Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 15, Issue 2 (December 2020), pp. 1396 – 1406

Vertical Motion of the Variable Infinitesimal Mass In the Circular Sitnikov Problem

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Received: May 12, 2020; Accepted: September 18, 2020

Abstract

The circular case of Sitnikov problem is studied here when the infinitesimal body varies its mass according to Jeans law and it is moving along the z-axis which is perpendicular to the orbital plane of the two equal spherical primaries. The two primaries are moving in $xy-$plane on the same circular path. These two primaries are imposing the Newtonian forces on the third variable mass body but not influenced by it. Stability of equilibrium points is examined followed by the derived equations of motion. The time-series solutions of the equation of motion are performed by using the Lindstedt-Poincaré method which is used to remove the secular term. We have numerically performed the time-series which shows that variation parameters have great impact on it.

Keywords: Circular Sitnikov problem; Variable mass; Meshcherskii transformation; Lindstedt-Poincaré method

MSC 2010 No.: 70F15, 70K42, 70F07

1. Introduction

The few body problem is an interesting problem in celestial mechanics and dynamical astronomy for the researchers. The most interesting and studied problem in this area is restricted three-body problem. It has many configuration like Lagrangian configuration, Euler configuration, Copenhagen configuration, Sitnikov configuration, Robes configuration, etc. In Lagrangian configuration, two bodies remain on straight line with moving on different circular or elliptic path and third body will move in space. In Euler configuration, all the bodies will remain on the same straight line. In Copenhagen configuration, two bodies remain on the same straight line but moving on the same circular path and the third body is moving in the space.

In Sitnikov configuration, two bodies remain on the same straight line and moving either in elliptic or circular path while third body is moving on the vertical line of the orbital plane of the last two bodies. In the Robes configuration, two bodies remain on the same straight line and moving either in elliptic or circular path and also the shape of one of the body is taken as spherical shell while the third body is moving inside the shell. These configurations are studied with different shapes of the bodies (as point mass, spherical shape, spherical shell, oblateness, triaxial body, heterogeneous body, homogeneous body, Roche-ellipsoid, finite-straight segment, cylindrical shape, etc.), variable mass, mobile coordinates, viscous force, Stroke force, Poynting-Robertson drag, resonance, coriolis and centrifugal forces, modified Newtonian potential, etc. Many researchers have studied these problems, some of them are as follows.

Jeans (1928) explained about variable mass in his book. Singh (2008) and Singh (2009) studied the effect of oblateness and variable parameters in the frame of circular restricted three-body problem. Perdios and Markellos (1988), Perdios and Markellos (2007), and Perdios and Kalantonis (2012) studied the restricted three- body problem in the Sitnikov configuration where they discussed the periodic motion, the stability and self resonant bifurcations. Faruque (2003) used a Lindstedt-Poincare perturbation method and Courant-Snyder transformation to evaluate the solution and compared with existing solution where he found satisfactory agreement. Kalantonis et al. (2008) studied the problem with Sitnikov configuration by supposing the radiated oblate-primaries. Zhang et al. (2012) investigated the effect of variable mass of the infinitesimal body in the frame of circular restricted three-body problem. Pandey and Ahmad (2013) studied the periodic orbits and bifurcation in the circular Sitnikov configuration by supposing all the primaries as oblate body.

Shahbaz Ullah et al. (2015) studied the series solution of the problem in the elliptic Sitnikov configuration when there are $N + 1$ bodies by following Giacaglia (1967). Ansari (2017), Ansari et al. (2018), Ansari et al. (2019a), Ansari et al. (2019b), Ansari et al. (2020) have studied restricted three-body problem by considering many factors and shown the effects of these factors analytically and numerically on the equilibrium points, regions of motion, Poincaré surfaces of section and basins of attraction.

This paper is organized as follows. Literature review is made in the first section. The configuration of the problem with derived equations of motion and equilibrium point are presented in second section. The third section contains stability examination of the equilibrium point. The fourth section shows the solution of the equations of motion. Numerical study is presented in the fifth section, and finally the paper ends with the conclusion in the sixth section.

2. Model Description with Equations of Motion

Let m_1, m_2 (with $m_1 = m_2$) and $m(t)$ be three masses, where m_1 and m_2 are moving on the same circular orbit around their common center of mass (i.e., these primaries are forming the Copenhagen configuration). And the mass of third particle $m(t)$ varies its mass according to Jeans law ($\dot{m}(t) = -\epsilon_1 m(t)$), where ϵ_1 is variational constant which is defined for all values of real number except zero, is moving in the vertical direction (along z-axis) of the orbital plane of the primaries (i.e., xy-coordinate system or synodic coordinate system), it will clearly visualise from Figure 1. For non-dimensional coordinates, we fixed the sum of masses of the primaries, the distance between them and gravitational constant as unity. Hence, the mean motion will be unity. Following the procedure given by Sitnikov (1961), Abouelmagd et al. (2015) and Abouelmagd et al. (2019), we can write the equations of motion under the assumption that the variation of mass emitted from one point has zero momenta as

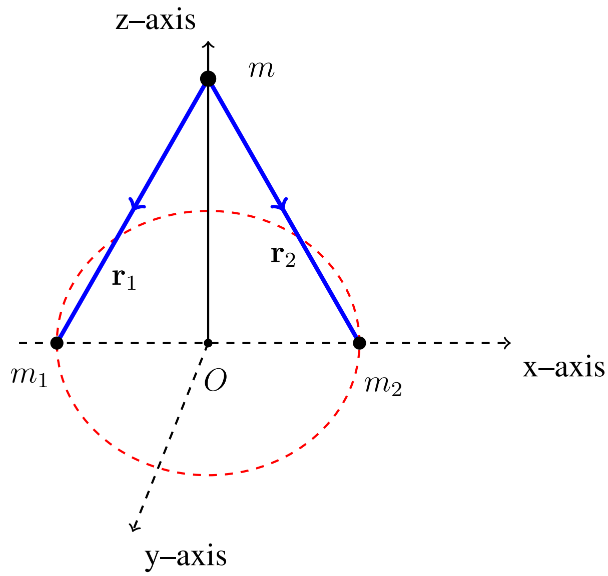


Figure 1. Geometric configuration for the vertical motion

$$\begin{cases} \ddot{x} - 2\dot{y} = \Psi_x, \\ \ddot{y} + 2\dot{x} = \Psi_y, \\ \ddot{z} \qquad = \Psi_z, \end{cases} \tag{1}$$

where

$$\begin{cases} \Psi \; = \dfrac{1}{2}(x^2+y^2) + \dfrac{\epsilon_1^2}{8}(x^2+y^2+z^2) + \dfrac{2\,\mu\,\epsilon_2^{3/2}}{\rho_1}, \\ \rho_i^2 = (x \pm \mu\,\epsilon_2^{1/2})^2 + y^2 + z^2, \quad i=1,2, \\ \dot{\epsilon_2} = -\,\epsilon_1\,\epsilon_2. \end{cases} \tag{2}$$

For the Sitnikov motion, we will put $x=0, y=0$ and $\mu=0.5$ in Equation (1), hence

$$\ddot{z} = \frac{\epsilon_1^2 z}{4} - \frac{\epsilon_2^{3/2} z}{(z^2+0.25\,\epsilon_2)^{3/2}}, \tag{3}$$

and the corresponding equilibrium point of Equation (3) is given by

$$(0,0,z) = \left(0,0,\frac{(4^{5/3}-\epsilon_1^{4/3})^{1/2}\sqrt{\epsilon_2}}{2\,\epsilon_1^{2/3}}\right). \tag{4}$$

This coordinate of equilibrium point clearly shows the dependency on the variable parameters ϵ_1 and ϵ_2.

3. Stability

The variational equations for the perturbed motion are

$$\begin{cases} \ddot{\xi} - 2\dot{\eta} = U_{xx}^0(z)\,\xi, \\ \ddot{\eta} + 2\dot{\xi} = U_{yy}^0(z)\,\eta, \\ \ddot{\zeta} \qquad\;\; = U_{zz}^0(z)\,\zeta, \end{cases} \tag{5}$$

where

$$U^0_{xx}(z) = 1 + \frac{\epsilon_1^2}{4} + \epsilon_2^{3/2}\left\{\frac{3\epsilon_2}{4(z^2+\frac{\epsilon_2}{4})^{5/2}} - \frac{1}{(z^2+\frac{\epsilon_2}{4})^{3/2}}\right\},$$

$$U^0_{yy}(z) = 1 + \frac{\epsilon_1^2}{4} - \frac{\epsilon_2^{3/2}}{(z^2+\frac{\epsilon_2}{4})^{3/2}},$$

$$U^0_{zz}(z) = \frac{\epsilon_1^2}{4} + \epsilon_2^{3/2}\left\{\frac{3z_1^2}{(z^2+\frac{\epsilon_2}{4})^{5/2}} - \frac{1}{(z^2+\frac{\epsilon_2}{4})^{3/2}}\right\},$$

$(\xi, \eta, \zeta) =$ the displacement from the equilibrium point $(0, 0, z)$,

where the super script zero means the value of the derivative of the potential function at the equilibrium point $(0, 0, z)$.

Let

$$\begin{cases} \dot\xi = u, \\ \dot\eta = v, \\ \dot\zeta = w, \\ \\ \dot u = U^0_{xx}(z)\,\xi + 2v, \\ \dot v = U^0_{yy}(z)\,\eta - 2u, \\ \dot w = U^0_{zz}(z)\,\zeta. \end{cases} \tag{6}$$

Due to variation of mass, we can not examine stability with general method; therefore, we will apply Meshcherskii inverse space time transformation as

$$\begin{cases} \alpha = \xi\,\epsilon_2^{-1/2}, \\ \eta = \eta\,\epsilon_2^{-1/2}, \\ \gamma = \zeta\,\epsilon_2^{-1/2}, \\ \alpha_1 = u\,\epsilon_2^{-1/2}, \\ \beta_1 = v\,\epsilon_2^{-1/2}, \\ \gamma_1 = w\,\epsilon_2^{-1/2}. \end{cases} \tag{7}$$

Using the above transformation, Equation (5) can be written as

$$\begin{bmatrix} \dot\alpha \\ \dot\beta \\ \dot\gamma \\ \dot\alpha_1 \\ \dot\beta_1 \\ \dot\gamma_1 \end{bmatrix} = M \times \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix}, \tag{8}$$

where

$$M = \begin{bmatrix} \frac{\epsilon_1}{2} & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{\epsilon_1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\epsilon_1}{2} & 0 & 0 & 1 \\ U^0_{xx} & 0 & 0 & \frac{\epsilon_1}{2} & 2 & 0 \\ 0 & U^0_{yy} & 0 & -2 & \frac{\epsilon_1}{2} & 0 \\ 0 & 0 & U^0_{zz} & 0 & 0 & \frac{\epsilon_1}{2} \end{bmatrix},$$

and the corresponding characteristic polynomial will be

$$f(\lambda) = \lambda^6 + C_5\lambda^5 + C_4\lambda^4 + C_3\lambda^3 + C_2\lambda^2 + C_1\lambda + C_0 \tag{9}$$

with

$$C_5 = -3\epsilon_1,$$

$$C_4 = 4 - W^0_{xx}(z) - W^0_{yy}(z) - W^0_{zz}(z) + \tfrac{15}{4}\epsilon_1^2,$$

$$C_3 = -\epsilon_1(8 - 2W^0_{xx}(z) - 2W^0_{yy}(z) - 2W^0_{zz}(z) + \tfrac{5}{2}\epsilon_1^2),$$

$$C_2 = W^0_{xx}(z)W^0_{yy}(z) + W^0_{zz}(z)(-4 + W^0_{xx}(z) + W^0_{yy}(z))$$

$$+\epsilon_1^2(6 - \tfrac{3}{2}W^0_{xx}(z) - \tfrac{3}{2}W^0_{yy}(z) - \tfrac{3}{2}W^0_{zz}(z) + \tfrac{15}{16}\epsilon_1^2),$$

$$C_1 = -W^0_{xx}(z)W^0_{yy}(z)\epsilon_1 + 4W^0_{zz}(z)\epsilon_1 - W^0_{xx}(z)W^0_{zz}(z)\epsilon_1$$

$$-W^0_{yy}(z)W^0_{zz}(z)\epsilon_1 - 2\epsilon_1^3 + \tfrac{1}{2}W^0_{xx}(z)\epsilon_1^3 + \tfrac{1}{2}W^0_{yy}(z)\epsilon_1^3 + \tfrac{1}{2}W^0_{zz}(z)\epsilon_1^3 - \tfrac{3}{16}\epsilon_1^5,$$

$$C_0 = -W^0_{xx}(z)W^0_{yy}(z)W^0_{zz}(z) + \tfrac{1}{4}W^0_{xx}(z)W^0_{yy}(z)\epsilon_1^2 - W^0_{zz}(z)\epsilon_1^2 + \tfrac{1}{4}W^0_{xx}(z)W^0_{zz}(z)\epsilon_1^2$$

$$+\tfrac{1}{4}W^0_{yy}(z)W^0_{zz}(z)\epsilon_1^2 + \tfrac{1}{4}\epsilon_1^4 - \tfrac{1}{16}W^0_{xx}(z)\epsilon_1^4 - \tfrac{1}{16}W^0_{yy}(z)\epsilon_1^4 - \tfrac{1}{16}W^0_{zz}(z)\epsilon_1^4 + \epsilon_1^6.$$

From Equation (9), $f(\lambda) \to \infty$ as $\lambda \to \infty$ and $f(0) = C_0 < 0$. Therefore, characteristic equation (9) has at least one positive root and hence the equilibrium point is unstable.

4. Solution by Lindstedt-Poincaré method

Rearranging Equation (3), we get

$$\ddot{z} + P_0\, z + Qz^3 = 0, \tag{10}$$

where

$$P_0 = \frac{32-\epsilon_1^2}{4},\, Q = -\frac{48}{\epsilon_2}.$$

We will use Lindstedt-Poincaré method to find the solution of Equation (10).

Let

$$\begin{cases} z \ = z_0 + Q\, z_1 + Q^2\, z_2 + Q^3\, z_3 + Q^4\, z_4 + \cdots\,, \\ s \ = P\, t, \\ P = P_0 + Q\, P_1 + Q^2\, P_2 + Q^3\, P_3 + Q^4\, P_4 + \cdots\,. \end{cases} \tag{11}$$

Hence from Equations (10, 11), we get

$$P_0^2(\frac{d^2 z_0}{ds^2} + z_0) = 0, \tag{12}$$

$$P_0^2(\frac{d^2 z_1}{ds^2} + z_1) + 2P_0 P_1 \frac{d^2 z_0}{ds^2} + z_0^3 = 0, \tag{13}$$

$$P_0^2(\frac{d^2 z_2}{ds^2} + z_2) + 2P_0 P_1 \frac{d^2 z_1}{ds^2} + (P_1^2 + 2P_0 P_2)\frac{d^2 z_0}{ds^2} + 3z_0^2 z_1 = 0, \tag{14}$$

$$\begin{aligned} &P_0^2(\frac{d^2 z_3}{ds^2} + z_3) + 2P_0 P_1 \frac{d^2 z_2}{ds^2} + (P_1^2 + 2P_0 P_2)\frac{d^2 z_1}{ds^2} + 2(P_1 P_2 + P_0 P_3)\frac{d^2 z_0}{ds^2} \\ &+3(z_0 z_1^2 + z_0^2 z_2) = \ 0. \end{aligned} \tag{15}$$

By taking initial value as $z_0(0) = S$ and $\dot{z}_0(0) = 0$, the solution of Equation (12) can find as

$$z_0 = S\, Cos(s). \tag{16}$$

The solution of Equation (13) by taking initial value as $z_1(0) = S$ and $\dot{z}_1(0) = 0$ will be

$$z_1 = S_0\, Cos(s) + A_0\, Cos(3s), \tag{17}$$

and also,

$$\begin{aligned} S_0 &= S - \frac{S^3}{32\, P_0^2}, \\ A_0 &= \frac{S^3}{32\, P_0^2}, \\ P_1 &= \frac{3S^2}{8\, P_0}. \end{aligned} \tag{18}$$

When initial value is taken as $z_2(0) = S$ and $\dot{z}_2(0) = 0$, then the solution of Equation (14) will be

$$z_2 = S_1\,Cos(s) + \tfrac{A_1}{8}Cos(3s) + \tfrac{A_2}{24}Cos(5s), \tag{19}$$

where

$$\begin{aligned} S_1 &= S - \frac{3S^3}{32\,P_0^2} - \frac{21S^5}{1024\,P_0^4}, \\ A_1 &= \frac{3S^3}{4\,P_0^2} + \frac{9S^5}{128\,P_0^4}, \\ A_2 &= \frac{9S^5}{32\,P_0^4}, \\ P_2 &= \frac{3S^2}{4\,P_0} - \frac{3S^4}{128\,P_0^3}. \end{aligned} \tag{20}$$

The solution of Equation (15) can be written by considering the initial value as $z_3(0) = S$ and $\dot{z}_3(0) = 0$ as

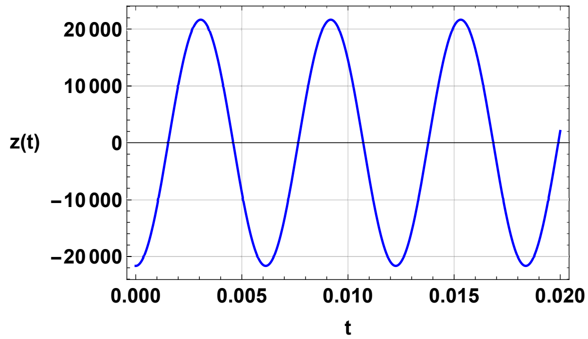
$$z_3 = S_2\,Cos(s) - \tfrac{B_1}{8}Cos(3s) - \tfrac{B_2}{24}Cos(5s) - \tfrac{B_3}{48}Cos(7s), \tag{21}$$

where

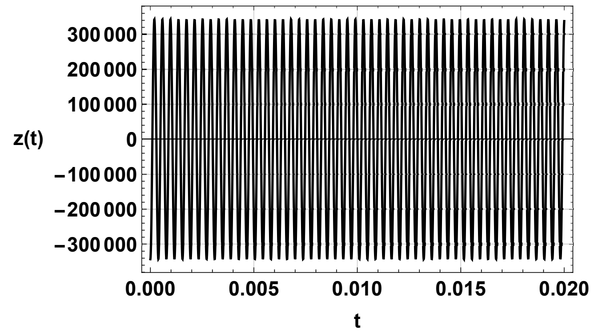
$$\begin{aligned} S_2 &= S + \frac{B_1}{8} + \frac{B_2}{24} + \frac{B_3}{48}, \\ B_1 &= -\frac{3S^3}{2\,P_0^2} + \frac{72S^5}{128\,P_0^4} + \frac{141S^7}{4096\,P_0^6}, \\ B_2 &= -\frac{3S^3}{32\,P_0^2} + \frac{9S^5}{1024\,P_0^4} + \frac{3175S^7}{32768\,P_0^6}, \\ B_3 &= -\frac{39S^7}{4096\,P_0^6}, \\ P_3 &= \frac{15S^2}{8\,P_0} - \frac{93S^4}{256\,P_0^3} - \frac{390S^6}{8192\,P_0^5}. \end{aligned} \tag{22}$$

Finally we get the solution of Equation (11) is

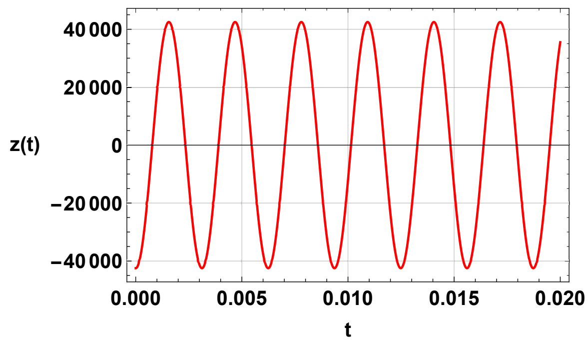
$$\begin{aligned} z(t) &= S\,Cos(P\,t) \\ &+ Q\,\{S_0\,Cos(P\,t) + A_0\,Cos(3\,P\,t)\} \\ &+ Q^2\,\{S_1\,Cos(P\,t) + \tfrac{A_1}{8}Cos(3\,P\,t) + \tfrac{A_2}{24}Cos(5\,P\,t)\} \\ &+ Q^3\,\{S_2\,Cos(P\,t) - \tfrac{B_1}{8}Cos(3\,P\,t) - \tfrac{B_2}{24}Cos(5\,P\,t) - \tfrac{B_3}{48}Cos(7\,P\,t)\} \\ &+ \cdots . \end{aligned} \tag{23}$$



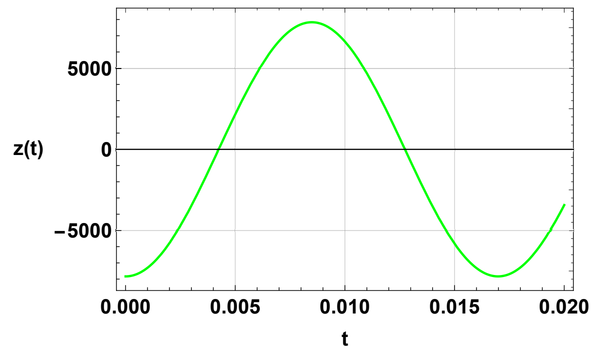
(a) For the evolution case at
$S = 0.2$, $\epsilon_1 = 0$, $\epsilon_2 = 1$



(b) For the evolution case at
$S = 0.2$, $\epsilon_1 = 0.2$, $\epsilon_2 = 0.4$

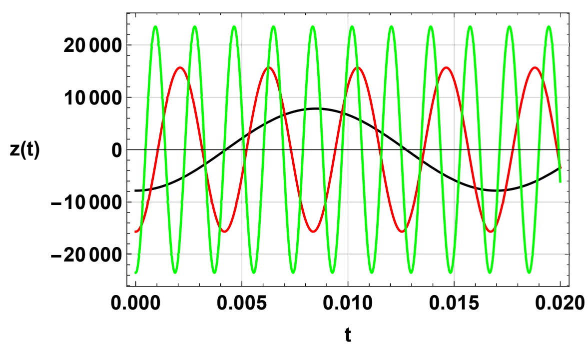


(c) For the evolution case at
$S = 0.2$, $\epsilon_1 = 0.2$, $\epsilon_2 = 0.8$

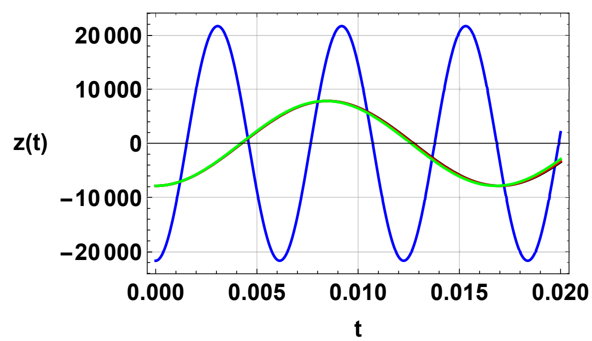


(d) For the evolution case at
$S = 0.2$, $\epsilon_1 = 0.2$, $\epsilon_2 = 1.4$

Figure 2. Time-series evolution for the different values of variation parameters



(a) Evolution for the different initial values
$S = 0.2$(Black), $S = 0.4$(Red), $S = 0.6$(Green)



(b) Evolution for the different values of
$\epsilon_1 = 0$(Blue), $\epsilon_1 = 0.2$(Black),
$\epsilon_1 = 0.4$(Red), $\epsilon_1 = 0.6$(Green)

Figure 3. Time-series evolution for the different values of S, ϵ_1 and fixed value of $\epsilon_2 = 1.4$

5. Numerical Analysis

To see the impact of the variational constant on the motion of the third body, we need to study numerically for the various values of parameters used with the help of equation (23) up-to third order of Q, and given in Figures 2 and 3. The value of S is taken from Shahbaz Ullah et al. (2015). From Figure (2 a), we observed that there is a sine series with amplitude approximately

20,000 units for time period around 0.0065 units when there is constant mass (i.e., ϵ_1 = 0, ϵ_2 = 1), but when we consider the effects of variation parameters we again get the sine series but with different amplitude and time period. Figure (2 b) (i.e., ϵ_1 = 0.2, ϵ_2 = 0.4), presents amplitude around 300000 units and time periods 0.0005 units. Figure (2 c) (i.e., ϵ_1 = 0.2, ϵ_2 = 0.8), presents amplitude around 40000 units and time periods 0.003 units. Figure (2 d) (i.e., ϵ_1 = 0.2, ϵ_2 = 1.4), presents amplitudes around 9000 units and time periods 0.0225 units. From all these figures we reveal that when we took the effect of variation parameters, the amplitude increases highly while the time period decreases. As we increase the value of parameter ϵ_2 from 0.4 to 1.4, we observed that amplitude decreases and time period increases.

From Figure (3 a), i.e., we observed when we increase the initial value, we revealed that amplitude increases and time period decreases. Similarly from Figure (3 b), we received that as we increase the value of ϵ_1 from 0 to 0.6, the amplitude decreases and time period increases. In this way we can say that the variation parameters have great impact on the motion of the infinitesimal body.

6. Conclusion

Here we have presented the Sitnikov problem with copenhagen configuration where infinitesimal mass is moving along the z-axis and varying its mass according to Jeans law. Our equations of motion are different from the original Sitnikov problem due to variation parameters ϵ_1 and ϵ_2. We have evaluated the equilibrium point and examined the stability of the equilibrium point and found that this equilibrium point is unstable. Further we have solved the equation of motion by Lindstedt-Poincaré Method analytically and solved numerically for the various values of the variation parameters. We observed that these parameters have great impact on the motion of the infinitesimal body.

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