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Approximate Solutions for the Nonlinear Systems of Algebraic Equations Using the Power Series Method

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Abstract

In this paper, the approximate solutions for systems of nonlinear algebraic equations by the power series method (PSM) are presented. Illustrative examples have been presented to demonstrate the efficiency of the proposed method. In addition, the obtained results are compared with those obtained from the standard Adomian decomposition method. It turns out that the convergence of the proposed algorithm is rapid.

Keywords: Power series method; Systems of nonlinear algebraic equations

MSC 2010 No.: 65N20; 41A30

1. Introduction

The nonlinear systems of algebraic equations (NSAE) often arise from the numerical modeling of problems in many branches of science and engineering (Brown and Saad (1990)). There are many real life problems in biology, physics and science that give us linear and nonlinear system of equations (El-Ajou et al. (2019), Goufoa et al. (2020), Khader and Adel (2018)-Kumar et al. (2013), Kumar et al. (2020), Sweilam et al. (2014)). Also, such these systems arise from the discretization of boundary value problems by finite difference or finite element methods with a huge sparse system of nonlinear algebraic equations. These equations more often are not solved analytically and there is no general theory for finding their solutions; hence, the resort to numerical solutions. More robust and efficient methods for solving NSAE are continuously being sought. Some available methods include variations of the Newton approach (Sundar et al. (2001)), the conjugate gradient method (Chronopoulos (1992), Daniel (1967), Fokkemma et al. (1996)) and the spectral methods (Brown and Saad (1990)). The Newton method is well-known for solving nonlinear systems of equations but at each step of the Newton method, it requires to solve a linear system of equations. However, solving a system of linear equations at each step becomes expensive if the number of unknowns is large and may not be justified when the iterative solution is far from a solution.

Recently, the PSM has been used for solving a wide range of problems (Ercan and Mustafa (2003), Ercan C. and Mustafa B. (2004), Liu and Megahed (2012)-Liu et al. (2012)). This new iterative method has proven rather success in dealing with linear as well as nonlinear problems. This method yields solutions for high accuracy and offers certain advantages over standard numerical methods. It is free from rounding off errors since it does not involve discretization and is computationally inexpensive.

Consider the following nonlinear system of algebraic equations:

$$F(X)=0, \qquad \text{or} \qquad f_i(x_1,x_1,...,x_n)=0, \qquad i=1,2,...,n, \tag{1}$$

where F and X are vector functions and $f_i:\Re^n \to \Re$.

The solutions of (1) can be assumed that:

$$x_i=\theta_i, \qquad \text{for some constants } \theta_i,\ \ i=1,2,...,n, \tag{2}$$

where $\theta=(\theta_i)$ is a vector value. Substituting from (2) into (1) and neglect higher-order term, we get a linear equation of θ in the form:

$$A\theta=b, \tag{3}$$

where A and b are constant matrices. By solving this equation (3), the coefficients of θ in (2) can be determined. By repeating the above procedure for a higher number of terms, we can get the arbitrary order power series of the solutions for (1).

2. Procedure solution by using the PSM

We define another type of power series in the form (Inc et al. (2016), Kumar et al. (2016)):

$$f=f_0+f_1+f_2+...+(f_n+p_1\theta_1+...+p_m\theta_m), \tag{4}$$

where $p_1,p_2,...,p_m$ are constants. $\theta_1,\theta_2,...,\theta_m$ are bases of vector θ; m is the size of a vector θ, and X is a vector with m elements in (2). Every element can be represented by the power series in

(4). Therefore, we can write:

$$x_i = x_{i,0} + x_{i,1} + x_{i,2} + ... + \theta_i, \tag{5}$$

where x_i is the $i-$th element of X. Substituting (5) into (1), we can get the following:

$$f_i = (f_{i,n} + p_{i,1}\theta_1 + ... + p_{i,m}\theta_m) + O(\theta_i^m), \tag{6}$$

where f_i is the $i-$th element of F in (1). From (6), we can determine the linear equation in (3) as follows:

$$A_{i,j} = P_{i,j}, \qquad b_i = -f_{i,n}. \tag{7}$$

By solving the linear equation (3), we have θ_i, $(i = 1, 2, ..., m)$. By substituting θ_i into (5), we have x_i, $(i = 1, 2, ..., m)$. By repeating this procedure from (5)-(7), we can get the arbitrary order power series of the solution for the nonlinear system of algebraic equations (1).

3. Illustrative numerical examples

To give a clear overview of the proposed method, we present the following examples. We apply the PSM and compare the results with the standard Adomian decomposition method (ADM).

Example 3.1.

Consider the following nonlinear system of equations,

$$\begin{aligned} &x_1^2 - 10\,x_1 + x_2^2 + 8 = 0, \\ &x_1\,x_2^2 + x_1 - 10\,x_2 + 8 = 0. \end{aligned} \tag{8}$$

The exact solution is $x_1 = x_2 = 1$. According to PSM, the solutions of (8) can be supposed as

$$x_1 = \theta_1, \qquad\qquad x_2 = \theta_2. \tag{9}$$

Substituting (9) into (8) and neglecting higher-order terms, we have:

$$\begin{aligned} &(8 - 10\,\theta_1) + O(\theta_1^2, \theta_2^2) = 0, \\ &(8 + \theta_1 - 10\,\theta_2) + O(\theta_1^2, \theta_2^2) = 0. \end{aligned} \tag{10}$$

The linear equation that corresponds to (10) can be given in the following form:

$$A\theta = b, \tag{11}$$

where

$$A = \begin{pmatrix} -10 & 0 \\ 1 & -10 \end{pmatrix}, \qquad b = \begin{pmatrix} -8 \\ -8 \end{pmatrix}, \qquad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

By solving this linear equation, we get $\theta = \begin{pmatrix} 0.8 \\ 0.88 \end{pmatrix}$.

From (9) we can obtain:

$$x_1 = 0.8, \qquad x_2 = 0.88. \tag{12}$$

From (12) the solutions of (8) can be supposed as:

$$x_1 = 0.8 + \theta_1, \qquad x_2 = 0.88 + \theta_2. \tag{13}$$

In a like manner, by substituting (13) into (8) and neglecting higher-order terms, we get:

$$\begin{aligned}&(1.4144 - 8.4\,\theta_1 + 1.76\,\theta_2) + O(\theta_1^2, \theta_2^2) = 0,\\ &(0.61952 + 1.7744\,\theta_1 - 8.592\,\theta_2) + O(\theta_1^2, \theta_2^2) = 0,\end{aligned} \tag{14}$$

where

$$A = \begin{pmatrix} -8.4000 & 1.7600 \\ 1.7744 & -8.5920 \end{pmatrix}, \qquad b = \begin{pmatrix} -1.4144 \\ -0.6195 \end{pmatrix}, \qquad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

By solving this linear equation, we obtain $\theta = \begin{pmatrix} 0.191787 \\ 0.111712 \end{pmatrix}$.

Therefore,

$$x_1 = 0.8 + 0.191787 = 0.991787, \qquad x_2 = 0.88 + 0.111712 = 0.991712. \tag{15}$$

From (15), the solutions for (8) can be supposed as:

$$x_1 = 0.991787 + \theta_1, \qquad x_2 = 0.991712 + \theta_2. \tag{16}$$

In a like manner, by substituting (16) into (8) and neglecting higher-order terms, we get

$$\begin{aligned}&(0.0492618503846 - 8.01642555779\,\theta_1 + 1.9834234742\,\theta_2) + O(\theta_1^2, \theta_2^2) = 0,\\ &(0.0500848159063 + 1.98349216949\,\theta_1 - 8.0328659443\,\theta_2) + O(\theta_1^2, \theta_2^2) = 0,\end{aligned} \tag{17}$$

where

$$A = \begin{pmatrix} -8.01642555779 & 1.9834234742 \\ 1.98349216949 & -8.0328659443 \end{pmatrix}, \qquad b = \begin{pmatrix} -0.0492618503846 \\ -0.0500848159063 \end{pmatrix}.$$

By solving this linear equation, we get $\theta = \begin{pmatrix} 0.0081880 \\ 0.0082568 \end{pmatrix}$. Therefore,

$$x_1 = 0.991787 + 0.0081880 = 0.999975, \qquad x_2 = 0.991712 + 0.0082568 = 0.999969. \tag{18}$$

By repeating the above procedure (only twice) we have:

$$x_1 = 1, \qquad x_2 = 1. \tag{19}$$

The solution of the system (8) by the standard ADM (Kaya and El-Sayed (2004)) (after 20 iterations) is

$$x_1 = 1.00000181935, \qquad x_2 = 1.00000224783.$$

For more details about the ADM, see Kaya and El-Sayed (2004).

Example 3.2.

Consider the following nonlinear system of equations,

$$\begin{aligned} x_1^3 + x_2^3 - 6\,x_1 &= -3, \\ x_1^3 - x_2^3 - 6\,x_2 &= -2. \end{aligned} \tag{20}$$

The exact solution is $x_1 = 0.5323642890259361,\ \ x_2 = 0.3512537227407807$.

According to the PSM, the solutions for (20) can be supposed as:

$$x_1 = \theta_1, \qquad x_2 = \theta_2. \tag{21}$$

By substituting (21) into (20) and neglecting higher-order terms, we get:

$$\begin{aligned} (3 - 6\,\theta_1) + O(\theta_1^2, \theta_2^2) &= 0, \\ (2 - 6\,\theta_2) + O(\theta_1^2, \theta_2^2) &= 0. \end{aligned} \tag{22}$$

The linear equation that corresponds to (22) can be given in the form (11), where

$$A = \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}, \qquad b = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \qquad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

By solving this linear equation, we obtain $\theta = \begin{pmatrix} 0.50000 \\ 0.333333 \end{pmatrix}$, and

$$x_1 = 0.50000, \qquad x_2 = 0.333333. \tag{23}$$

Using (23), the solutions for (20) can be supposed as:

$$x_1 = 0.50000 + \theta_1, \qquad x_2 = 0.333333 + \theta_2. \tag{24}$$

In like manner, by substituting (24) into (20) and neglecting higher-order terms, we get:

$$\begin{aligned} (0.162037037037037 - 5.25\,\theta_1 + 0.3333333333\,\theta_2) + O(\theta_1^2, \theta_2^2) &= 0, \\ (0.087962962962963 + 0.75\,\theta_1 - 6.3333333333\,\theta_2) + O(\theta_1^2, \theta_2^2) &= 0, \end{aligned} \tag{25}$$

where

$$A = \begin{pmatrix} -5.25 & 0.333333333 \\ 0.75 & -6.333333333 \end{pmatrix}, \qquad b = \begin{pmatrix} -0.162037037037 \\ -0.087962962963 \end{pmatrix}, \qquad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

By solving this linear equation, we get $\theta = \begin{pmatrix} 0.0319865 \\ 0.0176768 \end{pmatrix}$.

Therefore,

$$x_1 = 0.5 + 0.0319865 = 0.531987, \qquad x_2 = 0.333333 + 0.0176768 = 0.35101. \tag{26}$$

From (26), the solutions for (20) can be supposed as:

$$x_1 = 0.531987 + \theta_1, \qquad x_2 = 0.35101 + \theta_2. \tag{27}$$

In like manner, by substituting (27) into (20) and neglect higher-order terms, we find:

$$\begin{aligned}(0.00188542552824 - 5.15097098935\,\theta_1 + 0.3696242730\,\theta_2) + O(\theta_1^2, \theta_2^2) = 0,\\ (0.00124944244468 + 0.84902901065\,\theta_1 - 6.3696242730\,\theta_2) + O(\theta_1^2, \theta_2^2) = 0,\end{aligned} \tag{28}$$

where

$$A = \begin{pmatrix} -5.15097098935 & 0.3696242730 \\ 0.84902901065 & -6.3696242730 \end{pmatrix}, \qquad b = \begin{pmatrix} -0.00188542552824 \\ -0.00124944244468 \end{pmatrix}.$$

By solving this linear equation, we get $\theta = \begin{pmatrix} 0.00038378 \\ 0.00024731 \end{pmatrix}$. Therefore,

$$x_1 = 0.531987 + 0.00038378 = 0.53237, \quad x_2 = 0.35101 + 0.000247312 = 0.351257. \tag{29}$$

By repeating the above procedure (only twice) we obtain:

$$x_1 = 0.53237, \qquad x_2 = 0.351257. \tag{30}$$

The solution for the system (20) by the standard ADM (Kaya and El-Sayed (2004)) (after 8 iterations) is

$$x_1 = 0.532365, \qquad x_2 = 0.351254.$$

4. Conclusion and Discussion

The power series method is a powerful approach that yields a convergent series solution for a wide class of nonlinear problems. This method is better than the other numerical methods as it is free from rounding off errors, and does not require large computing. The proposed method yields a series of solutions which converges faster than the series obtained by standard ADM. Illustrative examples presented clearly to support this claim. Mathematica has been used for computations in this paper.

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