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Stability Analysis of Circular Robe's R3BP with Finite Straight Segment and Viscosity

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Abstract

In this paper, the effect of viscous force on the linear stability of equilibrium points of the circular Robe's restricted three-body problem (CRR3BP) with smaller primary as a finite straight segment is studied. The present model comprises of a bigger primary m_1^* which is a rigid spherical shell filled with a homogeneous incompressible fluid of density ρ_1 and the smaller primary m_2 lies outside the shell. The infinitesimal mass m_3 is a small solid sphere of density ρ_3 moving inside m_1^*. The pertinent equations of motion of m_3 are derived and solved for the equilibrium points. Routh-Hurwitz criterion is used to detect the stability of the obtained equilibrium points. The stability of the collinear equilibrium points has been studied systematically in the different regions for the various values of the parameters involved. These points are found to be conditionally stable, whereas the non-collinear and out-of-plane equilibrium points are always unstable for all the values of the parameters. We observed that viscosity has no effect on the location of equilibrium points. However, its effect along with the length parameter l is evident on the stability of equilibrium points.

Keywords: Circular Robe's restricted three-body problem; Finite straight segment; Viscosity; Stability

MSC 2010 No.: 37N05, 70F07, 70F15

1. Introduction

Restricted three-body problem (Szebehely (1967)) is one of the most popularly researched problem in celestial mechanics. In this problem two massive primaries move in circular orbits around their common center of mass and a third infinitesimal body being influenced by the primaries but not influencing them, moves in the plane of motion of the primaries. To describe the motion of the infinitesimal body under the Newtonian gravitational attraction is known as the restricted three-body problem.

Robe (1977) configured a new kind of restricted three-body problem by assuming the bigger primary m_1^* to be a spherical shell filled with a homogeneous incompressible fluid of density ρ_1 and the smaller one m_2 being a point mass that lies outside m_1^*. He considered the cases in which the orbit of m_2 around m_1^* is either circular or elliptic. He studied the motion of the infinitesimal body m_3 which is a small solid sphere of density ρ_3 lying inside m_1^*. The motion of m_3 is studied subject to the attraction of m_2, attraction of fluid of density ρ_1 and the buoyancy force of the fluid. He found that the centre of the shell $(-\mu, 0, 0)$ is the only equilibrium point and studied its stability in continuation.

The existence and linear stability of equilibrium points in the RR3BP has been studied by Hallan and Rana (2001a). They found that there exist other equilibrium points apart from the centre of the first primary $(-\mu, 0, 0)$. Later, Hallan and Rana (2003) studied the effect of perturbations in Coriolis and centrifugal forces on the location and stability of the equilibrium points in the circular Robe's R3BP. Plastino and Plastino (1995) revisited the RR3BP with the assumption that the hydrostatic equilibrium figure of m_1^* is a Roche ellipsoid (Chandrashekhar (1987)). They obtained the equations of motion governing the motion of m_3 and investigated the location and stability of equilibrium points.

The Robe's restricted $2+2$ body problem has been studied by Kaur and Aggarwal (2012); Aggarwal and Kaur (2014); Kaur and Aggarwal (2013a); Kaur and Aggarwal (2013b); Aggarwal et al. (2014); Kaur et al. (2016). Kaur and Aggarwal (2012) extended the RR3BP to the problem of 2+2 bodies. Kaur and Aggarwal (2013a) extended their problem by taking the hydrostatic equilibrium figure of m_1^* as a Roche ellipsoid. The Robe's restricted $2+2$ body problem when the bigger and smaller primaries are Roche ellipsoid and oblate spheroid respectively was studied by Kaur and Aggarwal (2013b). Aggarwal and Kaur (2014) examined the existence and stability of Robe's restricted problem of $2+2$ bodies with one of the primary as an oblate body. The perturbed version of the Robe's restricted problem of $2+2$ bodies when the primaries form a Roche ellipsoid-triaxial system has been studied by Kaur et al. (2016). They pointed out the effect of small perturbation in centrifugal force on the location of the equilibrium points, however the stability is being affected by the perturbation in Coriolis and centrifugal forces.

Jain and Sinha (2014b) investigated the stability of the equilibrium points in the R3BP when both the primaries are finite straight segments. They also obtained the possible regions of motion for m_3. Non-linear stability of equilibrium points for the same problem has also been studied by Jain and Sinha (2014a).

Chauhan et al. (2018) studied the restricted three-body under the assumption that the smaller primary is a finite straight segment. They derived the equations of motion of the infinitesimal mass under the influence of Albedo.

Kumar et al. (2019) extended the RR3BP by taking one of the primaries as a finite straight segment. The effect of length parameter has been perceived on the location and stability of the equilibrium points. The collinear equilibrium points are found to be conditionally stable for the density, mass and length parameters k, μ and l respectively. However, the non-collinear and out-of-plane equilibrium points are always unstable for every value of the parameters μ, k and l. Recently, they investigated their problem by considering the effect of small perturbations in the Coriolis and centrifugal forces in Kaur et al. (2020).

RR3BP has been studied with many variations in the configuration of involved bodies by many authors in the series of papers like Singh and Mohammed (2012); Singh and Sandah (2012); Singh (2012); Singh and Mohammed (2013); Singh and Omale (2014). Ansari et al. (2019a) studied the effect of oblateness and viscosity in the circular RR3BP. The bigger primary has been considered as a rigid spherical shell filled with homogeneous, incompressible and viscous fluid, and shape of the second primary is taken to be an oblate spheroid. They summarised that the viscosity of the fluid has no impact on the positions of the equilibrium points, however it has a subsequent effect on the stability of the obtained equilibrium points.

Ansari et al. (2019b) investigated the motion of m_3 in the perturbed CRR3BP. They assumed the shape of bigger primary as in Ansari et al. (2019a) and smaller primary to be a point mass. They discussed the problem with viscous force of the fluid and small perturbations in the Coriolis and centrifugal forces. They obtained the equilibrium points for their problem and systematically investigated their linear stability by using Routh-Hurwitz criterion.

Robe's model with the smaller primary as a finite straight segment has been studied by Kumar et al. (2019) without taking into consideration the effect of viscosity of the fluid ρ_1 on m_3. This effect of viscosity was considered by Ansari et al. (2019b). Motivated by Kumar et al. (2019) and Ansari et al. (2019b), we have made an effort to study the combined effects of viscosity and finite straight segment on the locations and stability of the equilibrium points. Routh-Hurwitz criterion has been used to determine the stability without factorizing the characteristic polynomial which is obtained by using the variational equations.

This paper is organized into five sections. Section 1 is an introductory in which the development of the problem is mentioned. The statement of the problem and the pertinent equations of motion are stated in Section 2. In Section 3, the collinear, non-collinear and out-of-plane equilibrium points are stated as in Kumar et al. (2019). Section 4 comprises of stability of equilibrium points by using the well established Routh-Hurwitz criterion. Section 5 includes the discussion and Section 6 concludes the paper.

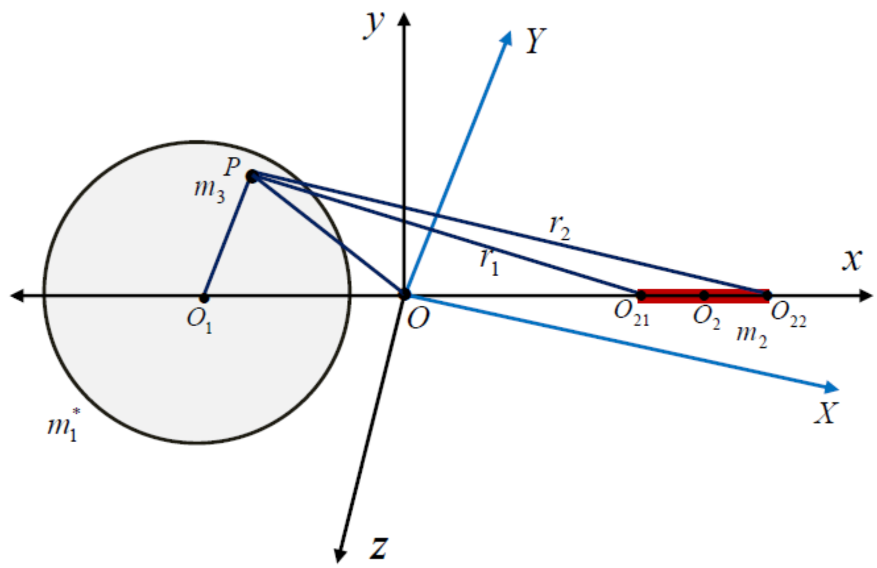

Figure 1. Robe's restricted three-body problem when smaller primary is a finite straight segment

2. Description of the dynamical system

Let there be two primaries of masses m_1^* and m_2, the bigger primary m_1^* be a rigid spherical shell filled with homogeneous incompressible fluid of density ρ_1 and m_2 be a finite straight segment of length $2l$ ($l << 1$, such that $o(l^3) \approx 0$) lying outside m_1 (as in Figure 1). They revolve in circular orbit with angular velocity ω. The line joining the centres of the primaries m_1^* and m_2 is taken as x-axis of the synodic co-ordinate system. The center of mass of the primaries is taken as origin O. The line perpendicular to the x-axis passing through O in the plane of motion of the primaries is taken as y-axis. The z-axis is the line perpendicular to the plane of motion of primaries through O. Let the synodic co-ordinate system, $Oxyz$ initially coincident with the inertial co-ordinate system $OXYZ$, rotate with the same angular velocity ω as that of the primaries.

Let the third body of mass m_3 be a small solid sphere of density ρ_3 inside the shell, with the assumption that its mass and radius are infinitesimal. We assume that the mass m_3 does not influence the motion of m_1^* and m_2 but is influenced by them. Kumar et al. (2019) studied this model without considering the force acting on m_3 due to viscosity of the fluid ρ_1. By introducing the parameters V_x, V_y and V_z due to viscosity, the equations of motion of m_3 in the uniformly rotating coordinate system in dimensionless variables (Kumar et al. (2019)) are

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} &= W_x + V_x, \\ \ddot{y} + 2\omega\dot{x} &= W_y + V_y, \\ \ddot{z} &= W_z + V_z, \end{aligned} \tag{1}$$

where

$$\begin{aligned} W(x,y,z) =&\frac{1}{2}\omega^2(x^2+y^2) - \frac{k}{2}[(x+\mu)^2+y^2+z^2] + \frac{\mu}{2l}\log\left(\frac{r_1+r_2+2l}{r_1+r_2-2l}\right), \\ k =&\frac{4\pi}{3}\rho_1\left(1-\frac{\rho_1}{\rho_3}\right),\ \omega^2 = 1+l^2,\ \mu = \frac{m_2}{m_1^*+m_2},\ \ 0<\mu<1, \\ r_1^2 =&(x-1+\mu+l)^2+y^2+z^2,\ r_2^2 = (x-1+\mu-l)^2+y^2+z^2, \end{aligned}$$

with

$$V_x = -\alpha\dot{x},\ \ V_y = -\alpha\dot{y},\ \ V_z = -\alpha\dot{z} \text{ (Ansari et al. (2019b))}$$

and α is a positive constant. W_x, W_y and W_z represent the partial derivatives of W with respect to x, y and z respectively. The dot $(\cdot)$ signifies the derivatives with respect to time t, where time t is chosen such that the value of gravitational constant G becomes unity.

3. The location of the equilibrium point

To determine the equilibrium points (x, y, z), we have to solve the following system of equations,

$$W_x(x,y,z) = 0, W_y(x,y,z) = 0, \text{ and } W_z(x,y,z) = 0$$

simultaneously, where

$$\begin{aligned} W_x(x,y,z) =&\omega^2 x - k(x+\mu) - \frac{2\mu}{[(r_1+r_2)^2-4l^2]}\left(\frac{(x-1+\mu+l)}{r_1} + \frac{(x-1+\mu-l)}{r_2}\right), \\ W_y(x,y,z) =&\omega^2 y - ky - \frac{2\mu}{[(r_1+r_2)^2-4l^2]}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)y, \\ W_z(x,y,z) =&z\left[k + \frac{2\mu}{[(r_1+r_2)^2-4l^2]}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right]. \end{aligned}$$

The equilibrium points as categorized by Kumar et al. (2019) are as follows.

- The points collinear with the centres of the primaries m_1^* and m_2 are collinear equilibrium points. The co-ordinates of the collinear equilibrium points are $(-\mu, 0, 0)$ and $(x_1, 0, 0)$, where

$$\begin{aligned} x_1 =&\frac{1}{2(k-1-l^2)}\Bigg[(1+l^2)(\mu-2) + 2k(1-\mu) - \Bigg\{\mu(-4+4k+\mu) + 2l^2\Bigg(2+2k^2 \\ &+ 2k(\mu-2) - 4\mu + \mu^2\Bigg)\Bigg\}^{1/2}\Bigg]. \end{aligned} \tag{2}$$

  The point $(-\mu, 0, 0)$ is always an equilibrium point, whereas $(x_1, 0, 0)$ will be an equilibrium point provided $k > 1 + l^2$.
- The points lying in xy-plane with $y \neq 0$ are non-collinear equilibrium points. These points are of the form $(x, y, 0)$, where x and y satisfy the equation

$$(1-\mu-x)^2 + y^2 = 1 - \frac{2}{3}l^2, \tag{3}$$

provided $k = 1 - \mu + l^2(1-\mu)$. The points lying on the above circle within the bigger primary m_1^* are infinite in number.

- The points in the xz-plane are the out-of-plane equilibrium points. These points are
$$\left(k(1-l^2), 0, \pm\sqrt{b_1^2 - a_1^2}\right),$$
for $k < 0$ and $k + \mu + 2\mu l^2 > 0$, where
$$a_1 = 1 - \mu - k \quad \text{and} \quad b_1 = \left(-\frac{\mu}{k} + \frac{\mu l^2}{k - 1 + \mu}\right)^{1/3}.$$

4. Linear stability of equilibrium points

In order to examine the stability of an equilibrium point, its position is slightly displaced via perturbations. If the resultant motion of the infinitesimal body is a rapid departure from its vicinity, such location of equilibrium point is called unstable, else stable. In the series of papers investigating the stability of equilibrium points, nature of characteristic roots of the variational equations, obtained by linearizing the equations of motion, establish the stability or unstability of the equilibrium point.

Let the third body be displaced to $(x_0 + \xi, y_0 + \eta, z_0 + \zeta)$ from its equilibrium position (x_0, y_0, z_0) with a small displacement (ξ, η, ζ). Substituting these values in system of Equations (1) by retaining only the linear terms of ξ, η, ζ, $\dot{\xi}$, $\dot{\eta}$ and $\dot{\zeta}$, we obtain the following variational equations
$$\begin{aligned}\ddot{\xi} - 2\omega\dot{\eta} &= V^0_{x\dot{x}}\dot{\xi} + V^0_{x\dot{y}}\dot{\eta} + V^0_{x\dot{z}}\dot{\zeta} + \left(W^0_{xx} + V^0_{xx}\right)\xi + \left(W^0_{xy} + V^0_{xy}\right)\eta + \left(W^0_{xz} + V^0_{xz}\right)\zeta,\\ \ddot{\eta} + 2\omega\dot{\xi} &= V^0_{y\dot{x}}\dot{\xi} + V^0_{y\dot{y}}\dot{\eta} + V^0_{y\dot{z}}\dot{\zeta} + \left(W^0_{yx} + V^0_{yx}\right)\xi + \left(W^0_{yy} + V^0_{yy}\right)\eta + \left(W^0_{yz} + V^0_{yz}\right)\zeta,\\ \ddot{\zeta} &= V^0_{z\dot{x}}\dot{\xi} + V^0_{z\dot{y}}\dot{\eta} + V^0_{z\dot{z}}\dot{\zeta} + \left(W^0_{zx} + V^0_{zx}\right)\xi + \left(W^0_{zy} + V^0_{zy}\right)\eta + \left(W^0_{zz} + V^0_{zz}\right)\zeta,\end{aligned} \tag{4}$$
where the $W^0_{ij}, i, j = x, y, z$ denotes the second order partial derivatives of W with respect to i and j; $V^0_{ij}, i = x, y, z$ and $j = x, y, z, \dot{x}, \dot{y}, \dot{z}$ denotes the first order partial derivative of V_i with respect to j, being evaluated at the point $(x_0, y_0, z_0, 0, 0, 0)$. Routh–Hurwitz stability criterion (Clark (1996)) has been used to investigate the stability of the obtained equilibrium points without factorizing the characteristic polynomial.

4.1. Linear stability of $(-\mu, 0, 0)$

For the equilibrium point $(-\mu, 0, 0)$, we have
$$W^0_{xx} = 1 + 2\mu - k + l^2(1+4\mu),\ W^0_{yy} = 1 - \mu - k + l^2(1-2\mu),\ W^0_{zz} = -\mu - k - 2\mu l^2,$$
$$W^0_{xy} = 0,\ W^0_{yz} = 0,\ W^0_{zx} = 0,\ V^0_{x\dot{x}} = V^0_{y\dot{y}} = V^0_{z\dot{z}} = -\alpha,$$
and other derivatives of V_x, V_y and V_z are zero. Therefore the system of Equations (4) becomes
$$\ddot{\xi} - 2\omega\dot{\eta} = -\alpha\dot{\xi} + \left(1 + 2\mu - k + l^2(1+4\mu)\right)\xi, \tag{5a}$$
$$\ddot{\eta} + 2\omega\dot{\xi} = -\alpha\dot{\eta} + \left(1 - \mu - k + l^2(1-2\mu)\right)\eta, \tag{5b}$$
$$\ddot{\zeta} = -\alpha\dot{\zeta} - \left(\mu + k + 2\mu l^2\right)\zeta. \tag{5c}$$

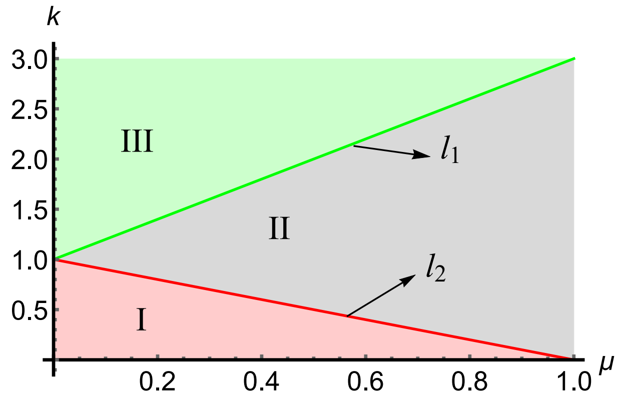

Figure 2. Geometry of the stability regions for $(-\mu, 0, 0)$ in μk-plane

From Equation (5c), we infer that the motion of m_3 parallel to the $z-$axis is stable when $\mu + k + 2\mu l^2 > 0$, that is, when m_3 is denser than the fluid of density ρ_1 ($\rho_3 > \rho_1$). To analyse the stability of the motion of m_3 in $xy-$plane, we take the trial solutions $\xi = A\exp(\lambda t)$ and $\eta = B\exp(\lambda t)$ of Equations (5a) and (5b), respectively. The characteristic equation corresponding to them is given by

$$\lambda^4 + p_1\lambda^3 + p_2\lambda^2 + p_3\lambda + p_4 = 0, \tag{6}$$

where

$$p_1 = 2\alpha,\ p_2 = \alpha^2 + 2 - \mu + 2k + 2l^2(1-\mu),\ p_3 = -\alpha\left(2 + \mu - 2k + l^2(2+2\mu)\right),$$
$$p_4 = \left(1 + 2\mu - k + l^2(1+4\mu)\right)\left(1 - \mu - k + l^2(1-2\mu)\right).$$

Now, we draw the regions in $\mu k-$plane to investigate the stability of the equilibrium points for the different values of μ and k. For a fixed value of l, $l_1 : 1 + 2\mu - k + l^2(1+4\mu) = 0$ and $l_2 : 1 - \mu - k + l^2(1-2\mu) = 0$ divide the strip $\{(\mu, k) :\ 0 < \mu < 1,\ \text{and}\ k > 0\}$ in the following regions,

$$\text{Region I} = \{(\mu, k) : 1 - \mu - k + l^2(1-2\mu) \geq 0\},$$
$$\text{Region II} = \{(\mu, k) : 1 - \mu - k + l^2(1-2\mu) < 0 \text{ and } 1 + 2\mu - k + l^2(1+4\mu) > 0\},$$
$$\text{Region III} = \{(\mu, k) : 1 + 2\mu - k + l^2(1+4\mu) < 0\},$$

and the line l_1 as shown in Figure 2. We conclude the following about the motion of m_3.

- In Region I, $p_3 < 0$. Therefore, by the Routh-Hurwitz criterion, the motion of m_3 is unstable.
- In Region II, the motion is unstable since $p_4 < 0$.
- In Region III, all the coefficients $p_i, i = 1, 2, 3, 4$ of Equation (6) are positive. In order to determine the stability, we form the Routh-Hurwitz array as follows

1	p_2	p_4
p_1	p_3	0
Δ_1	p_4	
Δ_2	0	
p_4		

,

where

$$\Delta_1 =3 + 3l^2 + \alpha^2 + \mu\left(l^2 + \frac{1}{2}\right),$$

$$\text{and } \Delta_2 =\frac{\alpha}{2\Delta_1}\left[9\mu^2\left(2l^2+1\right)^2 - 16\left\{1 + l^2 - k + \mu\left(l^2 + \frac{1}{2}\right)\right\}\left(1 + l^2 + \frac{\alpha^2}{4}\right)\right].$$

By Routh-Hurwitz criterion the equilibrium point $(-\mu, 0, 0)$ is asymptotically stable in this region since $\Delta_1 > 0$ and $\Delta_2 > 0$.

- When $k = 1 + 2\mu + l^2(1 + 4\mu)$, the coefficients $p_i > 0, i = 1, 2, 3$ and $p_4 = 0$. Thus, the characteristic Equation (6) reduces in the following form

$$\lambda(\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3) = 0. \tag{7}$$

One can notice that $\lambda = 0$ is one of the root of Equation (7). Other roots are the solutions of the equation

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0.$$

The Routh-Hurwitz array for the above equation is

1	p_2
p_1	p_3
Δ_3	0
p_3	

,

where $\Delta_3 = \frac{1}{2}\left[7(1 + l^2) + 2\alpha^2 + k + \mu(1 + 2l^2)\right] > 0$. Therefore, by the Routh-Hurwitz Criterion the equilibrium point $(-\mu, 0, 0)$ is marginally stable.

The stability regions of the equilibrium point $(-\mu, 0, 0)$ are represented in Figure 3. These regions are obtained for the increasing values of length parameter l. The lines for the fixed values of l, represented by different colours, shows the marginally stable regions. For a fixed value of l, the region shown by light green colour lying above that line is asymptotically stable region. It is clear from the Figure 3 that the stability region is affected by the length parameter l. The region moves upward as the value of the length parameter l increases. Thus, it can be inferred that for the stability of motion of m_3 at $(-\mu, 0, 0)$, more denser m_3 is required for larger value of the length parameter l.

4.2. Linear Stability of the equilibrium point $(x_1, 0, 0)$

For the equilibrium point $(x_1, 0, 0)$, we have

$$W^0_{xx} = 1 + l^2 - k + 4A_1 - 2A_2 = \wp_1,\ W^0_{yy} = 1 + l^2 - k - 2A_1 + A_2 = \wp_2,$$
$$W^0_{zz} = -k - 2A_1 + A_2 = \wp_3,\ W^0_{xy} = 0,\ W^0_{yz} = 0,\ W^0_{zx} = 0,$$
$$V^0_{x\dot{x}} = V^0_{y\dot{y}} = V^0_{z\dot{z}} = -\alpha,$$

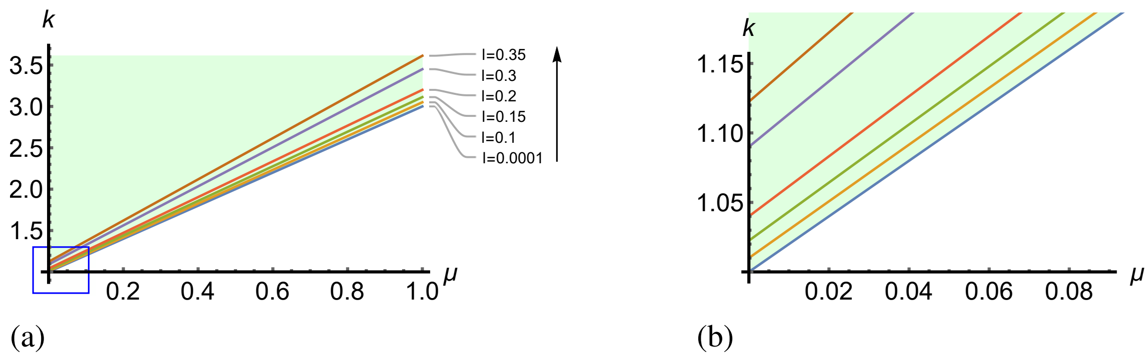

Figure 3. The regions of stability of $(-\mu, 0, 0)$ for the increasing values of $l = 0.0001$, 0.1, 0.15, 0.2, 0.3 and 0.35 are shown in (a). The zoomed portion of intersection of lines with k-axis is shown in (b)

where

$$A_1 = \frac{4\mu(k-1)^3}{\left(\mu + \sqrt{\mu(4k+\mu-4)}\right)^3},$$

$$A_2 = \frac{16(k-1)^2 l^2 \mu}{\sqrt{\mu(4k+\mu-4)}\left[\sqrt{\mu(4k+\mu-4)}+\mu\right]^5}\left[3(k-1)\mu\left\{k^2 + k\left(\sqrt{\mu(4k+\mu-4)}-2\right)+1\right\}\right.$$
$$\left. +3\left\{k\mu^3 + k\mu^2\left(\sqrt{\mu(4k+\mu-4)}+3k-3\right)\right\} + (k-1)^3\left\{-\sqrt{\mu(4k+\mu-4)}\right\}\right],$$

and rest of the derivatives of V_x, V_y and V_z are zero. Therefore, the system of Equations (4) becomes

$$\ddot{\xi} - 2\omega\dot{\eta} = -\alpha\dot{\xi} + \wp_1\xi, \tag{8}$$
$$\ddot{\eta} + 2\omega\dot{\xi} = -\alpha\dot{\eta} + \wp_2\eta, \tag{9}$$
$$\ddot{\zeta} = -\alpha\dot{\zeta} + \wp_3\zeta. \tag{10}$$

Since α and $-\wp_3$ are always positive, therefore, the Equation (10) shows that the motion of m_3 parallel to z-axis is always stable. The characteristic equation corresponding to the Equations (8) and (9) is

$$\lambda^4 + p_1\lambda^3 + p_2\lambda^2 + p_3\lambda + p_4 = 0, \tag{11}$$

where

$$p_1 = 2\alpha,\ p_2 = \alpha^2 + 4(1+l^2) - (\wp_1 + \wp_2),\ p_3 = -\alpha(\wp_1 + \wp_2),\ p_4 = \wp_1\wp_2.$$

The location of the roots of Equation (11) in argand plane will depend on the values of $\wp_1$ and $\wp_2$ in the $\wp_1\wp_2-$plane (shown in Figure 4). For the stability of the equilibrium point $(x_1, 0, 0)$, we have the following cases

- In the region $\Gamma_1 = \{(\wp_1, \wp_2) : \wp_1 \geq 0, \wp_2 > 0 \text{ or } \wp_1 > 0, \wp_2 \geq 0\}$, the coefficient $p_3 < 0$. Therefore, the motion of m_3 is unstable in this region.

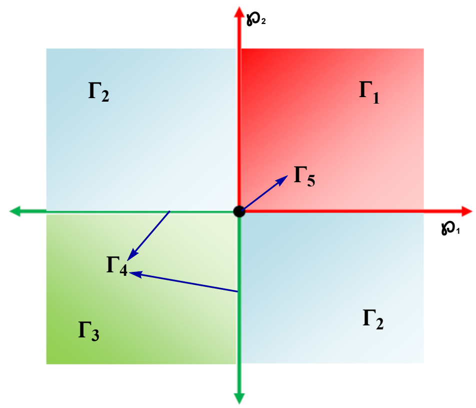

Figure 4. Geometry of the stability regions for $(x_1, 0, 0)$ in $\wp_1\wp_2$-plane

- In the region $\Gamma_2 = \{(\wp_1, \wp_2) : \wp_1 < 0, \wp_2 > 0 \text{ or } \wp_1 > 0, \wp_2 < 0\}$ the motion is unstable since $p_4 < 0$.
- At $\wp_1 = 0$ and $\wp_2 = 0$ both the coefficients p_3 and p_4 are zero, which results in unstable motion.
- In the region $\Gamma_3 = \{(\wp_1, \wp_2) : \wp_1 < 0 \text{ and } \wp_2 < 0\}$, all the coefficients $p_i, i = 1, 2, 3, 4$ of characteristic Equation (11) are positive. The Routh's array corresponding to the Equation (11) is given by

1	p_2	p_4
p_1	p_3	0
Δ_1	p_4	
Δ_2	0	
p_4		

,

where

$$\Delta_1 = \frac{1}{2}\Big[6(1 + l^2) + 2\alpha^2 + 2k - 2A_1 + A_2\Big],$$

$$\text{and } \Delta_2 = \frac{\alpha}{2\Delta_1}\Big[9(3A_1 - A_2)^2 - 2\Big\{2(1 + l^2) - 2k + 2A_1 - A_2\Big\}\Big\{4(1 + l^2) + \alpha^2\Big\}\Big].$$

Routh's criterion is satisfied since Δ_1 and Δ_2 are positive in this region. Therefore, the motion of m_3 is asymptotically stable.
The regions of asymptotic stability of the equilibrium point $(x_1, 0, 0)$ are shown in Figures 5, 6 and 7 by the light green colour. The regions of stability change for the different values of μ, k and l in their respective planes.

- In the region $\Gamma_4 = \{(\wp_1, \wp_2) : \wp_1 = 0, \wp_2 < 0 \text{ and } \wp_1 < 0, \wp_2 = 0\}$ the coefficients $p_i > 0, i = 1, 2, 3$ and $p_4 = 0$. The characteristic equation in this case becomes

$$\lambda(\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3) = 0. \tag{12}$$

$\lambda = 0$ is the root of Equation (12) and other roots are the solution of following equation

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0.$$

The Routh-Hurwitz array for the above equation is

1	p_2
p_1	p_3
Δ_3	0
p_3	

,

where $\Delta_3 = 3(1 + l^2) + \alpha^2 + k - A_1 + A_2/2$. In this region $\Delta_3 > 0$, therefore the equilibrium point $(x_1, 0, 0)$ is marginally stable.
The regions of marginal stability of the equilibrium point $(x_1, 0, 0)$ are represented by the light green coloured curves lying in the light pink regions in Figures 8, 9 and 10. These regions of stability changes for the different values of μ, k and l in their respective planes. Figures 5, 6, 7, 8, 9 and 10 explain that the stability region is affected by all the parameter involved in the problem. For example, from Figure 5(a) it is clear that for $\mu = 0.1$ the equilibrium point $(x_1, 0, 0)$ is unstable whenever $k \geq 1.15$, but from Figure 5(f) it is observed that there are some values of k greater than or equal to 1.15 for which the equilibrium point $(x_1, 0, 0)$ is stable provided $\mu = 0.3$. Thus, for $k \geq 1.15$ the stability of the equilibrium point $(x_1, 0, 0)$ depends on the all other parameter involved.

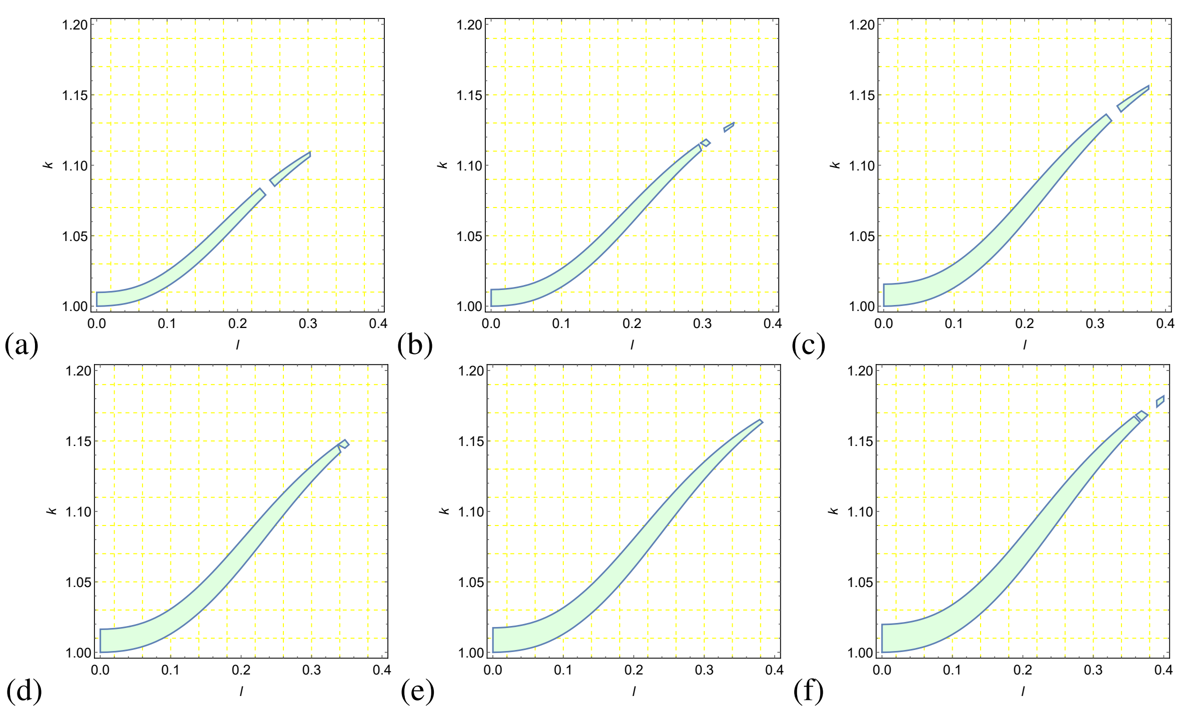

Figure 5. The regions of asymptotic stability of $(x_1, 0, 0)$ in the Region Γ_3 in $lk-$plane with $1.00000001 < k < 1.20$ and $0.001 < l < 0.4$ (a) for $\mu = 0.0001$, (b) for $\mu = 0.1$, (c) for $\mu = 0.15$, (d) for $\mu = 0.2$, (e) for $\mu = 0.3$ and (f) for $\mu = 0.35$

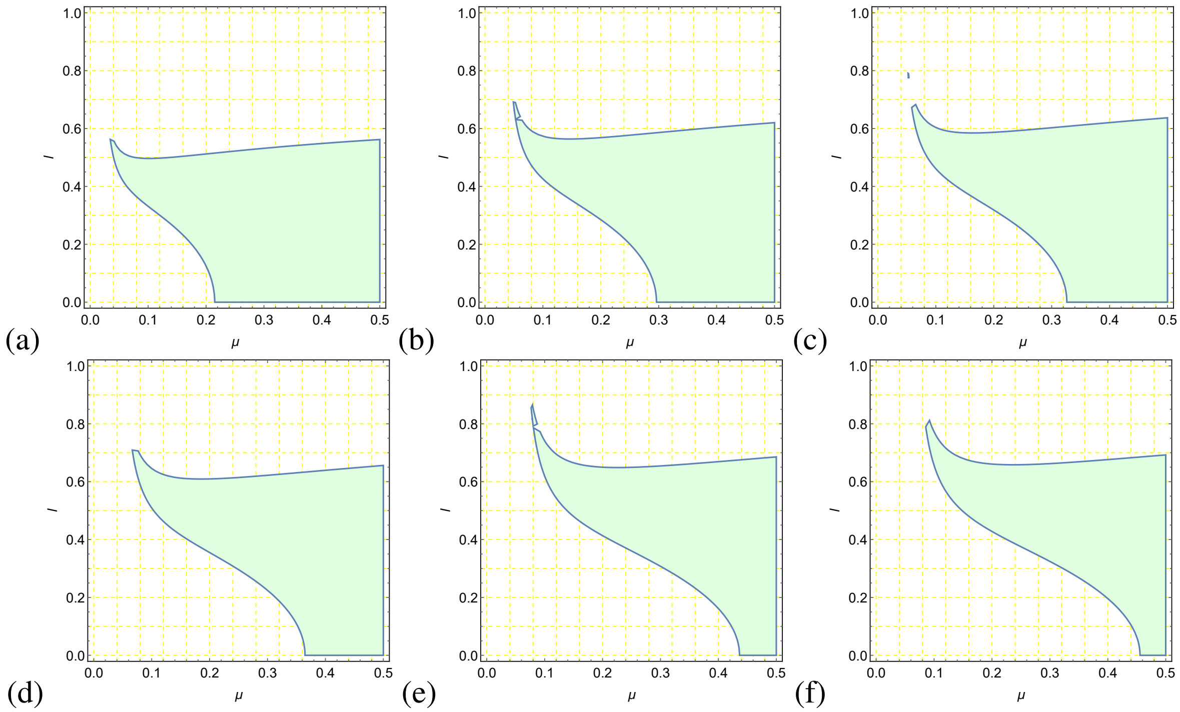

Figure 6. The regions of asymptotic stability of $(x_1, 0, 0)$ in the Region Γ_3 in $\mu l-$plane with $0 < \mu < 0.5$ and $0 < l < 1$ (a) for $k = 1.42985$, (b) for $k = 1.59234$, (c) for $k = 1.65241$, (d) for $k = 1.72943$, (e) for $k = 1.87295$ and (f) for $k = 1.91123$

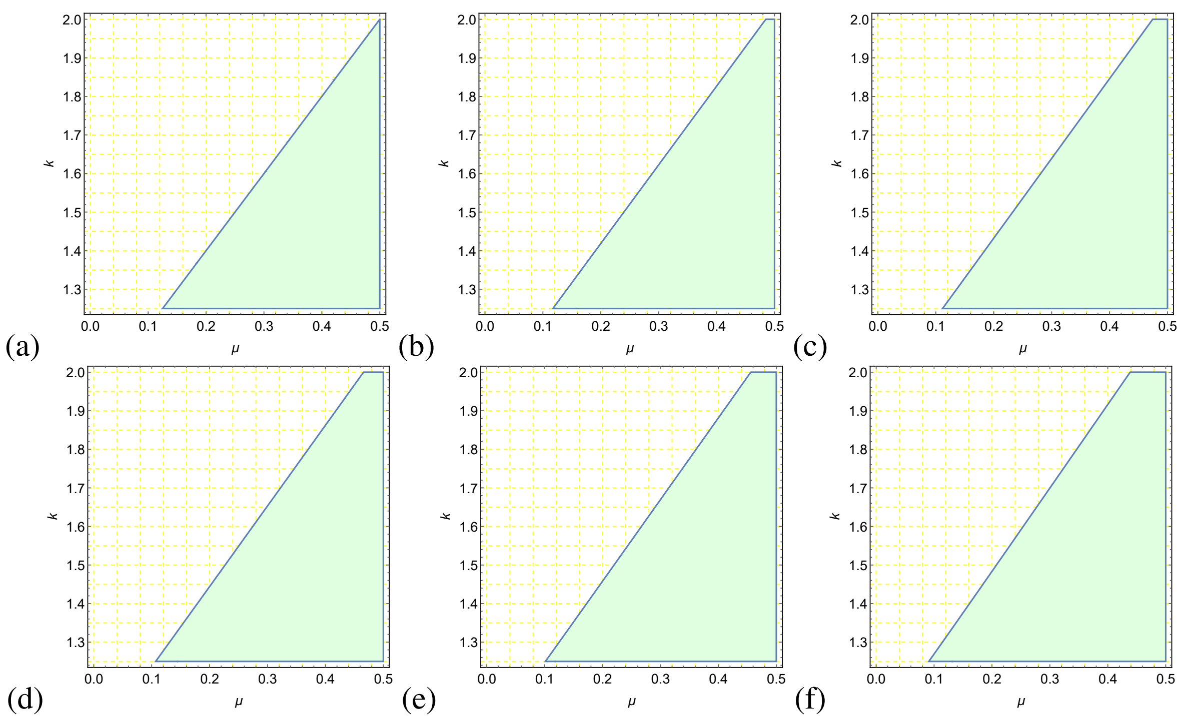

Figure 7. The regions of asymptotic stability of $(x_1, 0, 0)$ in the Region Γ_3 in $\mu k-$plane with $0 < \mu < 0.5$ and $1.25 < k < 2$ (a) for $l = 0.0001$, (b) for $l = 0.1$, (c) for $l = 0.13$, (d) for $l = 0.15$, (e) for $l = 0.17$ and (f) for $l = 0.2$

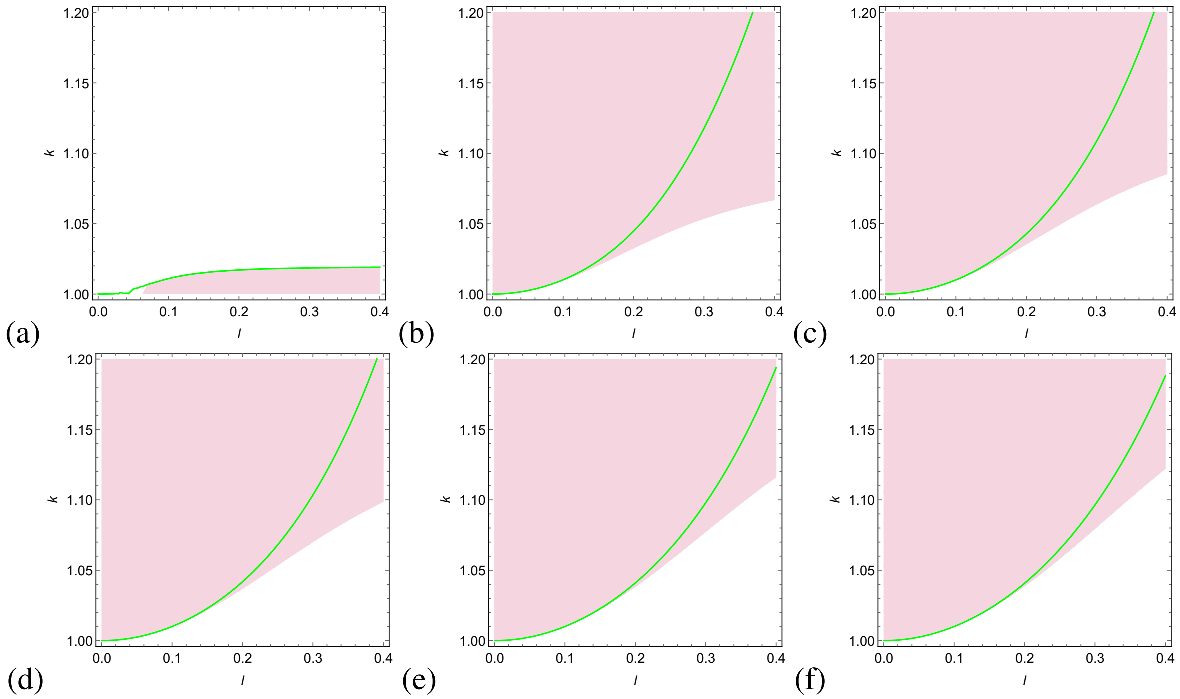

Figure 8. The regions of marginal stability of $(x_1, 0, 0)$ in the Region Γ_4 in $lk-$plane with $1.00000001 < k < 1.20$ and $0.001 < l < 0.4$ (a) for $\mu = 0.0001$, (b) for $\mu = 0.1$, (c) for $\mu = 0.15$, (d) for $\mu = 0.2$, (e) for $\mu = 0.3$ and (f) for $\mu = 0.35$

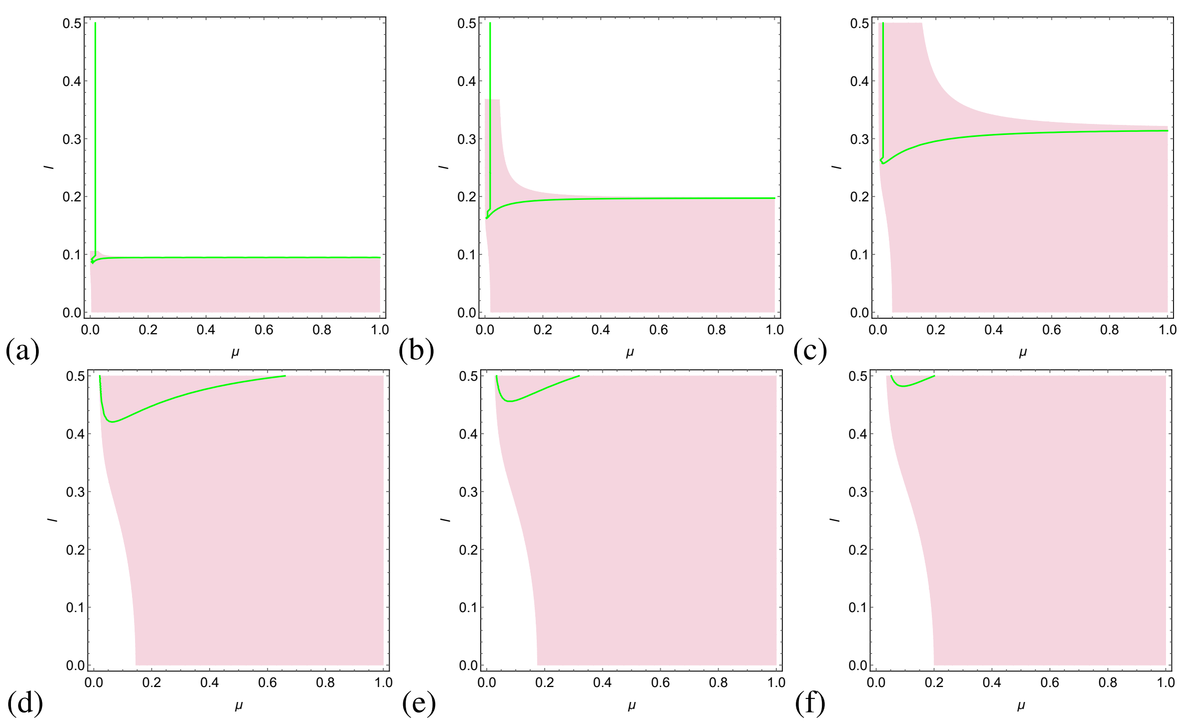

Figure 9. The regions of marginal stability of $(x_1, 0, 0)$ in the Region Γ_4 in $\mu l-$plane with $0 < \mu < 1$ and $0 < l < 0.5$ (a) for $k = 1.009$, (b) for $k = 1.039$, (c) for $k = 1.1$, (d) for $k = 1.29$, (e) for $k = 1.35$ and (f) for $k = 1.4$

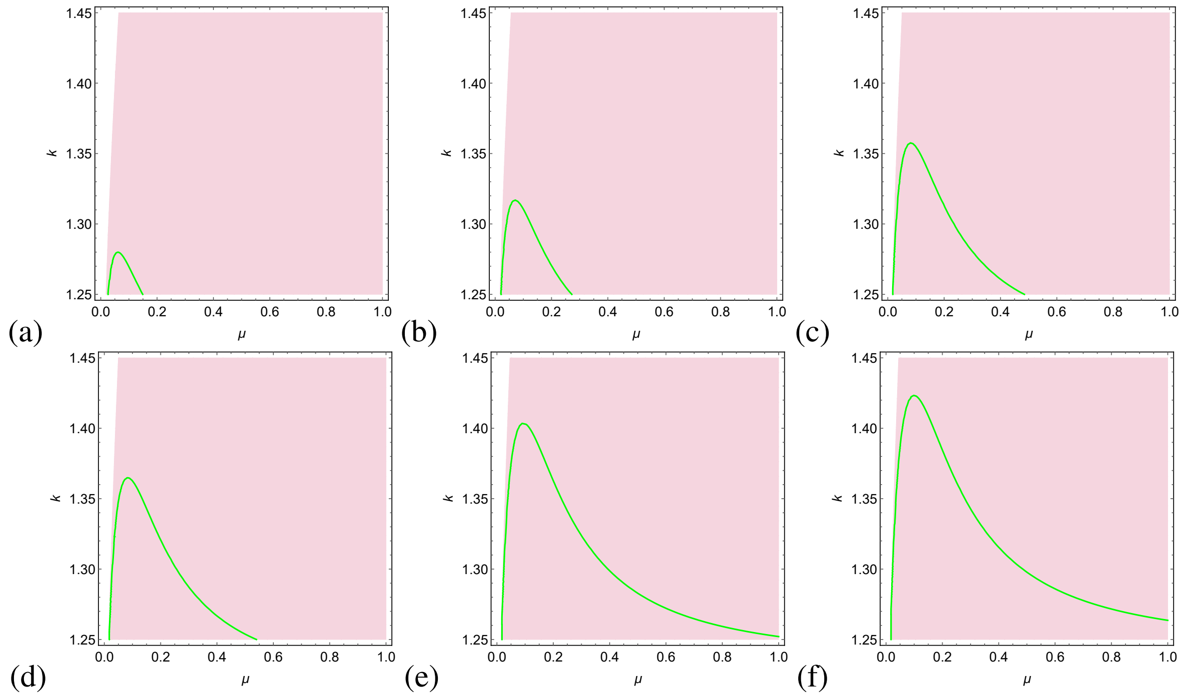

Figure 10. The regions of marginal stability of $(x_1, 0, 0)$ in the Region Γ_4 in $\mu l-$plane with $0 < \mu < 0.5$ and $0 < l < 1$ (a) for $k = 1.42985$, (b) for $k = 1.59234$, (c) for $k = 1.65241$, (d) for $k = 1.72943$, (e) for $k = 1.87295$ and (f) for $k = 1.91123$

4.3. Stability of the non-collinear equilibrium points

The coordinates of any point on the circle

$$(1-\mu-x)^2 + y^2 = 1 - \frac{2}{3}l^2,$$

are of the form

$$\left(1-\mu-\left(1-\frac{1}{3}l^2\right)\cos\theta, \left(1-\frac{1}{3}l^2\right)\sin\theta, 0\right).$$

At these point, we have

$$\begin{aligned}
W^0_{xx} =& 3\mu\cos^2\theta + \left\{\frac{35}{2}\mu\cos^4\theta - 12\mu\cos^2\theta + \frac{3\mu}{2}\right\}l^2 = \wp_1^{'},\\
W^0_{yy} =& 3\mu\sin^2\theta + \left\{\frac{1}{2}\mu\sin^2\theta - \frac{5}{2}\mu\cos^2\theta + \frac{35}{2}\mu\sin^2\theta\cos^2\theta + \frac{\mu}{2}\right\}l^2 = \wp_2^{'},\\
W^0_{xy} =& -3\mu\sin\theta\cos\theta + \left\{\frac{9}{2}\mu\sin\theta\cos\theta - \frac{35}{2}\mu\sin\theta\cos^3\theta\right\}l^2 = \wp_3^{'},\\
W^0_{zz} =& -1 - \left\{\frac{5}{2}\mu\cos^2\theta + 1 - \frac{\mu}{2}\right\}^2 < 0,
\end{aligned}$$

and $W^0_{yz} = W^0_{xz} = 0.$

Thus, the system of Equations (4) becomes

$$\ddot{\xi} - 2\omega\dot{\eta} = -\alpha\dot{\xi} + \wp_1^{'}\xi + \wp_3^{'}\eta, \tag{13}$$

$$\ddot{\eta} + 2\omega\dot{\xi} = -\alpha\dot{\eta} + \wp_3^{'}\xi + \wp_2^{'}\eta, \tag{14}$$

$$\ddot{\zeta} = -\alpha\dot{\zeta} + W_{zz}^{0}\zeta. \tag{15}$$

Since α and $-W_{zz}^0$ are positive for all the values of the parameters, therefore, the motion of m_3 parallel to z-axis is always stable. The characteristic equation corresponding to the Equations (13) and (14) is given by

$$\lambda^4 + p_1\lambda^3 + p_2\lambda^2 + p_3\lambda + p_4 = 0, \tag{16}$$

where

$$p_1 = 2\alpha,\ p_2 = \alpha^2 - 3\mu + 4 - \left\{\frac{5}{4}\mu\cos 2\theta + \frac{15}{4}\mu - 4\right\}l^2,$$

$$p_3 = -\alpha\left\{3\mu + \left(\frac{5}{4}\cos 2\theta + \frac{15}{4}\right)\mu l^2\right\},\ p_4 = -\frac{1}{4}\mu^2\left\{21\cos 2\theta + 3\right\}l^2.$$

For the stability of non-collinear equilibrium points, we study the following cases in $\wp_1^{'}\wp_2^{'}-$plane (Figure 11)

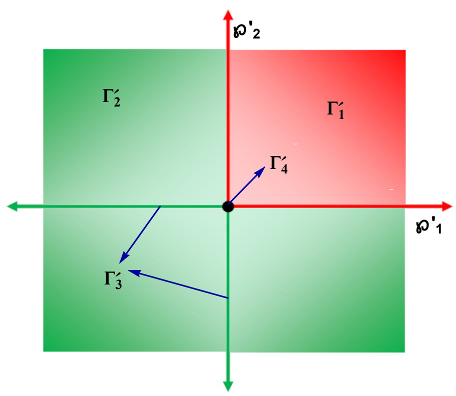

Figure 11. Geometry of the stability regions for non-collinear equilibrium points in $\wp_1^{'}\wp_2^{'}$-plane

- In the Region $\Gamma_1^{'} = \{(\wp_1^{'}, \wp_2^{'}) : \wp_1^{'} \geq 0, \wp_2^{'} > 0 \text{ or } \wp_1^{'} > 0, \wp_2^{'} \geq 0\}$, the coefficient $p_3 < 0$. Thus, the motion of m_3 is unstable.
- In the Region $\Gamma_2^{'} = \{(\wp_1^{'}, \wp_2^{'}) : \wp_1^{'} > 0, \wp_2^{'} < 0 \text{ or } \wp_1^{'} < 0, \wp_2^{'} > 0 \text{ or } \wp_1^{'} < 0, \wp_2^{'} < 0\}$, the coefficient $p_4 < 0$. Thus, the motion of m_3 is unstable.
- In the Region $\Gamma_3^{'} = \{(\wp_1^{'}, \wp_2^{'}) : \wp_1^{'} < 0, \wp_2^{'} = 0 \text{ or } \wp_1^{'} = 0, \wp_2^{'} < 0\}$, we have $\wp_3^{'} \neq 0$. If $\wp_3^{'} = 0$, then $\wp_1^{'} > 0$, which is not possible to happen in this region. Therefore, $p_4 < 0$ which implies that the motion of m_3 is unstable.
- In the Region $\Gamma_4^{'} = \{(\wp_1^{'}, \wp_2^{'}) : \wp_1^{'} = 0, \text{and } \wp_2^{'} = 0\}$, the coefficient $p_3 = 0$. Next, we have two cases.
  - When $\wp_3^{'} \neq 0$, the coefficient $p_4 < 0$ which results in unstable motion.
  - When $\wp_3^{'} = 0$, then the characteristic Equation (16) reduces in the form $\lambda^2(\lambda^2 + p_1\lambda + p_2) = 0$. Thus, the motion of m_3 is unstable.

4.4. Linear Stability of the out-of-plane equilibrium points

The coordinates of the out-of-plane equilibrium points are $\left(k(1-l^2), 0, \pm\sqrt{b_1^2 - a_1^2}\right)$. At these points, we have

$$W_{xx}^0 = 1 - \frac{3a_1^2 k}{b_1^2},\ W_{yy}^0 = 1 + l^2,\ \ W_{zz}^0 = 1 + \frac{3k}{b_1^2}(a_1^2 - b_1^2) + l^2,\ \ W_{xz}^0 = \pm\frac{-3ka_1\sqrt{b_1^2 - a_1^2}}{b_1^2},$$
$$W_{xy}^0 = W_{yz}^0 = 0.$$

Thus, the system of Equations (4) becomes

$$\begin{aligned} \ddot{\xi} - 2\omega\dot{\eta} &= \left(1 - \frac{3a_1^2 k}{b_1^2}\right)\xi \pm \left(\frac{3ka_1\sqrt{b_1^2 - a_1^2}}{b_1^2}\right)\zeta - \alpha\dot{\xi}, \\ \ddot{\eta} + 2\omega\dot{\xi} &= \left(1 + l^2\right)\eta - \alpha\dot{\eta}, \\ \ddot{\zeta} &= \left(\pm\frac{3ka_1\sqrt{b_1^2 - a_1^2}}{b_1^2}\right)\xi + \left\{1\frac{3k}{b_1^2}\left(a_1^2 - b_1^2\right) + l^2\right\}\zeta - \alpha\dot{\zeta}. \end{aligned} \tag{17}$$

The characteristic equation corresponding to the system of Equations (17) is

$$\lambda^6 + p_1\lambda^5 + p_2\lambda^4 + p_3\lambda^3 + p_4\lambda^2 + p_5\lambda + p_6 = 0, \tag{18}$$

where

$$p_1 = 3\alpha,\ p_2 = -3 + 3k - 2l^2 + 4\omega^2 + 3\alpha^2,\ p_3 = 4\alpha + 6\alpha(k-1) + \alpha^3,$$
$$\begin{aligned} p_4 =& \frac{1}{b_1^2}\Big\{3b_1^2 - 6kb_1^2 + 4b_1^2 l^2 - 3a_1^2 kl^2 - 3b_1^2 kl^2 - 4b_1^2\omega^2 - 12a_1^2 k\omega^2 + 12b_1^2 k\omega^2 - 4b_1^2 l^2\omega^2 - 3b_1^2\alpha^2 \\ &+ 3b_1^2 k\alpha^2 - 2b_1^2 l^2\alpha^2\Big\}, \end{aligned}$$
$$p_5 = \frac{1}{b_1^2}\Big\{3b_1^2\alpha - 6b_1^2 k\alpha + 4b_1^2 l^2\alpha - 3ka_1^2 l^2\alpha - 3b_1^2 kl^2\alpha\Big\},$$
$$p_6 = \frac{1}{b_1^2}\Big\{(3k-1)b_1^2 + (3kb_1^2 + 3ka_1^2 - 2b_1^2)l^2\Big\}.$$

By Descartes rule of sign there is at least one positive root of the Equation (18) since $p_6 < 0$. Therefore the equilibrium points are unstable.

5. Discussion

In the present paper, we have discussed the effect of viscosity and finite straight segment simultaneously in the CRR3BP. We have obtained collinear, non-collinear and out of plane equilibrium points. It has been found that there exist two equilibrium points (-μ,0,0) and (x_1,0,0) collinear with the centres of the primaries m_1^* and m_2. The point (-μ,0,0) is always an equilibrium point whereas the point (x_1,0,0) is an equilibrium point provided $k > 1 + l^2$. The non-collinear equilibrium points are infinite in number and lie on a circle of radius $\sqrt{1 - \frac{2}{3}l^2}$ with center at the mid-point of smaller primary m_2. These points exist only when $k = 1 - \mu + (1 - \mu)l^2$. The two out of plane equilibrium points $(k(1 - l^2), 0, \pm\sqrt{b_1^2 - a_1^2})$ exist only when $k < 0$ and $k + \mu + 2\mu l^2 > 0$. These are the same as those obtained by Kumar et al. (2019) in which they have not considered the viscous force.

Further, we have analyzed the stability of all these equilibrium points. We have found that the equilibrium point $(-\mu, 0, 0)$ is asymptotically stable for $k > 1 + 2\mu + (1 + 4\mu)l^2$ and marginally stable for $k = 1 + 2\mu + (1 + 4\mu)l^2$. The equilibrium point $(x_1, 0, 0)$ is asymptotically stable in the region Γ_3 and marginally stable in the region Γ_4. It has been noticed that the viscosity has a significant effect on stability. We observed that due to the presence of viscous force and length parameter, the stability of equilibrium points changes its nature from being marginally stable to asymptotically stable. All the other equilibrium points remain unstable whenever they exist.

On eliminating the effect of viscosity in our present model, the results of Kumar et al. (2019) can be obtained. By taking $l = 0 = \alpha$, the result of Hallan and Rana (2001a) are obtained, in which the equilibrium points $(-\mu, 0, 0)$ and $(x_1, 0, 0)$ are marginally stable. Our results are in tuned with Ansari et al. (2019a) in the absence of oblateness of the bigger primary in their paper and considering m_2 to be a point mass in the present paper. The results of our present work are in accordance with those of Ansari et al. (2019b) in absence of perturbation in Coriolis and centrifugal forces in their paper and considering m_2 as a point mass in our work.

Moreover, we have discussed the regions of stability for the equilibrium points $(-\mu, 0, 0)$ and $(x_1, 0, 0)$. For the equilibrium point $(-\mu, 0, 0)$, we have drawn the regions of stability in μk- plane (Figure 3) for different values of length parameter l. In Figure 3 the lines for fixed values of l, represented by different colours, show the marginally stable regions. For fixed values of l the asymptotically stable region are represented by light green colour lying above the respective lines. For the equilibrium point $(x_1, 0, 0)$, the regions of stability in the lk-plane, μl-plane and μk-plane are drawn for the different values of μ, k and l (Figures 5, 6, 7). Here, the asymptotically stable regions change in accordance with the change in the values of μ, k and l. Similarly, the marginally stable regions change for the different values of μ, k and l as depicted in Figures 8, 9 and 10, respectively.

6. Conclusion

The combined effect of viscosity and finite straight segment on the motion of the infinitesimal body in CRR3BP has been studied. The bigger primary is considered as a rigid spherical shell filled with a homogeneous incompressible fluid and the smaller one a finite straight segment that lies outside the shell. There is an infinitesimal body that lies inside the bigger primary. For the present model, two collinear, infinite number of non-collinear and two out of plane equilibrium points are obtained. It is noticed that all the equilibrium points are affected by the length parameter, but there is no impact of the viscosity parameter on them. Furthermore, the linear stability of the obtained equilibrium points is also studied. The collinear equilibrium points are found to be conditionally stable. It is observed that both length and viscosity parameters have subsequent effect on the stability of the collinear equilibrium points. The non-collinear and out-of-plane equilibrium points are always unstable. It is also observed that the unstable nature of these points are not affected by the viscosity parameter.

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