times b : a\in A, b\in B\}.$$

Lemma 3.6.

Let N be a nexus, and a and b are two addresses in N.

(i) If $a\le b$ then $a\times b=a$.

(ii) Suppose that N is a cyclic nexus then, the product of two addresses in N is equal to the smaller one.

Proof:

(i) Suppose that $a=(a_1, a_2, ..., a_n)$. Since $a \leq b$ so b can be considered as $b=(a_1, a_2, ..., a_n+k, b_1, b_2, ..., b_m)$ for some $k \in \mathbb{N}^*$. Therefore $a \times b = a$.

(ii) Since every cyclic nexus is a chain, so, all the addresses are comparable. Now, the proof is the result of the part (i).

Theorem 3.7.

Let N be an $\mathbb{N}^\infty$-moduloid and let K and M be two subsets of N.

(iii) If there is an address a in K and an address b in M such that $a \leq b$, then,

$$(K \times M) \cap K \neq \phi.$$

(iv) If there is an address a in K such that a is greater than or equal to any address in M, then, M is a subset of $K \times M$.

Proof:

(v) Since $a \in K$ and $b \in M$. then $a \times b \in K \times M$. Also, by Lemma 3.6 $b \leq a$ implies that $a \times b = b$. Consequently, $(K \times M) \cap M \neq \phi$.

(ii) Since a is greater than or equal to any address in M then by Lemma 3.6, any address b in M can be written in the form $b = a \times b$. Therefore, for any b in M, $b \in K \times M$, Thus, M is a subset of $K \times M$.

Theorem 3.8.

Let N be an $\mathbb{N}^\infty$-moduloid and let a and b be two addresses in N. Then,

$$<a>_m \times <b>_m = \{x \times y : x \in <a>_m, y \in <b>_m\},$$

is an $\mathbb{N}^\infty$-submoduloid of N.

Proof:

Suppose that, $a=(a_1, a_2, ..., a_n)$ and $b=(b_1, b_2, ..., b_m)$. With the loss of generality, assume that $n \leq m$ and $index_N(a \times b) = k$. Every non-empty address x in $<a>_m$ or y in $<b>_m$ is of the form, $x=(a_1, a_2, ..., a_i)$ for $i = 1, 2, \ldots, n$ or $y=(b_1, b_2, ..., b_j)$ for $j = 1, 2, \ldots, m$ respectively. Therefore,

$$x \times y = (a_1, a_2, ..., a_i) \times (b_1, b_2, ..., b_j) \textit{ for } i = 1, 2, ..., n \qquad \textit{for } j = 1, 2, ..., m.$$

So,

$$M = \langle a \rangle_m \times \langle b \rangle_m = \{(), (a_1 \wedge b_1), (a_1 \wedge b_1, a_2 \wedge b_2), ..., (a_1 \wedge b_1, a_2 \wedge b_2, ..., a_k \wedge b_k)\}.$$

Since M is a chain, M is closed under the '+' operation. Also, M is closed under the dot product. Thus, M is an $\mathbb{N}^\infty$-submoduloid of N.

Remark.

In the above theorem, if $index_N(a \times b)$, that is, k is less than n and m, then M does not contain a and b. Also, the level of $a \times b$ is equal to the minimum of the level a, the level b and the index of $a \times b$, that is,

$$l(a \times b) = min\{l(a), l(b), index_N(a \times b)\}.$$

Example 3.10.

Suppose that $a = (2, 3, 1, 4, 5, 3)$, $b = (4, 1, 7, 2, 6)$ and $index_N(a \times b) = 4$. By assumption,

$$\langle a \rangle_m = \{(), (2), (2, 3), (2, 3, 1), (2, 3, 1, 4), (2, 3, 1, 4, 5), (2, 3, 1, 4, 5, 3)\},$$

and

$$\langle b \rangle_m = \{(), (4), (4, 1), (4, 1, 7), (4, 1, 7, 2), (4, 1, 7, 2, 6)\}.$$

Now, one may obtain

$$M = \langle a \rangle_m \times \langle b \rangle_m = \{x \times y : x \in \langle a \rangle_m, y \in \langle b \rangle_m\}.$$

As follows:

$(2) \times (4) = (2 \wedge 4)$
$(2) \times (4, 1) = (2 \wedge 4)$
$(2) \times (4, 1, 7) = (2 \wedge 4)$
$(2) \times (4, 1, 7, 2) = (2 \wedge 4)$
$(2) \times (4, 1, 7, 2, 6) = (2 \wedge 4)$
$(2, 3) \times (4) = (2 \wedge 3)$
$(2, 3) \times (4, 1) = (2 \wedge 4, 3 \wedge 1)$
$(2, 3) \times (4, 1, 7) = (2 \wedge 4, 3 \wedge 1)$
$(2, 3) \times (4, 1, 7, 2) = (2 \wedge 4, 3 \wedge 1)$

$(2,\ 3)\times(4,\ 1,\ 7,\ 2,\ 6)=(2\wedge 4,\ 3\wedge 1)$

...

...

...

$(2,\ 3,\ 1,\ 4,\ 5,\ 3)\times(4)=(2\wedge 4)$

$(2,\ 3,\ 1,\ 4,\ 5,\ 3)\times(4,\ 1)=(2\wedge 4, 3\wedge 1)$

$(2,\ 3, 1,\ 4,\ 5,\ 3)\times(4,\ 1,\ 7)=(2\wedge 4,\ 3\wedge 1,\ 1\wedge 7)$

$(2,\ 3,\ 1,\ 4,\ 5,\ 3)\times(4,\ 1,\ 7,\ 2)=(2\wedge 4,\ 3\wedge 1,\ 1\wedge 7,\ 4\wedge 2)$

$(2,\ 3,\ 1,\ 4, 5,\ 3)\times(4,\ 1,\ 7,\ 2,\ 6)=(2\wedge 4,\ 3\wedge 1,\ 1\wedge 7,\ 4\wedge 2)$,

since $index_N(a\times b)=4$, So, the product of every address in $<a>_m$ to every address in $<b>_m$ has four terms. As one can see, any address in $<a>_m\times<b>_m$ is of the form

$$(), (2\wedge 4), (2\wedge 4, 3\wedge 1), (2\wedge 4,\ 3\wedge 1,\ 1\wedge 7),\ (2\wedge 4, 3\wedge 1, 1\wedge 7,\ 4\wedge 2).$$

Therefore,

$$M=<a>_m\times<b>_m=\{(),\ (2\wedge 4),\ (2\wedge 4,\ 3\wedge 1),\ (2\wedge 4,\ 3\wedge 1,\ 1\wedge 7),\ (2\wedge 4,\ 3\wedge 1,\ 1\wedge 7,\ 4\wedge 2)\}.$$

Also, since $index_N(a\times b)=4$, is less than $l(a)=6$ and $l(b)=5$, then M does not contain a and b.

Corollary 3.11.

By all assumptions in Theorem 3.8, the below equations hold:

$$<a>_m\times<b>_m=\{a\}\times<b>_m=<a>_m\times\{b\}=<a\times b>_m .$$

Proof:

By Theorem 3.8,

$$<a>_m\times<b>_m=\{(),\ (a_1\wedge b_1),\ (a_1\wedge b_1,\ a_2\wedge b_2,\ ...,\ (a_1\wedge b_1),\ a_2\wedge b_2,\ ...,\ a_k\wedge b_k)\}.$$

Now, consider the product of $\{a\}$ and $<b>_m$, that is

$$\{a\}\times<b>_m=\{a\times y: y\in<b>_m\}=\{(a_1,a_2,...,a_n)\times(b_1,b_2,...,b_j): j=1,2,...,m\}$$

$$=\{(a_1,\ a_2,\ ...,\ a_n)\times(b_1,\ b_2,\ ...,\ b_j): j=1,\ 2,\ ...,\ k\}$$

$$=\{(), (a_1\wedge b_1), (a_1\wedge b_1, a_2\wedge b_2), ..., (a_1\wedge b_1, a_2\wedge b_2, ..., a_k\wedge b_k)\}.$$

Similarly,

$$<a>_m\times\{b\}=\{x\times b: x\in<a>_m\}=\{(a_1, a_2, ..., a_i)\times(b_1, b_2, ..., b_m): i=1, 2, ..., n\}$$

$$=\{(\mathrm{a}_1, \mathrm{a}_2, ..., \mathrm{a}_\mathrm{i})\times(\mathrm{b}_1, \mathrm{b}_2, ..., \mathrm{b}_\mathrm{k}): \mathrm{i}=1, 2, ..., \mathrm{k}\}$$

$$=\{(), (a_1\wedge b_1), (a_1\wedge b_1, a_2\wedge b_2, ..., (a_1\wedge b_1), a_2\wedge b_2, ..., a_k\wedge b_k)\}.$$

Finally, since $index_N(a\times b)=k$,

$$a\times b=(a_1\wedge b_1, a_2\wedge b_2, ..., a_k\wedge b_k).$$

Therefore,

$$<a\times b>_m=\{(), (a_1\wedge b_1), (a_1\wedge b_1, a_2\wedge b_2, ..., (a_1\wedge b_1), a_2\wedge b_2, ..., a_k\wedge b_k)\},$$

and the proof is completed.

Definition 3.12.

Let N be a nexus and let $a=(a_1, a_2, ..., a_n)$ be an address in N. Then the subset

$$B_a=\{x\in N: a\times x=x\},$$

of N, is called the bounded set by a.

Example 3.13.

Suppose that

$$N=\{(), (1), (2), (3), (1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (1, 2, 3), (2, 2, 1), (2, 2, 2), (2, 2, 3), (1, 1, 1, 1), (1, 1, 1, 2), (2, 2, 1, 1), (2, 2, 1, 2)\},$$

and $a=(2, 2, 2)$. The dendrogram of N is shown in Figure 4. Then,

$$B_a=\{(), (1), (2), (1, 1), (1, 2), (2, 1), (2, 2), (1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 2, 1), (2, 2, 2)\}.$$

The addresses of B_a in Figure 4, are shown by thick lines.

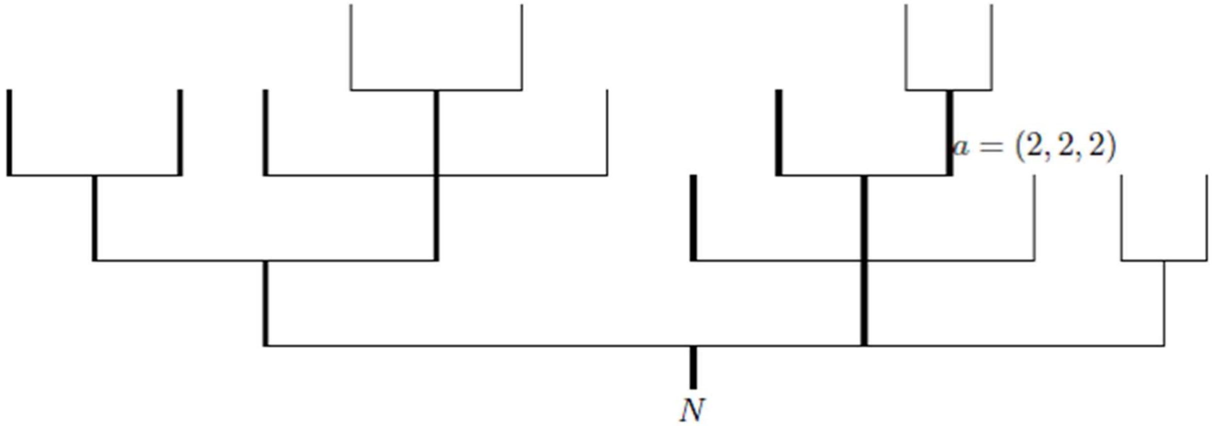

Figure 4. Dendrogram of N

Lemma 3.14.

Let N be a nexus and let $a=(a_1, a_2, ..., a_n)$ be an address in N. Then,

(i) $b\in B_a$ if and only if $l(b)\le l(a)$ and $b_i \le a_i$ for $i=1,2,..,l(b)$.

(ii) If $b\in B_a$ and $c\le b$, then $c\in B_a$.

(iii) B_a is closed under the '$\times$' operation.

Proof:

(i) Suppose that, $b=(b_1, b_2, ..., b_m)$ and $b\in B_a$. Then, by definition of the '$\times$' operation, $l(a)\le l(b)$ and $a\times b=b$. So,

$$(a_1, a_2, ..., a_n)\times(b_1, b_2, ..., b_m)=(a_1\wedge b_1, a_2\wedge b_2, ..., a_m\wedge b_m)=(b_1, b_2, ..., b_m).$$

Therefore, $a_i\wedge b_i=b_i$ for all $i=1,2,...,m$. This implies that $a_i\le b_i$ for all $i=1,2,...,m$. Conversely, suppose b is an arbitrarily address in N such that, $l(b)\le l(a)$ and $b_i\le a_i$ for all $i=1,2,...,l(b)$. Then,

$$a\times b=(a_1, a_2, ..., a_n)\times(b_1, b_2, ..., b_m)=(a_1\wedge b_1, a_2\wedge b_2, ..., a_m\wedge b_m)=(b_1, b_2, ..., b_m)=b.$$

So, $b\in B_a$.

(ii) Since $c \le b$, so, $c=(b_1, b_2, ..., b_{j-t})$ for $j=1,2,\ldots,m$, where t is a non-negative integer. Based on the hypothesis we have, $b \in B_a$. Therefore, $l(c) \le l(b) \le l(a)$ and $b_i \le a_i$ for $i=1,2,\ldots,m$. Thus, $c \in B_a$

(iii) suppose that $x=(x_1, x_2, ..., x_p), y=(y_1, y_2, ..., y_q) \in B_a$. So, $l(x), l(y) \le l(a)$ and $x_i \le a_i, y_j \le a_j$ for all $i=1, 2, ..., p$ and $j=1, 2, ..., q$. Therefore, $x_i \wedge y_i \le a_i$ for all $i=1, 2, ..., min\{p,q\}$. Thus,

$$x \times y=(x_1 \wedge y_1, x_2 \wedge y_2, ..., x_i \wedge y_i),$$

where $i=index_N(x \times y)$. Consequently,

$$a \times (x \times y) = x \times y.$$

So, $x \times y \in B_a$.

Theorem 3.15.

Let N be an $\mathbb{N}^\infty$-submoduloid and let $a=(a_1, a_2, ..., a_n)$ be an address in N.

(i) B_a is a subnexus of N.

(ii) B_a is an $\mathbb{N}^\infty$-submoduloid of N.

Proof:

(ii) By Theorem 2.7, one must show that if $b \in B_a$ and $c \in N$ such that $c \le b$ then $c \in B_a$. By using part (ii) of Lemma 3.14, one can show that $c \in B_a$.

(ii) Firstly, it is require to show that B_a is closed under the '+' operation. Suppose that, $x=(x_1, x_2, ..., x_p), y=(y_1, y_2, ..., y_q) \in B_a$. So, $l(x), l(y) \le l(a)$ and $x_i \le a_i, y_j \le a_j$ for all $i=1, 2, ..., p$ and $j=1, 2, ..., q$. Therefore, $x_i \vee y_i \le a_i$ for all $i=1, 2, ..., min\{p,q\}$. Now, note that, $x+y$ is one of the following three forms

$$x+y=(x_1 \vee y_1, x_2 \vee y_2, ..., x_k \vee y_k),$$

$$x+y=(x_1 \vee y_1, x_2 \vee y_2, ..., x_p \vee y_p, y_{p+1},.., y_k) \qquad p<q,$$

or

$$x+y=(x_1\vee y_1,\ x_2\vee y_2,\ ...,\ x_p\vee y_p,\ x_{q+1},\ ...,\ x_k) \qquad p>q,$$

where $k=index_N(x+y)$.

Consequently, $a\times(x+y)=x+y$. So, $x+y\in B_a$. Also, it is easy to see that if $x=(x_1,\ x_2,...,x_p)\in B_a$, then

$$kx=k(x_1,\ x_2,\ ...,\ x_p)=(x_1,\ x_2,\ ...,\ x_k)\in B_a.$$

So, B_a is closed under the dot product and now, by Theorem 2.18, the proof is completed.

Theorem 3.16.

Let N be an $\mathbb{N}^\infty$-moduloid and let $a=(a_1,\ a_2,\ ...,\ a_n)$ be an address in N. Then

(i) $Q_a+B_a=Q_a$.

(ii) $Q_a\times B_a=B_a$.

(iii) $Q_a\cap B_a=a$.

(iv) $Q_a\cup B_a=<Q_a\cup B_a>_m$, which is an $\mathbb{N}^\infty$-submoduloid generated by $Q_a\cup B_a$.

Proof:

(i) Suppose that x and y are two arbitrary addresses in Q_a and B_a, respectively. Then $x=(a_1,\ a_2,\ ...,\ a_n+r,\ x_{n+1},\ ...,\ x_m)$, where r is a non-negative integer and $y=(y_1,y_2,...,y_t)$, where $t\le n$ and $y_i\le a_i$ for $i=1,\ 2,\ ...,\ n$. Therefore,

$$x+y=(a_1,a_2,\ ...,\ a_n+r,\ x_{n+1},\ ...,\ x_m)+(y_1,\ y_2,\ ...,\ y_t)$$

$$=(a_1\vee y_1,\ a_2\vee y_2,\ ...,\ a_t\vee y_t,\ ...,\ a_n+r,\ x_{n+1},\ ...,\ x_m)=x\in Q_a.$$

The result is that $Q_a+B_a\subseteq Q_a$. Since $()$ is an address in B_a, so, every address x in Q_a can be written as $()+x=x$. which results in, $Q_a\subseteq Q_a+B_a$. Consequently $Q_a+B_a=Q_a$.

(ii) Suppose that x and y are the two arbitrarily addresses in Q_a and B_a, respectively. Then

$$x\times y=(a_1,\ a_2,\ ...,\ a_n+r,\ x_{n+1},\ ...,\ x_m)\times(y_1,\ y_2,\ ...,\ y_t)=(a_1\wedge y_1,\ a_2\wedge y_2,\ ...,\ a_t\wedge y_t)=y\in B_a.$$

Thus, $Q_a \times B_a \subseteq B_a$. Since a is an address in Q_a, so, every address y in B_a can be written of the form $a \times y = y$. Therefore, $B_a \subseteq Q_a \times B_a$. Consequently, $Q_a \times B_a = B_a$.

(iii) By definitions of Q_a and B_a, the proof is clear.

(iv) First, it will be shown that $Q_a \cup B_a$ is an $\mathbb{N}^\infty$-submoduloid of N. To do this, one may prove that $Q_a \cup B_a$ is closed under the '+' operation and dot product. Now, suppose that x and y are two addresses in $Q_a \cup B_a$. Consider two cases:

Case 1: x and y are in the same set, that is $x, y \in Q_a$ or $x, y \in B_a$. Since Q_a and B_a are closed under the '+' operation, so $x + y \in Q_a$ or $x + y \in B_a$. Therefore, $x + y \in Q_a \cup B_a$.
Case 2: x and y are in the different set. For instance, $x \in Q_a$ and $y \in B_a$. So, x and y can be written as

$$x = (a_1, a_2, ..., a_n + r, x_{n+1}, ..., x_m),$$

where r is non-negative integer and

$$y = (y_1, y_2, ..., y_t),$$

where $t \leq n$ and $y_i \leq a_i$ for $i = 1, 2, ..., n$. Therefore, $x + y = x \in Q_a$. This leads us to $x + y \in Q_a \cup B_a$. Consequently, $Q_a \cup B_a$ is closed under the '+' operation. Now, one may show that $Q_a \cup B_a$ is closed under the dot product. If $x \in B_a$, since B_a is an $\mathbb{N}^\infty$-submoduloid, so, $kx \in B_a$ for all $k \in \mathbb{N}^*$. Now, assume that $x \in Q_a$. Then, x is an adress of the form $x = (a_1, a_2, ..., a_n + r, x_{n+1}, ..., x_m)$. Now, we consider kx and get desirable result. If $k < n$ then $kx \in B_a$ and if $k \geq n$ then $kx \in Q_a$. Thus, $kx \in Q_a \cup B_a$ for all $k \in \mathbb{N}^*$. Consequently, $Q_a \cup B_a$ is closed under dot product. Therefore, $Q_a \cup B_a$ is an $\mathbb{N}^\infty$-submoduloid containing $Q_a \cup B_a$. Thus $Q_a \cup B_a \subseteq < Q_a \cup B_a >_m$. But $< Q_a \cup B_a >_m$ is the smallest $\mathbb{N}^\infty$-submoduloid containing $Q_a \cup B_a$. So, $< Q_a \cup B_a >_m \subseteq Q_a \cup B_a$, therefore, $< Q_a \cup B_a >_m = Q_a \cup B_a$.

Remark.

The above theorem is still true if Q_a is replaced by q_a. However, note that in part (iii), $q_a \bigcap B_a = \phi$.

Definition 3.18.

Suppose that A is a subset of $\mathbb{N}^\infty$-moduloid N. Then,

$$B_A = \bigcup_{a \in A} B_a = \bigcup_{a \in A} \{x \in N; a \times x = x\}.$$

Theorem 3.19.

Let N be an $\mathbb{N}^\infty$-moduloid and A be a subset of N. Suppose that A is closed under '$+$' operation. Then B_A is an $\mathbb{N}^\infty$-submoduloid of N.

Proof:

Suppose that $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_m)$ are two addresses in B_A. Now, one must show that $x + y \in B_A$. Since $x, y \in B_A$, so, there exist $b, c \in A$ such that $x \in B_b$ and $y \in B_c$. Therefore, $n \le l(b)$, $x_i \le b_i$ for $i = 1, 2, ..., n$ and $m \le l(c), y_j \le c_j$ for $j = 1, 2, ..., m$. Now, consider $x + y$ and $b + c$. Since $l(x + y) = max\{l(x), l(y)\}$ and $l(b + c) = max\{l(b), l(c)\}$, so, $l(x + y) \le l(b + c)$. Since $x_i \le b_i$ and $y_i \le c_i$. So, $x_i + y_i \le b_i + c_i$ for $i = 1, 2, ..., l(x + y)$. Therefore, by Lemma 3.14 part (i), $x + y \in B_{b+c}$. Since A is closed under '$+$' operation, so,

$$x + y \in B_{b+c} \subseteq \bigcup_{a \in A} B_a = B_A.$$

Now, suppose that, $x \in B_A$ and $k \in \mathbb{N}^*$. Since $x \in B_A$, so, there exists $b \in A$ such that $x \in B_b$. By Theorem 3.15, B_a is an $\mathbb{N}^\infty$-submoduloid of N, so $kx \in B_a$ for all $k \in \mathbb{N}^*$. Thus

$$kx \in B_a \subseteq \bigcup_{a \in A} B_a = B_A.$$

Consequently, B_A is closed under '$+$' operation and dot product. So, B_A is an $\mathbb{N}^\infty$-submoduloid of N.

Corollary 3.20.

Let N be an $\mathbb{N}^\infty$-moduloid, and let A be a subset of N, a be an address in N.

(i) If A is a chain then B_A is an $\mathbb{N}^\infty$-submoduloid of N.

(ii) B_{Q_a} is an $\mathbb{N}^\infty$-submoduloid of N.

Proof:

(i) Since A is a chain, so, all addresses in N are comparable. Now, suppose that a and b are two arbitrary addresses in A and $a \le b$. Then $a + b = b \in A$. Therefore, A is closed under '$+$' operation. By Theorem 3.19, B_A is an $\mathbb{N}^\infty$-submoduloid of N.

(ii) By Lemma 3.4 part (ii), Q_a is closed under '+' operation. So, B_{Q_a} is an $\mathbb{N}^\infty$-submoduloid of N.

Theorem 3.21.

Let N be an $\mathbb{N}^\infty$-moduloid and $a=(a_1, a_2, ..., a_n)$ be an address in N. Suppose that, $b=(b_1, b_2, ..., b_m)$ is an address in B_a. Now, consider the set

$$K=\{x=(x_1, x_2, ..., x_p)\in B_a;\ b_i \le x_i \le a_i,\ i=1,\ 2,\ ...,\ p\}.$$

Then K is an $\mathbb{N}^\infty$-submoduloid of N.

Proof:

Let $x=(x_1, x_2, ..., x_p)$ and $y=(y_1, y_2, ..., y_q)$ be two addresses in K. So, $b_i \le x_i \le a_i,\ i=1,\ 2,\ ...,\ p$ and $b_j \le y_j \le a_j, j=1,2,...,q$. Without loss of generality, suppose that $p\le q$. The moduloid sum of x and y is one of two below forms

$$x+y=(x_1\vee y_1,\ x_2\vee y_2,\ ...,\ x_k\vee y_k),$$

or

$$x+y=(x_1\vee y_1,\ x_2\vee y_2,\ ...,\ x_p\vee y_p,\ y_{p+1},\ ...,\ y_k),$$

Where $k=index_N(x+y)$. Since $b_i\le x_i\le a_i$ and $b_i\le y_i\le a_i$, then, $b_i\le x_i\vee y_i\le a_i$. Therefore, the i^{th} term of $x+y$ is between b_i and a_i, in both cases, for $i=1,\ 2,\ ...,\ k$. It means that $x+y\in K$. Now, suppose that, $x=(x_1, x_2, ..., x_p)\in K$ and $n\in\mathbb{N}^*$. Therefore, $b_i\le x_i\le a_i,\ i=1,\ 2,\ ...,\ p$. If $n\ge p$, then $nx=x\in K$ and if $n<p$ then $nx=(x_1, x_2, ..., x_n)$, where $b_i\le x_i\le a_i,\ i=1,\ 2,\ ...,\ n.$ Therefore, $nx\in K$. So, K is closed under dot product. Consequently, K is an $\mathbb{N}^\infty$-submoduloid of N.

Theorem 3.22.

Let N be an $\mathbb{N}^\infty$-moduloid, and let $a=(a_1, a_2, ..., a_n)$ be an address in N. The number of the addresses in B_a is less than or equal to $\sum_{i=1}^{n}\prod_{j=1}^{i}a_j=a_1+a_1\times a_2+...+a_1\times a_2\times...\times a_n$.

Proof:

Let $x\in B_a$. So, $l(a)\le n.$ Therefore, there are n possibilities for the level of x as addresses in B_a, that is, $l(x)=1,\ 2,\ ...,\ n$. Now suppose that, $x\in B_a$ and $l(x)=p\le n$. So, $x=(x_1, x_2, ..., x_p)$ and

$x_i \le a_i$ for all $i = 1,\ 2,\ ...,\ p$. Now, note that there are a_1 possibilities for the value of x_i, that is, $1, 2, ..., a_1$ and there are a_2 possibilities for value of x_2, that is, $1, 2, ..., a_2$. In general, there are a_i possibilities for value of x_i, that is, $1,\ 2,\ ...,\ a_i$ for $i = 1,\ 2,\ ...,\ p$. Therefore, there are

$$\prod_{i=1}^{p} a_i = a_1 \times a_2 \times ... \times a_p,$$

possibilities for the values of $x \in B_a$, where $l(x) = p$. Therefore, the number of possibilities of x with any level less than or equal to n, is

$$\sum_{i=1}^{n} \prod_{j=1}^{i} a_j = a_1 + a_1 \times a_2 + ... + a_1 \times a_2 \times ... \times a_n.$$

However, some possible value of x may be not an address in N. So, the number of addresses in B_a is less than or equal to $\sum_{i=1}^{n} \prod_{j=1}^{i} a_j$.

Theorem 3.23.

Let N be a moduloid and let $a = (a_1,\ a_2,\ ...,\ a_n)$ and $b = (b_1,\ b_2,\ ...,\ b_n)$ be two addresses in N. Also, suppose that, $a \times b$ is an address in N. Then,

$$B_a \cap B_b = B_{a \times b},$$

is a subnexus and $\mathbb{N}^\infty$-submoduloid of N.

Proof:

Let $x = (x_1,\ x_2,\ ...,\ x_p) \in B_a \cap B_b$. So, $x \in B_a$ and $x \in B_b$. Therefore, $p \le min\{l(a),\ l(b)\}$ and $x_i \le a_i \wedge b_i$ for $i = 1,\ 2,\ ...,\ p$. Thus, by definition 3.5, $x \in B_{a \times b}$ and $B_a \cap B_b \subseteq B_{a \times b}$. Now, conversely suppose that $x = (x_1,\ x_2,\ ...,\ x_p) \in B_{a \times b}$. So, we have $p \le min\{l(a),\ l(b)\}$ and $x_i \le a_i \wedge b_i$ for $i = 1, 2, ..., p$. Therefore, $x \in B_a$ and $x \in B_b$. Thus, $x \in B_a \cap B_b$. This means that $B_{a \times b} \subseteq B_a \bigcap B_b$. Consequently, $B_a \cap B_b = B_{a \times b}$. Now, by Theorem 3.15, $B_{a \times b}$ is a subnexus and $\mathbb{N}^\infty$-submoduloid of N.

4. The Fraction of $\mathbb{N}^\infty$-moduloids

In this section, the fractions of an $\mathbb{N}^\infty$-moduloid N is defined and denoted by $S^{-1}N$, where S is a meet closed subset of $\mathbb{N}^\infty$

Remark.

Let $\phi \neq S \subseteq \mathbb{N}^{\infty}$ and $0 \notin S$. Then, S is meet closed subset of $\mathbb{N}^{\infty}$. Because for all $a,b \in S$ implies $a \wedge b \in S$. If S is meet closed subset of $\mathbb{N}^{\infty}$, then we write S a m.c.s. of $\mathbb{N}^{\infty}$

Definition 4.2.

Let S be a m.c.s. of $\mathbb{N}^{\infty}$ On $N \times S$ we define a binary relation by the following:

$$(a,t) \sim_s (b,r) \Leftrightarrow \exists k \in S, (k \wedge r).a = (k \wedge t).b.$$

Theorem 4.3.

'$\sim_s$' is an equivalence relation on $N \times S$.

Proof:

It is clear that '$\sim_s$' is reflexive and symmetric on $N \times S$. Now, let $(a,\ r) \sim (b,\ s)$ and $(b,\ s) \sim (c,\ t)$. Then, there exists $m,n \in S$ such that $(m \wedge s).a = (m \wedge r).b$ and $(n \wedge t).b = (n \wedge s).c$. Hence, we have $(m \wedge n \wedge t \wedge s).a = (m \wedge n \wedge t \wedge r).b$ and $(m \wedge n \wedge t \wedge r).b = (m \wedge n \wedge s \wedge r).c$. So, $(m \wedge n \wedge t \wedge s).a = (m \wedge n \wedge s \wedge r).c$. Since $m,n,s \in S$. So, $m \wedge n \wedge s \in S$. Thus $(a,\ r) \sim (c,\ t)$.

Remark.

The set of all equivalence class '$\sim$' on $N \times S$ is denoted by $S^{-1}N$ (i.e. $N \times S = \{\frac{a}{t} \mid (a,t) \in N \times S\}$) and called the fraction of N associated to S.

Example 4.5.

Consider nexus N={(), (1), (2), (1, 1), (1, 1, 1), (1, 1, 2), (1, 1, 2, 1), (2, 1), (2, 1, 1), (2, 1, 2)} and $S = \{2,\ 3\}$. So,

$$\frac{()}{2} = \{\frac{()}{2},\ \frac{()}{3}\},$$

$$\frac{(1)}{2} = \{\frac{(1)}{2},\ \frac{(1)}{3}\},$$

$$\frac{(2)}{2} = \{\frac{(2)}{2},\ \frac{(2)}{3}\},$$

$$\frac{(1,\ 1)}{2} = \{\frac{(1,\ 1)}{2},\ \frac{(1,\ 1)}{3},\ \frac{(1,\ 1,\ 1)}{2},\ \frac{(1,\ 1,\ 1)}{3},\ \frac{(1,\ 1,\ 2)}{2},\ \frac{(1,\ 1,\ 2)}{3},\ \frac{(1,\ 1,\ 2,\ 1)}{2},\ \frac{(1,\ 1,\ 2,\ 1)}{3}\},$$

$$\frac{(2,1)}{2} = \{\frac{(2,\,1)}{2},\ \frac{(2,\,1)}{3},\ \frac{(2,\,1,\,1)}{2},\ \frac{(2,\,1,\,1)}{3},\ \frac{(2,\,1,\,2)}{2},\ \frac{(2,\,1,\,2)}{3}\}.$$

So that $S^{-1}N = \{\frac{()}{2},\ \frac{(1)}{2},\ \frac{(2)}{2},\ \frac{(1,1)}{2},\ \frac{(2,1)}{2}\}$.

Theorem 4.6.

Let N be an $\mathbb{N}^{\infty}$-moduloid, S be a m.c.s. of $\mathbb{N}^{\infty}$ and $k = \bigwedge_{s\in S} S$. For every $\frac{a}{t} \in S^{-1}N$, we have $\frac{a}{t} = \frac{a}{k}$.

Proof:

The proof is trivial.

Definition 4.7.

Let N be an $\mathbb{N}^{\infty}$-moduloid, S be a m.c.s. of $\mathbb{N}^{\infty}$ and $\frac{a}{r}$, $\frac{b}{s}$ be two element of $S^{-1}N$. Define the relation $'\leq'$ on $S^{-1}N$ by the following:

$$\frac{a}{r} \leq \frac{b}{s} \quad \text{if and only if} \quad \exists t \in S,\ (t \wedge r \wedge s).\, a \leq (t \wedge r \wedge s).b.$$

Theorem 4.8.

Let N be an $\mathbb{N}^{\infty}$-moduloid, S be a m.c.s. of $\mathbb{N}^{\infty}$. Then $(S^{-1}N, \leq)$ is a partially ordered set.

Proof:

The proof is trivial.

Theorem 4.9.

Let N be an $\mathbb{N}^{\infty}$-moduloid, S be a m.c.s. of $\mathbb{N}^{\infty}$ and $k = \bigwedge_{s\in S} S$.

(i) For every $a \in N$ and $r, s \in S$ $(a, r) \sim_{\{k\}} (a, s)$.

(ii)

$$S^{-1}N \cong \{k\}^{-1}N.$$

Proof:

(ii) We have $(k \wedge r).a = k.a = (k \wedge s).a$. So, $(a,r) \sim_{\{k\}} (a,s)$.

(ii) We define the map φ by the following

$$\varphi : S^{-1}N \to \{k\}^{-1}N,$$

by $\varphi(\frac{a}{r}) = \frac{a}{k}$. First of all, one should show that φ is well-defined. Let $\frac{a_1}{r_1}$ and $\frac{a_2}{r_2}$ are two elements of $S^{-1}N$ such that $\frac{a_1}{r_1} = \frac{a_2}{r_2}$. Thus, there exists $t \in S; (r_2 \wedge t).a_1 = (r_1 \wedge t).a_2$. Thus, $(r_2 \wedge t \wedge k).a_1 = (r_1 \wedge t \wedge k).a_2$ and then we get $k.a_1 = k.a_2$. This means $\frac{a_1}{r_1} = \frac{a_2}{r_2}$. So, $\varphi(\frac{a_1}{r_1}) = \varphi(\frac{a_2}{r_2})$. Now let $\frac{a_1}{r_1} = \frac{a_2}{r_2}$, so that $\frac{a_1}{k} = \frac{a_2}{k}$ and hence $k.a_1 = k.a_2$. Thus, we have $(r_2 \wedge k).a_1 = (r_1 \wedge k).a_2$ that implies $\frac{a_1}{r_1} = \frac{a_2}{r_2}$ and φ is one-to-one. If $\frac{a}{k} \in \{k\}^{-1}N$, then $\varphi(\frac{a}{k}) = \frac{a}{k}$ and this means φ is onto. If $\frac{a_1}{r_1} \le \frac{a_2}{r_2}$, then there exists $t \in S; a_1.(t \wedge r_1) \le a_2.(t \wedge r_2)$ and hence $a_1.(k \wedge t \wedge r_1) \le a_2.(k \wedge t \wedge r_2)$. Thus, $a_1.k \le a_2.k$ and so $\varphi(\frac{a_1}{r_1}) \le \varphi(\frac{a_2}{r_2})$ and this means φ is a homomorphism. If $\frac{a_1}{r_1} = \frac{a_2}{r_2}$ then we have $\varphi^{-1}(\frac{a_1}{r_1}) \le \varphi^{-1}(\frac{a_2}{r_2})$ and thus φ^{-1} is a homomorphism and the proof is completed.

Corollary 4.10.

Let S_1 and S_2 are two m.c.s. of N, such that $\bigwedge_{\omega \in S_1} \omega = \bigwedge_{\upsilon \in S_2} \upsilon$. Then, $S_1^{-1}N = S_2^{-1}N$.

Proof:

The proof follows from Theorem 4.9.

Theorem 4.11.

Let $k = \bigwedge_{x \in S} x$. Then, $\{k\}^{-1}N \cong koN$.

Proof:

We define the map φ by the following:

$$\varphi : \{k\}^{-1}N \to koN, \qquad \varphi(\frac{a}{k}) = k.a,$$

Let $\frac{a_1}{k}, \frac{a_2}{k} \in \{k\}^{-1}N$, since

$$\frac{a_1}{k} = \frac{a_2}{k} \Leftrightarrow k.a_1 = k.a_2 \Leftrightarrow \varphi(\frac{a_1}{k}) = \varphi(\frac{a_2}{k}).$$

Thus φ is well-defined and one-to-one. If $\frac{a_1}{k} \le \frac{a_2}{k}$, then $k.a_1 \le k.a_2$ resulting in $(\varphi\frac{a_1}{k}) \le \varphi(\frac{a_2}{k})$. Thus φ is a homomorphism. If x and y are two elements of koN and $x \le y$, then $k.x \le k.y$ and so $\frac{x}{k} \le \frac{y}{k}$. On the other hand, $x, y \in koN$ implies $x = k.x$ and $y = k.y$. Thus $\varphi^{-1}(x) = \frac{x}{k}$ and $\varphi^{-1}(y) = \frac{y}{k}$ and hence $\varphi^{-1}(x) \le \varphi^{-1}(y)$. Therefore φ is an isomorphism.

Corollary 4.12.

$(S^{-1}N, \le)$ is isomorphic with an $\mathbb{N}^{\infty}$-submoduloid of N.

Proof:

By Theorem 4.9 and Theorem 4.11, we have

$$S^{-1}N \cong \{k\}N \cong koN.$$

Thus, we get the desired result.

5. Conclusions

The notions of generating $\mathbb{N}^{\infty}$-submoduloids, bounded sets, fractions of $\mathbb{N}^{\infty}$-submoduloid of N and the relationship between them are defined and investigated. In particular, it was shown that: (i) For address a in $\mathbb{N}^{\infty}$-moduloid Q_a and q_a are closed under the $+$ operation (Theorem 3.4). (ii) Every bounded set of $\mathbb{N}^{\infty}$ -moduloid N is an $\mathbb{N}^{\infty}$-submoduloid of N (Theorem 3.15). (iii) The number of addresses in B_a of an $\mathbb{N}^{\infty}$-moduloid (Theorem 3.22). (iv) The intersection $B_a \cap B_b$ of two sets B_a and B_b is the bounded set $B_{a \times b}$. (Theorem 3.23). (v) $(S^{-1}N, \le)$ is isomorphic with an $\mathbb{N}^{\infty}$-submoduloid of N (Corollary 4.12).

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