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Replenishment Policy for Pareto Type Deteriorating Items With Quadratic Demand under Partial Backlogging And Delay in Payments

[1*]**Ganesh Kumar,** [2]**Ramesh Inaniyan and** [3]**Sunita**

[1,2,3]Department of Mathematics
University of Rajasthan
Jaipur-302004
Rajasthan, India
[1]ganeshmandha1988@gmail.com; [2]rameshinaniyan05@gmail.com; [3]sunitarolan95@gmail.com

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Abstract

The present model develops a replenishment policy in which the demand rate is quadratic polynomial-time function. Deterioration rate is a Pareto type function. Shortages are partial backlogging and delay in payments are allowed. Holding cost is a linear function of time. The backlogging rate varies with the waiting duration for the next replenishment. The present paper determines the optimal policy for the individual by minimizing the total cost. The optimization procedure has been explained by a numerical example and a detailed sensitivity analysis of the optimal solution has been carried out to display the effect of various parameters.

Keywords: Inventory model; Instantaneous deterioration; Pareto type function; EOQ model and partial backlogging

MSC 2010 No.: 90B05, 93A30

1. Introduction

One of the key issues for any industry is how to manage and sustain the inventories of products. The first quantity inventory model to support business in reducing overall inventory costs was developed by Harris (1915). Later, several researchers established EOQ models by taking time or selling price or market price-dependent demand. Several researchers have done work in this area. Goyal (1985), Covert and Philip (1973), Shah (1993), Amutha and Chandrasekaran (2013), Dari and Sani (2020), Rangarajan and Karthikeyan (2016), and Tadikamalla (1978) assumed the EOQ model at constant demand rate. Inventory models for deteriorating items with time-varying demand and partial backlogging were established by Teng et al. (2003). Jamel et al. (2000) provided a note within the delay of payments by the wholesaler. Khanra et al. (2011), Jaggi et al. (2015), Singh et al. (2010), Gothi et al. (2016), Rao and Rao (2016), and Tripathi (2019) considered that demand rate is a function of time. Mondal et al. (2000), De and Sana (2015), Udayakumar and Geeta (2017), Indrajitsingha et al. (2019), Sahoo et al. (2019) assumed demand rate selling price dependent. Sarkar and Sarkar (2013), Singh et al. (2011), Jain et al. (2014), and Pal et al. (2014) developed inventory models with stock dependent demand rate. Ghare and Schrader (1963), Covert and Philip (1973), Agrawal and Jaggi (1995) investigated the ordering policies for deteriorating items under permissible delay in payments.

Several researchers such as Pal et al. (2014), Liao et al. (2001), Saha et al. (2018), and Gothi et al. (2016) considered constant deterioration rate. Covert and Philip (1973), Singh and Pattnayak (2013), Singh et al. (2010), Khurana et al. (2018), Vandana (2018), and Sahoo et al. (2019) considered time-varying deterioration rate. An EOQ model under instantaneous replenishment with zero lead time was developed by Padmanabhan and Vrat (1995). Mondal et al. (2000) addressed the inventory model with an imperfect production process.

Most of the researchers developed their models using a different type of deterioration rates. Most of them ignored the Pareto type deterioration rate. To our best knowledge, such types of models for quadratic demand, partial backlogging, and delay in payments with Pareto type deterioration rate are new.

This paper introduces an inventory model for instantaneous deteriorating items when shortages are allowed and are partially backlogged. The deterioration rate is considered the Pareto type function. The quadratic polynomial-time function demand rate is assumed. Delay in payments is allowed. The optimal policies are determined by minimizing the total cost. The optimization procedure has been illustrated by a numerical example and the detailed sensitivity analysis has been carried out to display the effect of various parameters.

This paper is organized as follows. Some basic assumptions and notations are presented in Sections 2 and 3, respectively. In Section 4, the mathematical formulation and solution of the model are discussed. To illustrate the proposed model a numerical example is discussed in Section 5. Sensitivity analysis of optimal time, economic order quantity, and total cost with respect to different parameters are presented in Section 6. In Section 7 observations are discussed. Conclusions are made in Section 8.

2. Assumptions

i. Lead time is assigned to zero.

ii. Replenishment rate is infinite, i.e., it is instantaneous.
iii. The finite planning horizon is reckoned.
iv. The demand rate, which is positive and consecutive, is a quadratic function of time.
v. Several types of deterioration rates are available in the literature. In the present study, we assumed that commodity life deterioration time t has the probability density function,

$$f(t) = \begin{cases} \frac{1}{\theta_1}(1 - \frac{\theta_2 t}{\theta_1})^{\frac{1}{\theta_2}-1}, & \theta_2 \neq 0, \\ \frac{1}{\theta_1} e^{\frac{-t}{\theta_1}}, & \theta_2 = 0, \end{cases}$$

where θ_1 and θ_2 are parameters. Distributions corresponding to this pdf are called Pareto type, which is taken as deterioration rate. Deterioration rate $\theta(t)$ is given as follows:

$$\theta(t) = \frac{f(t)}{1 - \int_0^t f(y)dy} = \frac{1}{\theta_1 - \theta_2 t}.$$

vi. Backorder starts after time t_1, which is decision variable.
vii. Supplier didn't have provided the replacement or return strategy.
viii. Entities that have terminated will be demolished.
ix. A single type of item is considered in this model.
x. Shortages are tolerated and partially backlogged. The backlogging rate is variable and is dependent on the length of the waiting time for the next replenishment. The backlogging rate is distinguished by $B(t) = \frac{1}{e^{\delta(T-t)}}$, where $\delta > 0$ is backlogging parameter and $t_1 \leq t \leq T$.

3. Notations

The following notations have been used for compiling this inventory model:

$D(t)$: The demand rate at time t.
$\theta(t)$: Deterioration rate.
HC : Holding cost has been taken as a linear function of time.
t_1 : Time to exhaust stock within a replenishment cycle.
T : Length of a replenishment cycle.
M : Permissible delay period.
R : The maximum inventory level at the scheduling period.
P : The maximum amount of demand backlogged.
A : Ordering cost.
$p(t)$: The selling price at time t.
C_b : Unit shortage cost of an item.
C_d : Unit purchasing cost of an item.
C_l : Unit lost sale cost of an item.
I_p : The interest charged per unit of money per year.
I_e : The interest earned per unit of money per year.

4. Mathematical Formulation and Solution of the model

In the proposed model, we have assumed that R is the initial inventory level. The inventory level is decreasing due to the quadratic time-dependent demand rate and deterioration rate (Pareto type) during the time interval $[0, t_1]$ and reaches zero level at the time t_1. Further, during the time interval $[t_1, T]$ the shortages occur, and demand is taken to be partially backlogged. The present model is developed for instantaneous deteriorating items, i.e., deterioration of inventory starts at $t = 0$. The breakdown of goods is the primary concern in numerous inventory-based networks. The deterioration is known to be damaged, vaporization, dryness, spoilage, etc. Blood supply, unreliable materials, medications, and consumer goods are deteriorating inventories decaying throughout their storage time. In this model shortages occurs during the period $[t_1, T]$, which are considered partially backlogged, and during the drought, demand is assumed to be constant D_0. For the depletion of the inventory system, see Figure 1 in the Appendix.

Let the instantaneous inventory level is $I(t)$ at any time 't' during the cycle time $[0, t_1]$, and demand rate is $D(t)$. Then the inventory system is governed by the differential equation

$$\frac{dI(t)}{dt} + \theta(t)I(t) = -D(t),\ 0 \le t \le t_1,$$
$$\implies \frac{dI(t)}{dt} + \left(\frac{1}{\theta_1 - \theta_2 t}\right) I(t) = -(a + bt + ct^2),\ 0 \le t \le t_1. \tag{1}$$

The solution of Equation (1) with boundary condition $I(t_1) = 0$, is

$$\begin{aligned} I(t) =& \frac{1}{\theta_2^2}\left[\frac{a\theta_2^2 + b\theta_1\theta_2 + c\theta_1^2}{\theta_2 - 1}\left\{(\theta_1 - \theta_2 t) - (\theta_1 - \theta_2 t_1)^{1-\frac{1}{\theta_2}}(\theta_1 - \theta_2 t)^{\frac{1}{\theta_2}}\right\}\right. \\ & - \frac{b\theta_2 + 2c\theta_1}{2\theta_2 - 1}\left\{(\theta_1 - \theta_2 t)^2 - (\theta_1 - \theta_2 t_1)^{2-\frac{1}{\theta_2}}(\theta_1 - \theta_2 t)^{\frac{1}{\theta_2}}\right\} \\ & \left. + \frac{c}{3\theta_2 - 1}\left\{(\theta_1 - \theta_2 t)^3 - (\theta_1 - \theta_2 t_1)^{3-\frac{1}{\theta_2}}(\theta_1 - \theta_2 t)^{\frac{1}{\theta_2}}\right\}\right],\ 0 \le t \le t_1. \end{aligned} \tag{2}$$

Using boundary condition $I(0) = R$, the maximum positive inventory is

$$\begin{aligned} R =& \frac{1}{\theta_2^2}\left[\frac{a\theta_2^2 + b\theta_1\theta_2 + c\theta_1^2}{\theta_2 - 1}\left\{\theta_1 - (\theta_1 - \theta_2 t_1)^{1-\frac{1}{\theta_2}}{\theta_1}^{\frac{1}{\theta_2}}\right\}\right. \\ & - \frac{b\theta_2 + 2c\theta_1}{2\theta_2 - 1}\left\{{\theta_1}^2 - (\theta_1 - \theta_2 t_1)^{2-\frac{1}{\theta_2}}{\theta_1}^{\frac{1}{\theta_2}}\right\} \\ & \left. + \frac{c}{3\theta_2 - 1}\left\{{\theta_1}^3 - (\theta_1 - \theta_2 t_1)^{3-\frac{1}{\theta_2}}{\theta_1}^{\frac{1}{\theta_2}}\right\}\right]. \end{aligned} \tag{3}$$

4.1. Partial Backlogging Model

The governing differential equation representing instantaneous inventory level $I_1(t)$, at any time $'t'$ during the shortage period $[t_1, T]$, is

$$\frac{dI_1(t)}{dt} = -\frac{D_0}{e^{\delta(T-t)}}, t_1 \leq t \leq T. \tag{4}$$

The solution of the equation (4) with boundary conditions $I_1(t_1) = 0$, is

$$I_1(t) = \frac{D_0}{\delta}[e^{-\delta(T-t_1)} - e^{-\delta(T-t)}], t_1 \leq t \leq T. \tag{5}$$

Using boundary condition $-I_1(T) = P$, negative inventory is

$$I_1(T) = P = \frac{D_0}{\delta}[1 - e^{-\delta(T-t_1)}]. \tag{6}$$

Total inventory is $Q = R + P$,

$$\begin{aligned} Q =& \frac{1}{\theta_2^2}\left[\frac{a\theta_2^2 + b\theta_1\theta_2 + c\theta_1^2}{\theta_2 - 1}\left\{\theta_1 - (\theta_1 - \theta_2 t_1)^{1-\frac{1}{\theta_2}}\theta_1^{\frac{1}{\theta_2}}\right\}\right. \\ &-\frac{b\theta_2 + 2c\theta_1}{2\theta_2 - 1}\left\{{\theta_1}^2 - (\theta_1 - \theta_2 t_1)^{2-\frac{1}{\theta_2}}\theta_1^{\frac{1}{\theta_2}}\right\} \\ &\left.+\frac{c}{3\theta_2 - 1}\left\{{\theta_1}^3 - (\theta_1 - \theta_2 t_1)^{3-\frac{1}{\theta_2}}\theta_1^{\frac{1}{\theta_2}}\right\}\right] + \frac{D_0}{\delta}[1 - e^{-\delta(T-t_1)}]. \end{aligned} \tag{7}$$

The total average inventory cost (TC) per cycle consists of the following costs.

i. **ordering cost per cycle**

$$OC = \frac{A}{T}. \tag{8}$$

ii. **Holding cost per cycle**

$$\begin{aligned} HC =& \frac{1}{T}\int_0^{t_1}(\alpha + \beta t)I(t)dt \\ =& \frac{\alpha}{T\theta_2^2}\left[\frac{a\theta_2^2 + b\theta_1\theta_2 + c\theta_1^2}{\theta_2 - 1}\left\{\frac{(\theta_2 - 1)(\theta_1 - \theta_2 t_1)^2}{2\theta_2(\theta_2 + 1)} + \frac{\theta_1^2}{2\theta_2}\right.\right. \\ &\left.-\frac{(\theta_1 - \theta_2 t_1)^{1-\frac{1}{\theta_2}}{\theta_1}^{1+\frac{1}{\theta_2}}}{\theta_2 + 1}\right\} - \frac{b\theta_2 + 2c\theta_1}{2\theta_2 - 1}\left\{\frac{(2\theta_2 - 1)(\theta_1 - \theta_2 t_1)^3}{3\theta_2(\theta_2 + 1)}\right. \\ &\left.+\frac{\theta_1^3}{3\theta_2} - \frac{(\theta_1 - \theta_2 t_1)^{2-\frac{1}{\theta_2}}{\theta_1}^{1+\frac{1}{\theta_2}}}{\theta_2 + 1}\right\} + \frac{c}{3\theta_2 - 1}\left\{\frac{(3\theta_2 - 1)(\theta_1 - \theta_2 t_1)^4}{4\theta_2(\theta_2 + 1)}\right. \\ &\left.\left.+\frac{\theta_1^4}{4\theta_2} - \frac{(\theta_1 - \theta_2 t_1)^{3-\frac{1}{\theta_2}}{\theta_1}^{1+\frac{1}{\theta_2}}}{\theta_2 + 1}\right\}\right] \end{aligned} \tag{9}$$

$$+\frac{\beta}{T\theta_2^2}\left[\frac{a\theta_2^2+b\theta_1\theta_2+c\theta_1^2}{\theta_2-1}\left\{\frac{(4\theta_2+1)(\theta_2-1)(\theta_1-\theta_2t_1)^3}{6\theta_2^2(\theta_2+1)(2\theta_2+1)}+\frac{\theta_1^3}{6\theta_2}\right.\right.$$
$$\left.-\frac{(\theta_1-\theta_2t_1)^{1-\frac{1}{\theta_2}}{\theta_1}^{2+\frac{1}{\theta_2}}}{(\theta_2+1)(2\theta_2+1)}\right\}-\frac{b\theta_2+2c\theta_1}{2\theta_2-1}\left\{\frac{(5\theta_2+1)(2\theta_2-1)(\theta_1-\theta_2t_1)^4}{12\theta_2^2(\theta_2+1)(2\theta_2+1)}\right.$$
$$\left.+\frac{\theta_1^4}{12\theta_2}-\frac{(\theta_1-\theta_2t_1)^{2-\frac{1}{\theta_2}}{\theta_1}^{2+\frac{1}{\theta_2}}}{(\theta_2+1)(2\theta_2+1)}\right\}$$
$$+\frac{c}{3\theta_2-1}\left\{\frac{(6\theta_2+1)(3\theta_2-1)(\theta_1-\theta_2t_1)^5}{20\theta_2^2(\theta_2+1)(2\theta_2+1)}+\frac{\theta_1^5}{20\theta_2^2}\right.$$
$$\left.\left.-\frac{(\theta_1-\theta_2t_1)^{3-\frac{1}{\theta_2}}{\theta_1}^{2+\frac{1}{\theta_2}}}{(\theta_2+1)(2\theta_2+1)}\right\}\right]. \tag{10}$$

iii. **Shortage cost per cycle**

$$SC=\frac{C_b}{T}\int_{t_1}^{T}[-I_1(t)]\,dt$$
$$=\frac{C_bD_0}{\delta^2T}\left[1-e^{-\delta(T-t_1)}\{1+\delta(T-t_1)\}\right]. \tag{11}$$

iv. **Deterioration cost per cycle**

$$DC=\frac{C_d}{T}\left\{R-\int_0^{t_1}D(t)\,dt\right\}$$
$$=\frac{C_d}{T}\left[\frac{1}{\theta_2^2}\left[\frac{a\theta_2^2+b\theta_1\theta_2+c\theta_1^2}{\theta_2-1}\left\{\theta_1-(\theta_1-\theta_2t_1)^{1-\frac{1}{\theta_2}}{\theta_1}^{\frac{1}{\theta_2}}\right\}\right.\right.$$
$$-\frac{b\theta_2+2c\theta_1}{2\theta_2-1}\left\{{\theta_1}^2-(\theta_1-\theta_2t_1)^{2-\frac{1}{\theta_2}}{\theta_1}^{\frac{1}{\theta_2}}\right\}$$
$$\left.+\frac{c}{3\theta_2-1}\left\{{\theta_1}^3-(\theta_1-\theta_2t_1)^{3-\frac{1}{\theta_2}}{\theta_1}^{\frac{1}{\theta_2}}\right\}\right]$$
$$\left.-(at_1+\frac{bt_1^2}{2}+\frac{ct_1^3}{3})\right]. \tag{12}$$

v. **Cost due to lost sales per cycle**

$$CLS=\frac{C_lD_0}{T}\int_{t_1}^{T}\left[1-\frac{1}{1+\delta(T-t)}\right]dt$$
$$=\frac{C_lD_0}{T}\left[T-t_1-\frac{1}{\delta}\log\{1+\delta(T-t_1)\}\right]. \tag{13}$$

vi. **Interest earned per cycle** $(t_1<M)$

$$I_nE=\frac{pI_e}{T}\int_0^{M}tD(t)\,dt$$
$$=\frac{pI_e}{T}\left[\frac{at_1^2}{2}+\frac{bt_1^3}{3}+\frac{ct_1^4}{4}+\frac{D_0}{2}(M^2-t_1^2)\right]. \tag{14}$$

vii. **Interest payable per cycle** $(t_1 \geq M)$

$$
\begin{aligned}
I_nP_1 &= \frac{C_pI_p}{T}\int_M^{t_1} I(t)\,dt \\
&= \frac{C_pI_p}{T}\Bigg[\frac{a\theta_2^2+b\theta_1\theta_2+c\theta_1^2}{2\theta_2(1+\theta_2)}(\theta_1-\theta_2t_1)^2-\frac{b\theta_2+2c\theta_1}{3\theta_2(1+\theta_2)}(\theta_1-\theta_2t_1)^3 \\
&\quad +\frac{c}{4\theta_2(1+\theta_2)}(\theta_1-\theta_2t_1)^4+(\theta_1-\theta_2M)^{1+\frac{1}{\theta_2}}\Bigg[\frac{a\theta_2^2+b\theta_1\theta_2+c\theta_1^2}{\theta_2-1} \\
&\quad \left\{\frac{1}{2\theta_2}(\theta_1-\theta_2M)^{1-\frac{1}{\theta_2}}-\frac{1}{1+\theta_2}(\theta_1-\theta_2t_1)^{1-\frac{1}{\theta_2}}\right\} \\
&\quad -\frac{b\theta_2+2c\theta_1}{2\theta_2-1}\left\{\frac{1}{3\theta_2}(\theta_1-\theta_2M)^{2-\frac{1}{\theta_2}}-\frac{1}{1+\theta_2}(\theta_1-\theta_2t_1)^{2-\frac{1}{\theta_2}}\right\} \\
&\quad +\frac{c}{3\theta_2-1}\left\{\frac{1}{4\theta_2}(\theta_1-\theta_2M)^{3-\frac{1}{\theta_2}}-\frac{1}{1+\theta_2}(\theta_1-\theta_2t_1)^{3-\frac{1}{\theta_2}}\right\}\Bigg]\Bigg].
\end{aligned}
\tag{15}
$$

The total cost is

$$
TC = OC + HC + SC + DC + CLS + I_nP_1 - I_nE,
$$

$$
\begin{aligned}
TC &= \frac{A}{T}+\frac{\alpha}{T\theta_2^2}\Bigg[\frac{a\theta_2^2+b\theta_1\theta_2+c\theta_1^2}{\theta_2-1}\Bigg\{\frac{(\theta_2-1)(\theta_1-\theta_2t_1)^2}{2\theta_2(\theta_2+1)}+\frac{\theta_1^2}{2\theta_2} \\
&\quad -\frac{(\theta_1-\theta_2t_1)^{1-\frac{1}{\theta_2}}{\theta_1}^{1+\frac{1}{\theta_2}}}{\theta_2+1}\Bigg\}-\frac{b\theta_2+2c\theta_1}{2\theta_2-1}\Bigg\{\frac{(2\theta_2-1)(\theta_1-\theta_2t_1)^3}{3\theta_2(\theta_2+1)} \\
&\quad +\frac{\theta_1^3}{3\theta_2}-\frac{(\theta_1-\theta_2t_1)^{2-\frac{1}{\theta_2}}{\theta_1}^{1+\frac{1}{\theta_2}}}{\theta_2+1}\Bigg\}+\frac{c}{3\theta_2-1}\Bigg\{\frac{(3\theta_2-1)(\theta_1-\theta_2t_1)^4}{4\theta_2(\theta_2+1)} \\
&\quad +\frac{\theta_1^4}{4\theta_2}-\frac{(\theta_1-\theta_2t_1)^{3-\frac{1}{\theta_2}}{\theta_1}^{1+\frac{1}{\theta_2}}}{\theta_2+1}\Bigg\}\Bigg] \\
&\quad +\frac{\beta}{T\theta_2^2}\Bigg[\frac{a\theta_2^2+b\theta_1\theta_2+c\theta_1^2}{\theta_2-1}\Bigg\{\frac{(4\theta_2+1)(\theta_2-1)(\theta_1-\theta_2t_1)^3}{6\theta_2^2(\theta_2+1)(2\theta_2+1)}+\frac{\theta_1^3}{6\theta_2} \\
&\quad -\frac{(\theta_1-\theta_2t_1)^{1-\frac{1}{\theta_2}}{\theta_1}^{2+\frac{1}{\theta_2}}}{(\theta_2+1)(2\theta_2+1)}\Bigg\}-\frac{b\theta_2+2c\theta_1}{2\theta_2-1}\Bigg\{\frac{(5\theta_2+1)(2\theta_2-1)(\theta_1-\theta_2t_1)^4}{12\theta_2^2(\theta_2+1)(2\theta_2+1)} \\
&\quad +\frac{\theta_1^4}{12\theta_2}-\frac{(\theta_1-\theta_2t_1)^{2-\frac{1}{\theta_2}}{\theta_1}^{2+\frac{1}{\theta_2}}}{(\theta_2+1)(2\theta_2+1)}\Bigg\} \\
&\quad +\frac{c}{3\theta_2-1}\Bigg\{\frac{(6\theta_2+1)(3\theta_2-1)(\theta_1-\theta_2t_1)^5}{20\theta_2^2(\theta_2+1)(2\theta_2+1)}+\frac{\theta_1^5}{20\theta_2^2} \\
&\quad -\frac{(\theta_1-\theta_2t_1)^{3-\frac{1}{\theta_2}}{\theta_1}^{2+\frac{1}{\theta_2}}}{(\theta_2+1)(2\theta_2+1)}\Bigg\}\Bigg]
\end{aligned}
\tag{16}
$$

$$+\frac{C_d}{T}\left[\frac{1}{\theta_2^2}\left[\frac{a\theta_2^2+b\theta_1\theta_2+c\theta_1^2}{\theta_2-1}\left\{\theta_1-(\theta_1-\theta_2t_1)^{1-\frac{1}{\theta_2}}\theta_1{}^{\frac{1}{\theta_2}}\right\}\right.\right.$$
$$-\frac{b\theta_2+2c\theta_1}{2\theta_2-1}\left\{\theta_1{}^2-(\theta_1-\theta_2t_1)^{2-\frac{1}{\theta_2}}\theta_1{}^{\frac{1}{\theta_2}}\right\}$$
$$\left.\left.+\frac{c}{3\theta_2-1}\left\{\theta_1{}^3-(\theta_1-\theta_2t_1)^{3-\frac{1}{\theta_2}}\theta_1{}^{\frac{1}{\theta_2}}\right\}\right]-(at_1+\frac{bt_1^2}{2}+\frac{ct_1^3}{3})\right]$$
$$-\frac{pI_e}{T}\left[\frac{at_1^2}{2}+\frac{bt_1^3}{3}+\frac{ct_1^4}{4}+\frac{D_0}{2}(M^2-t_1^2)\right]$$
$$+\frac{C_pI_p}{T}\left[\frac{a\theta_2^2+b\theta_1\theta_2+c\theta_1^2}{2\theta_2(1+\theta_2)}(\theta_1-\theta_2t_1)^2-\frac{b\theta_2+2c\theta_1}{3\theta_2(1+\theta_2)}(\theta_1-\theta_2t_1)^3\right.$$
$$+\frac{c}{4\theta_2(1+\theta_2)}(\theta_1-\theta_2t_1)^4+(\theta_1-\theta_2M)^{1+\frac{1}{\theta_2}}\left[\frac{a\theta_2^2+b\theta_1\theta_2+c\theta_1^2}{\theta_2-1}*\right.$$
$$\left\{\frac{1}{2\theta_2}(\theta_1-\theta_2M)^{1-\frac{1}{\theta_2}}-\frac{1}{1+\theta_2}(\theta_1-\theta_2t_1)^{1-\frac{1}{\theta_2}}\right\}$$
$$-\frac{b\theta_2+2c\theta_1}{2\theta_2-1}\left\{\frac{1}{3\theta_2}(\theta_1-\theta_2M)^{2-\frac{1}{\theta_2}}-\frac{1}{1+\theta_2}(\theta_1-\theta_2t_1)^{2-\frac{1}{\theta_2}}\right\}$$
$$\left.\left.+\frac{c}{3\theta_2-1}\left\{\frac{1}{4\theta_2}(\theta_1-\theta_2M)^{3-\frac{1}{\theta_2}}-\frac{1}{1+\theta_2}(\theta_1-\theta_2t_1)^{3-\frac{1}{\theta_2}}\right\}\right]\right]. \tag{17}$$

Our objective is to minimize the total average inventory cost per cycle.

4.2. Solution Procedure

Step 1: First of all differentiate equation (15) with respect to decision variable t_1.
Step 2: Set the derivative equal to zero, i.e., $\frac{d(TC)}{dt_1}=0$.
Step 3: Find possible values of t_1.
Step 4: Find second order derivative of Equation (15) with respect to t_1.
Step 5: Check the sign of $\frac{d^2(TC)}{dt_1^2}$ at t_1.
Step 6: If $\frac{d^2(TC)}{dt_1^2}>0$, then TC will be minimum.

By using MATLAB software, the optimal value of t_1 which is denoted by t_1^* can be obtained. Then from Equations (7) and (15), the optimal values of economic order quantity Q^* and optimal total cost TC^* can be calculated. For this numerical process, we assume suitable values for A, T, a, b, c, α, β, p, θ_1, θ_2, D_0, δ, M, C_b, C_d, C_l, C_p, I_e, and I_p with appropriate units.

5. Numerical Example

Suppose that there is a product with deteriorating function $\theta(t)=\frac{1}{\theta_1-\theta_2t}$, where $\theta_1>0$ and $0<\theta_2<1$.

The parameters of the inventory system are

$$A = 500,\ T = 2,\ \ a = 5, b = 4, c = 3, \alpha = 2, \beta = 0.5, p = 40,$$
$$\theta_1 = 0.40, \theta_2 = 0.16, D_0 = 10, \delta = 0.5, M = 0.5,$$
$$C_b = 2, C_d = 3, C_l = 4, C_p = 5, I_e = 0.1, I_p = 0.2.$$

Using MATLAB software, optimum solutions are

Optimal time $(t_1^*) = 0.3379$.
Optimal total cost $(TC^*) = 265.5413$.
Economic order quantity $(Q^*) = 14.5101$.

6. Sensivity Analysis and Observations

In every decision-making case, the values of the parameters can be adjusted. Sensitivity analysis can be a tremendous beneficiary in decision-making to analyze the consequences of these adjustments.

In this section, we study the effects of changes in the system parameters a, b, c, θ_1, θ_2, δ, α, β, and M on the optimal values of t_1^*, TC^*, and Q^*. The sensitivity is analyzed by changing each of the parameters by$+50\%$, $+25\%$, -25% and -50%, taking one parameter as a variable at a time and keeping the remaining parameters unchanged. The results based on the above example are shown in Table 1.

7. Observations

i. From Table 1, as we change the value of parameter a from -50% to +50%, t_1^* decreases from 0.3978 to 0.2961, TC^* increases from 264.7562 to 266.0608 and Q^* increases from 13.5949 to 15.1653.
ii. From Table 1, as we change the value of parameter b from -50% to +50%, t_1^* decreases from 0.3451 to 0.3319, TC^* decreases from 265.7644 to 265.3136 and Q^* increases from 14.3759 to 14.6392.
iii. From Table 1, as we change the value of parameter c from -50% to +50%, t_1^* increases from 0.3275 to 0.3477, TC^* decreases from 266.0561 to 265.0157 and Q^* increases from 14.3522 to 14.6762.
iv. From Table 1, as we change the value of parameter δ from -50% to +50%, t_1^* decreases from 0.3427 to 0.3336, TC^* increases from 264.3611 to 266.3511 and Q^* decreases from 16.8712 to 12.6635.
v. From Table 1, as we change the value of parameter θ_1 from -50% to +50%, t_1^* increases from 0.1798 to 0.5484, TC^* decreases from 268.3715 to 258.2450 and Q^* increases from 13.5877 to 16.4666.
vi. From Table 1, as we change the value of parameter θ_2 from -50% to +50%, t_1^* decreases from 0.4171 to 0.3140, TC^* increases from 262.5172 to 266.3203 and Q^* decreases from 15.5842 to 14.2705.

Table 1. Sensitivity analysis of partial backlogging inventory model

Changing parameter	Changing %	Optimal values		
		t_1^*	TC^*	Q^*
a	-50%	0.3978	264.7562	13.5949
	-25%	0.3649	265.1937	14.0961
	+25%	0.3153	265.8247	14.8612
	+50%	0.2961	266.0608	15.1653
b	-50%	0.3451	265.7644	14.3759
	-25%	0.3413	265.6535	14.4438
	+25%	0.3348	265.4279	14.5752
	+50%	0.3319	265.3136	14.6392
c	-50%	0.3275	266.0561	14.3522
	-25%	0.3328	265.8001	14.4301
	+25%	0.3429	265.2798	14.5922
	+50%	0.3477	265.0157	14.6762
δ	-50%	0.3427	264.3611	16.8712
	-25%	0.3405	265.0205	15.6196
	+25%	0.3355	265.9744	13.5292
	+50%	0.3336	266.3511	12.6635
θ_1	-50%	0.1798	268.3715	13.5877
	-25%	0.2566	267.2358	13.9967
	+25%	0.4327	262.7856	15.2660
	+50%	0.5484	258.2450	16.4666
θ_2	-50%	0.4171	262.5172	15.5842
	-25%	0.3617	264.7497	14.7892
	+25%	0.3239	265.9752	14.3641
	+50%	0.3140	266.3203	14.2705
α	-50%	0.3380	266.0901	14.5117
	-25%	0.3380	265.8157	14.5109
	+25%	0.3379	265.2668	14.5094
	+50%	0.3378	264.9924	14.5087
β	-50%	0.3341	265.7163	14.4643
	-25%	0.3360	265.6289	14.4874
	+25%	0.3397	265.4532	14.5327
	+50%	0.3416	265.3648	14.5551
M	-50%	0.3377	267.4156	14.5078
	-25%	0.3378	266.6341	14.5092
	+25%	0.3380	264.1370	14.5108
	+50%	0.3380	262.4209	14.5112

vii. From Table 1, as we change the value of parameter α from -50% to +50%, t_1^* decreases from 0.3380 to 0.3378, TC^* decreases from 266.0901 to 264.9924 and Q^* decreases from 14.5117 to 14.5087.

viii. From Table 1, as we change the value of parameter β from -50% to +50%, t_1^* increases from 0.3341 to 0.3416, TC^* decreases from 265.7163 to 265.3648 and Q^* increases from 14.4643 to 14.5551.

ix. From Table 1, as we change the value of parameter M from -50% to +50%, t_1^* increases from 0.3377 to 0.3380, TC^* decreases from 267.4156 to 262.4209 and Q^* increases from 14.5078 to 14.5112.

8. Conclusion

In this study, a replenishment policy is developed. The demand rate is the quadratic polynomial-time function, deterioration rate is Pareto type, shortages are partially backlogged, and delay in payments is allowed. A mathematical model has been developed to minimize the total optimal cost. Several parameters in our model play an important role in the decision-making process. Decisions can be made by setting the parameters a, b, c, M, α, θ_1, θ_2, δ and β accordingly. Therefore, the model appears to be a decision-maker in such competing scenarios. The study concludes with a numerical example and detailed sensitivity analysis to provide certain significant organizational consequences. For further studies, the existing model can be extended to consider the demand rate as a function of the selling price or stock dependent, Weibull distribution type demand rate, inflation, etc.

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Appendix

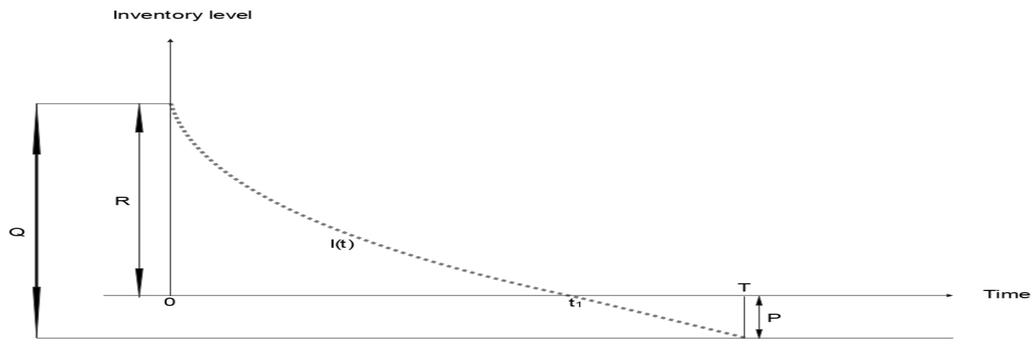

Figure 1. Inventory system for instantaneous deteriorating items with shortages

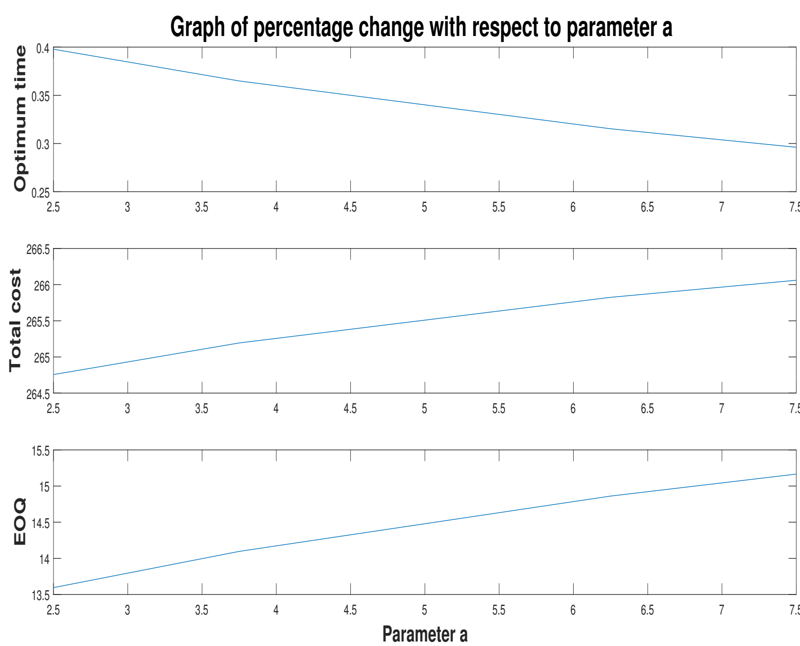

Figure 2. From the above graph we observe that with the increase in values of parameter a, the value of optimum time t_1^* decreases approximately linear, total cost TC^* increases approximately linear and economic order quantity Q^* increases approximately linear

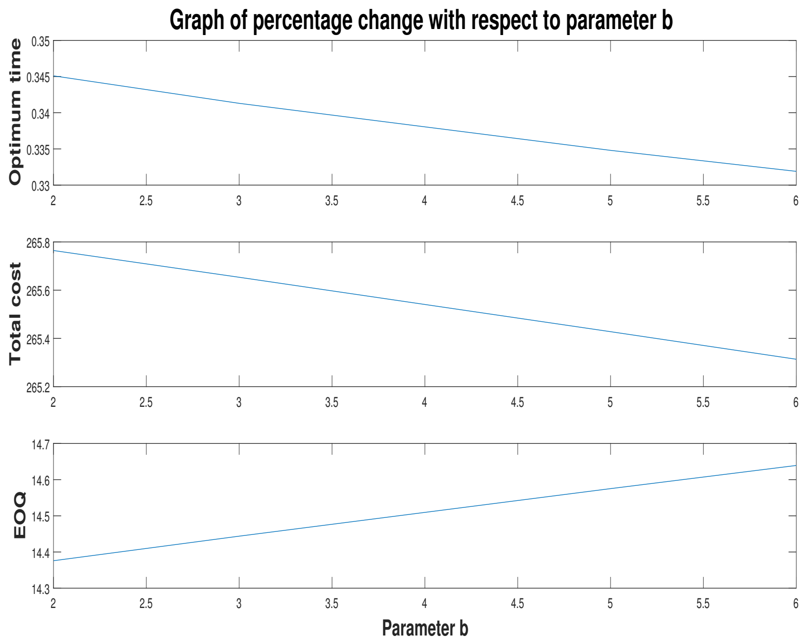

Figure 3. From the above graph we observe that with the increase in values of parameter b, the value of optimum time t_1^* decreases linearly, total cost TC^* decreases linearly and economic order quantity Q^* increases linearly

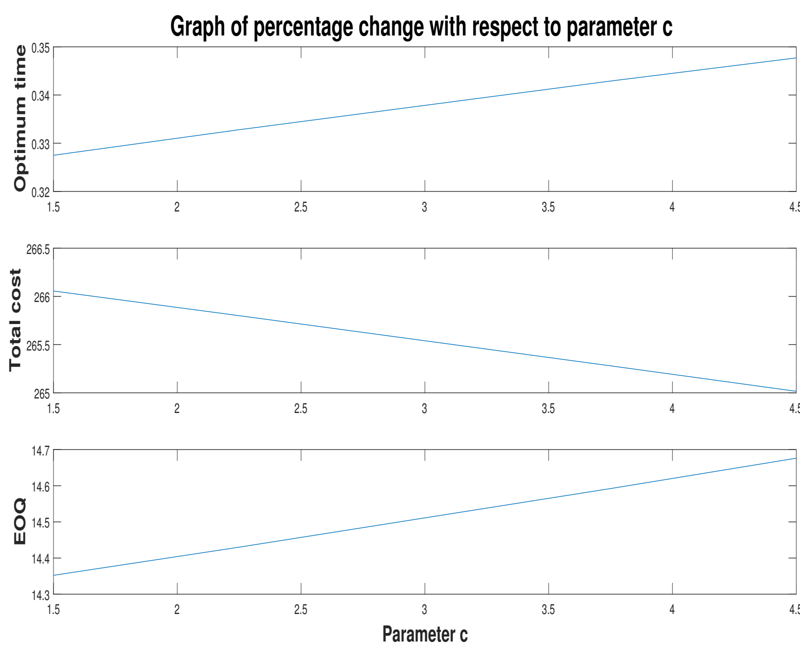

Figure 4. From the above graph we observe that with the increase in values of parameter c, the value of optimum time t_1^* increases linearly, total cost TC^* decreases linearly and economic order quantity Q^* increases linearly

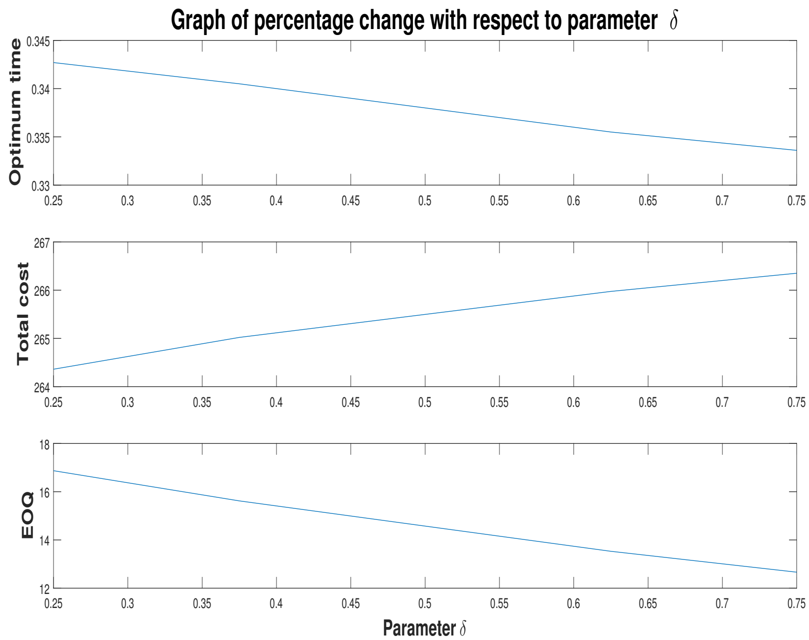

Figure 5. From the above graph we observe that with the increase in values of parameter δ, the value of optimum time t_1^* decreases approximately linear, total cost TC^* increases approximately linear and economic order quantity Q^* decreases approximately linear

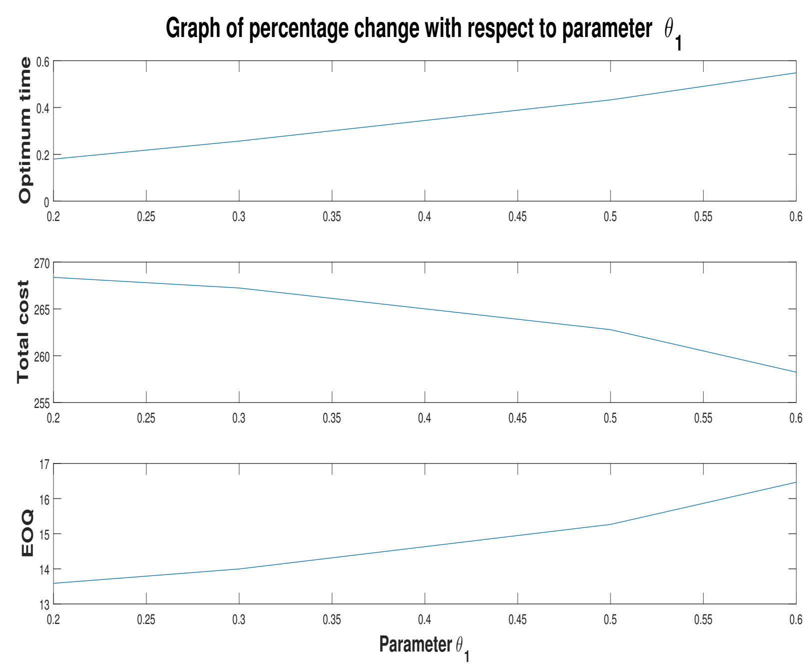

Figure 6. From the above graph we observe that with the increase in values of parameter θ_1, the value of optimum time t_1^* increases approximately linear, total cost TC^* initially decreases slowly and then decreases rapidly and economic order quantity Q^* initially increases slowly and then increases rapidly

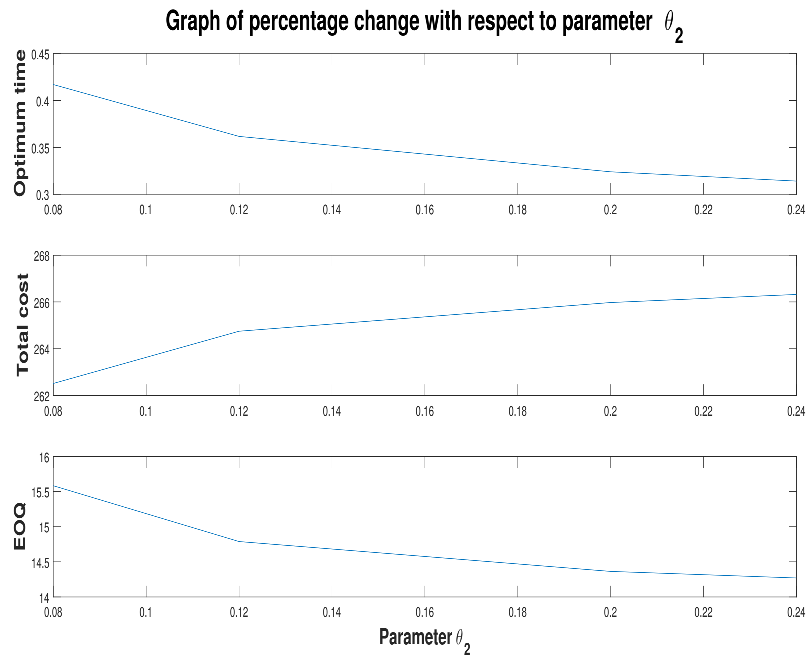

Figure 7. From the above graph we observe that with the increase in values of parameter θ_2, the value of optimum time t_1^* initially decreases rapidly and then decreases slowly, total cost TC^* initially increases rapidly and then increases slowly and economic order quantity Q^* initially decreases rapidly and then decreases slowly

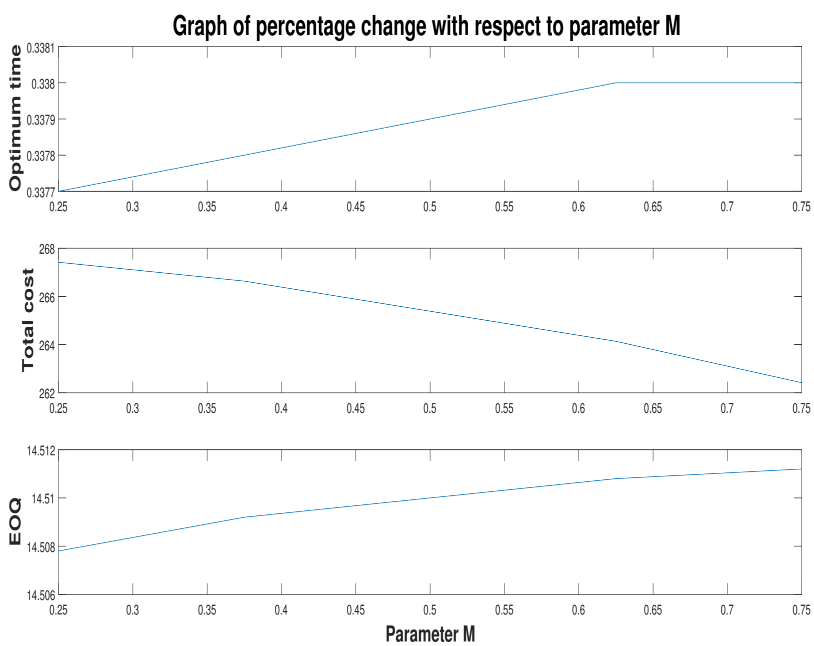

Figure 8. From the above graph we observe that with the increase in values of parameter M, the value of optimum time t_1^* increases linearly, total cost TC^* decreases approximately linear and economic order quantity Q^* increases approximately linear

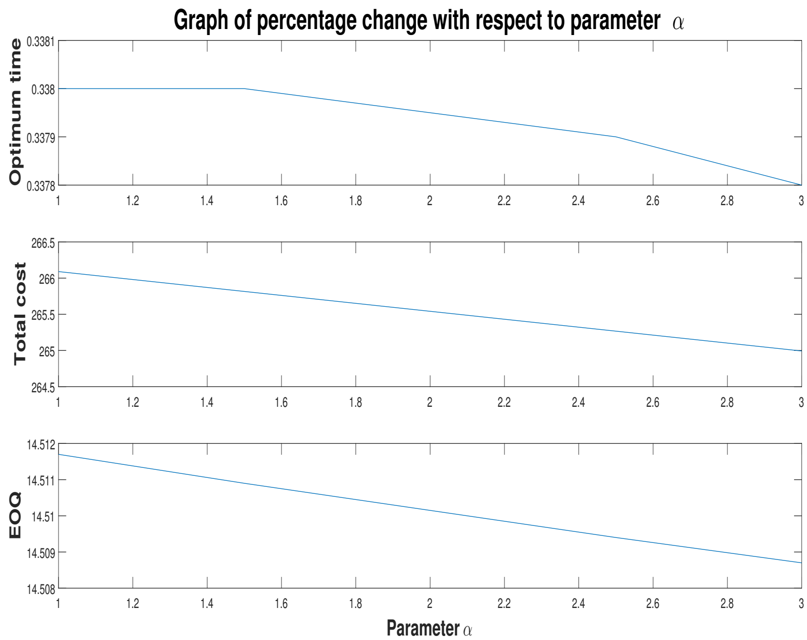

Figure 9. From the above graph we observe that with the increase in values of parameter α, the value of optimum time t_1^* initially remains constant, decreases after that,again becomes constant and finally decreases linearly. Total cost TC^* decreases linearly and economic order quantity Q^* decreases linearly

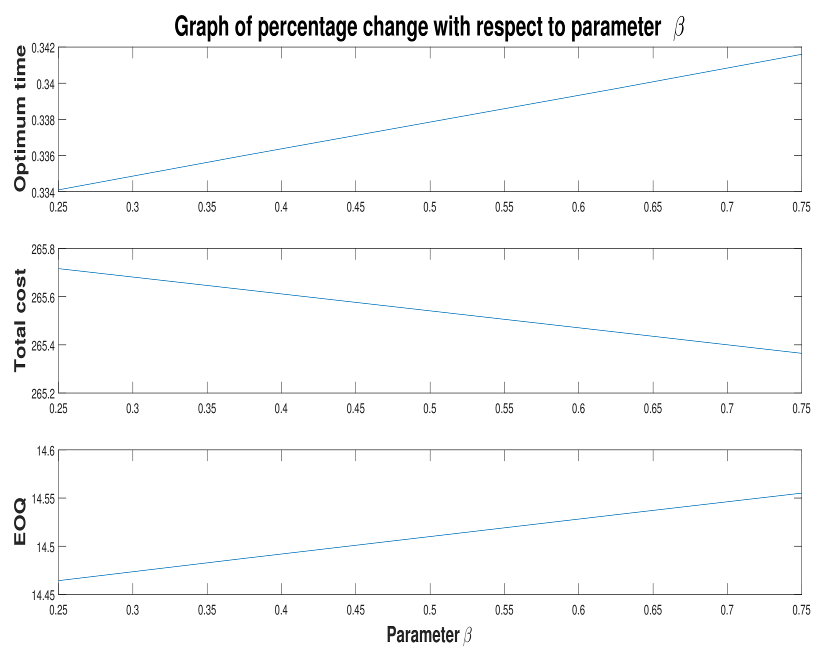

Figure 10. From the above graph we observe that with the increase in values of parameter β, the value of optimum time t_1^* increases linearly, total cost TC^* decreases linearly and economic order quantity Q^* increases linearly