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Accurate Spectral Algorithms for Solving Variable-order Fractional Percolation Equations

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Abstract

A high accurate spectral algorithm for one-dimensional variable-order fractional percolation equations (VO-FPEs) is considered. We propose a shifted Legendre Gauss-Lobatto collocation (SL-GL-C) method in conjunction with shifted Chebyshev Gauss-Radau collocation (SC-GR-C) method to solve the proposed problem. Firstly, the solution and its space fractional derivatives are expanded as shifted Legendre polynomials series. Then, we determine the expansion coefficients by reducing the VO-FPEs and its conditions to a system of ordinary differential equations (SODEs) in time. The numerical approximation of SODEs is achieved by means of the SC-GR-C method. The under-study's problem subjected to the Dirichlet or non-local boundary conditions is presented and compared with the results in literature, which reveals wonderful results.

Keywords: Collocation method; Shifted Legendre-Gauss-Lobatto quadrature; Shifted Chebyshev Gauss-Radau quadrature; Fractional derivatives; Variable-order fractional nonlinear percolation equations

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1. Introduction

Several numerical methods are presented for acquiring hight accurate solutions for fractional differential equations (Doha et al. (2012) and Bhrawy et al. (2012)). Spectral methods (Canuto et al. (2006), Khader and Megahed (2015), Abdelkawy and Taha (2014)) have been widely used in many fields in the last four decades. In the early times, the spectral methods based on Fourier expansion have been used in few fields such as a simple geometric field and periodic boundary conditions. Recently, they have been developed theoretically and used as powerful techniques to solve various kinds of problems. Based on the accuracy and the exponential rates of convergence, spectral methods have an excellent reputation when compared with other numerical methods. The expression of the problem solution as a finite series of some functions is the major step of all types of spectral methods. Then, the coefficients will be chosen such that the absolute error is diminished as well as possible while the numerical solution will be enforced to almost satisfy the partial differential equations (PDEs) in spectral collocation method (Bhrawy (2014), Khader (2018), Bhrawy and Abdelkawy (2015)). In other words, the residuals may be permitted to be zero at chosen points.

Fractional calculus is a division of calculus theory. That makes PDEs more relevant to represent many phenomena in different fields like fluid mechanics, biology, chemistry, viscoelasticity, engineering, finance, and physics fields. The concept of the variable-order fractional allows the power of the fractional operator to be a function of the independent variable. The early studies of variable-order fractional is firstly discussed by Samko and Ross (1993), Lorenzo and Hartley (2002) and Lorenzo (2007). Several phenomena may be more accurately represented via variable-order fractional operators. Mechanical (Coimbra (2003)), diffusion (Chen et al. (2012)), FIR filters (Tseng (2006)) multi-fractional Gaussian noises (Sheng et al. (2011)) and physical models (Sheng et al. (2010)) can be more accurately described by variable order derivatives mathematical models.

Few numerical methods are introduced and discussed to solve numerically the variable-order fractional problems. Convergence and stability of the explicit finite-difference method have been studied in Lin et al. (2009) to solve the variable-order nonlinear fractional diffusion equation. By means of Fourier analysis, Chen (2013) obtained numerical solutions for the two-dimensional variable-order modified diffusion equations. Finite difference techniques (Chen et al. (2011), Zhang et al. (2014)) are proposed for treatment of PDEs with fractional variable-order. Zhao et al. (2015), introduced two algorithms of the second-order approach to solve the time variable-order fractional problem. Also, the finite difference method has been applied by Xu and Ertürk (2014) to fractional integro-differential equations with variable-order.

By means of the SL-GL-C and SC-GR-C schemes, the numerical solutions of the VO-FPEs are obtained. Several studies based on percolation flow models have been listed in many fields including groundwater hydraulics, groundwater dynamics, seepage hydraulics, and fluid dynamics in porous media. For the temporal and spatial discretizations, we used the spectral collocation approach. The SL-GL-C with a suitable treatment of the boundary or non-local conditions is firstly applied for spatial discretization. This modification greatly improves the accuracy of our scheme. Then, the VO-FPE is transformed to SODEs with initial valued vector. Secondly, the temporal discretization has been achieved by means of the SC-GR-C. As a result, a system of algebraic equations is

obtained.

This paper is arranged as follows. Some revelent properties of Riemann-Liouville fractional derivatives (R-LFDs), shifted Legendre polynomials and shifted Chebyshev polynomials are listed in the coming section. The third section deals with one-dimensional fractional percolation problems with classical and non-classical boundary conditions. In Section 4, three numerical examples are tested. Remarks are included in the last section.

2. Preliminaries and notation

2.1. Riemann-Liouville fractional derivative

The fractional integration of order $\nu > 0$ is exist in different formulae (Miller and Ross (1993)). Riemann-Liouville formula, the most common and widely used, is defined as:

$$\begin{aligned} &J^{\mu} f(\zeta) = \frac{1}{\Gamma(\mu)} \int_0^{\zeta} (\zeta - \tau)^{\mu-1} f(\tau) d\tau, \quad \mu > 0, \quad \zeta > 0, \\ &J^0 f(\zeta) = f(\zeta). \end{aligned} \tag{1}$$

Thus,

$$J^{\mu} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\mu)}\, x^{\gamma+\mu}. \tag{2}$$

The R-LFD of order μ is

$$D^{\mu} f(x) = \frac{d^m}{dx^m}(J^{m-\mu} f(x)), \tag{3}$$

where $m-1 < \mu \leq m,\ m \in \mathbb{N}$ and m is the smallest integer greater than μ.

Lemma 2.1.

If $m-1 < \mu \leq m,\ m \in \mathbb{N}$, then,

$$D^{\nu} J^{\mu} f(x) = f(x), \qquad J^{\mu} D^{\mu} f(x) = f(x) - \sum_{i=0}^{m-1} f^{(i)}(0^+) \frac{x^i}{i!}, \quad x > 0. \tag{4}$$

2.2. Shifted Legendre Gauss-Lobatto interpolation

In this subsection, we list some approximation results for the shifted Legendre Gauss-Lobatto (SL-GL) interpolation. The Legendre polynomials $P_k(x)$ $(k = 0, 1, 2, \ldots,)$ satisfy the Rodrigue's formula

$$P_k(x) = \frac{(-1)^k}{2^k k!} D^k((1-x^2)^k). \tag{5}$$

Next, denoting by $\|u\|$ and (u, v) the norm and inner product of space $\ell^2[-1, 1]$. The set of $P_k(x)$ is a complete orthogonal system in $\ell^2[-1, 1]$,

$$(P_j(x), P_k(x)) = \int_{-1}^{1} P_j(x)P_k(x)\, dx = h_k\delta_{jk}, \tag{6}$$

where $h_k = \frac{2}{2k+1}$ and δ_{jk} is the Dirac function. Thus, for any $v \in \ell^2[-1, 1]$,

$$v(x) = \sum_{i=0}^{\infty} a_i P_i(x), \qquad a_i = \frac{1}{h_i}\int_{-1}^{1} v(x)P_i(x)\, dx. \tag{7}$$

Let $S_N[-1, 1]$ be the set of all polynomials of degree at most N ($N \geq 0$). Thus, for any $\varphi \in S_{2N-1}[-1, 1]$ we have

$$\int_{-1}^{1} \varphi(x)dx = \sum_{i=0}^{N} \varpi_{N,i}\varphi(x_{N,i}), \tag{8}$$

where $x_{N,i}$ ($0 \leq i \leq N$) and $\varpi_{N,i}$ ($0 \leq i \leq N$) are denoted to the nodes and Christoffel numbers of Legendre Gauss-Lobatto (L-GL) interpolation on the classical interval $[-1, 1]$, respectively. The norm and discrete inner product are defined as:

$$\|u\|_w = (u, u)_w^{\frac{1}{2}}, \qquad (u, v)_w = \sum_{j=0}^{N} u(x_{N,j})\, v(x_{N,j})\, \varpi_{N,j}. \tag{9}$$

Let us denote by $P_{\ell,k}(x)$ the shifted Legendre polynomials which defined on the interval $[0, \ell]$. These polynomials can be engendered from the recurrence relation:

$$(k+1)P_{\ell,k+1}(x) = (2k+1)(\frac{2x}{\ell} - 1)P_{\ell,k}(x) - kP_{\ell,k-1}(x), \qquad k = 1, 2, \cdots. \tag{10}$$

The analytic form of $P_{\ell,i}(x)$ may be written as:

$$P_{\ell,j}(x) = \sum_{k=0}^{j} (-1)^{j+k} \frac{(j+k)!}{(j-k)!\,(k!)^2\,\ell^k}\, x^k. \tag{11}$$

The $J^\nu P_{\ell,j}(x)$ may be obtained from:

$$\begin{aligned} J^\nu P_{\ell,j}(x) &= \sum_{k=0}^{j} (-1)^{k+j}\, \frac{(k+j)!}{(-k+j)!\,(k!)^2\,\ell^k}\, J^\nu x^k \\ &= \sum_{k=0}^{j} (-1)^{k+j}\, \frac{(k+j)!\,k!}{(-k+j)!\,(k!)^2\,\ell^k\,\Gamma(k+\nu+1)}\, x^{k+\nu}, \quad j = 0, 1, \cdots, N, \end{aligned} \tag{12}$$

where $P_{\ell,j}(0) = (-1)^j$. The orthogonality condition is

$$\int_0^\ell P_{\ell,j}(x)P_{\ell,k}(x)w_\ell(x)\,dx = h_k^\ell\,\delta_{jk}, \tag{13}$$

where $w_L(x) = 1$ and $h_k^\ell = \dfrac{\ell}{2k+1}$.

If function $u(t) \in L^2[0,\ell]$. Then, one can express it by means of $P_{\ell,i}(t)$ as

$$u(t) = \sum_{i=0}^{\infty} c_i P_{\ell,i}(t),$$

where c_i is given by

$$c_i = \frac{1}{h_i^\ell}\int_0^\ell u(t)P_{\ell,i}(t)dt, \quad i = 0,1,2,\cdots. \tag{14}$$

The approximation of $u(t)$ may be expanded as

$$u_N(t) \simeq \sum_{i=0}^{N} c_i P_{\ell,i}(t). \tag{15}$$

2.3. Shifted Chebyshev Gauss-Radau interpolation

The Chebyshev polynomials are defined on the interval $[-1,1]$, by

$$T_k(t) = \cos(k\arccos(t)), \quad k \geq 0. \tag{16}$$

Also,

$$T_k(\pm 1) = (\pm 1)^k, \quad T_k(-t) = (-1)^k T_k(t). \tag{17}$$

Let $w^c(t) = \dfrac{1}{\sqrt{1-t^2}}$, then, we introduce the following norm and inner product of the the weighted space $L^2_{w^c}$ as:

$$\|u\|_{w^c} = (u,u)_{w^c}^{\frac{1}{2}}, \qquad (u,v)_{w^c} = \int_{-1}^{1} u(t)\,v(t)w^c(t)dt. \tag{18}$$

The set of Chebyshev polynomials satisfies:

$$\int_{-1}^{1} T_k(t)T_j(t)w(t)dt = h_k^c\delta_{kj} = \frac{\varsigma_k}{2}\pi\delta_{kj},\ \varsigma_0 = 2, \quad \varsigma_k = 1, \quad k \geq 1. \tag{19}$$

Now, we define the following norm and discrete inner product

$$\|u\|_{w^c} = (u,u)_{w^c}^{\frac{1}{2}}, \qquad (u,v)_{w^c} = \sum_{j=0}^{N} u(t_{N,j})\, v(t_{N,j})\, \varpi_{N,j}^{c}. \tag{20}$$

Let us denote by $T_{\tau,n}(t)$ the shifted Chebyshev polynomials which defined on the interval $[0,\tau]$. The analytic form of $T_{\tau,n}(t)$ is obtained from

$$T_{\tau,n}(t) = n \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+k-1)!\, 2^{2k}}{(n-k)!\, (2k)!\, \tau^k}\, t^k, \tag{21}$$

where $T_{\tau,n}(0) = (-1)^n$ and $T_{\tau,n}(\tau) = 1$.

The orthogonality condition is

$$\int_0^{\tau} T_{\tau,m}(t) T_{\tau,n}(t) w_T(t) dt = \delta_{mn}\, {}^c h_n^{\tau}, \tag{22}$$

where $w_\tau(t) = \dfrac{1}{\sqrt{\tau t - t^2}}$ and ${}^c h_n^{\tau} = \dfrac{c_n}{2}\pi,$ with $\quad c_0 = 2,\ c_i = 1,\ i \geq 1.$

As in the previous subsection, if $u(t) \in L^2_{w_T(t)}[0,\tau]$. Then, one can express it by means of $T_{\tau,j}(t)$ as

$$u(t) = \sum_{j=0}^{\infty} a_j T_{\tau,j}(t), \tag{23}$$

where,

$$a_j = \frac{1}{{}^c h_j^{\tau}} \int_0^{\tau} u(t) T_{\tau,j}(t) w_{\tau(t)} dt, \quad j = 0,1,2,\cdots. \tag{24}$$

3. One-dimensional space VO-FPEs

Based on SL-GL-C and SC-GR-C methods, three algorithms have been derived for the numerical treatment of VO-FPEs with different types of boundary conditions. The main objective of this method is to discretize the VO-FPE in the spatial direction with a modification treatment of the conditions, to generate a SODEs of the unknown coefficients of spectral expansion in time direction. Then, SC-GR-C method is applied to solve this system.

3.1. Initial-boundary conditions

In this subsection, an algorithm is introduced to treat with the following VO-FPE

$$\frac{\partial \phi(x,t)}{\partial t} = \frac{\partial}{\partial x}\Big(g_1(x)\frac{\partial^{\nu(x)}\phi(x,t)}{\partial x^{\nu(x)}}\Big) + \chi(x,t), \quad (x,t) \in [0,\ell] \times [0,\tau], \tag{25}$$

subject to the initial condition

$$\phi(x,0) = g_2(x), \quad x \in [0,\ell], \tag{26}$$

and boundary conditions

$$\phi(0,t) = g_3(t), \quad \phi(\ell,t) = g_4(t), \quad t \in [0,\tau], \tag{27}$$

where, $\chi(x,t), \quad g_1(x), \quad g_2(x), \quad g_3(t)$ and $g_4(t)$ are given functions. The equation (25) may be restated as

$$\frac{\partial \phi(x,t)}{\partial t} = g_5(x)\frac{\partial^{\nu(x)}\phi(x,t)}{\partial x^{\nu(x)}} + g_1(x)\frac{\partial}{\partial x}\Big(\frac{\partial^{\nu(x)}\phi(x,t)}{\partial x^{\nu(x)}}\Big) + \chi(x,t), \quad (x,t) \in [0,\ell] \times [0,\tau], \tag{28}$$

where, $g_5(x) = \frac{\partial g_1(x)}{\partial x}$. Now, the SL-GL-C method has been used to transform the VO-FPE (25)-(27) into a SODEs. Thus, we collocate the spatial variable by means of the SL-GL-C method at $x^{\ell}_{N,i}$ points. We list the major procedure of the SL-GL-C method to solve VO-FPE related to initial-boundary conditions. Thus, the approximate solution is presented as

$$\phi_N(x,t) = \sum_{r=0}^{N} a_r(t) P_{\ell,r}(x), \tag{29}$$

where, $a_r(t)$ are approximated as

$$a_r(t) = \frac{1}{h^{\ell}_r}\sum_{s=0}^{N} P_{\ell,r}(x^{\ell}_{N,s})\varpi^{\ell}_{N,s}\,\phi(x^{\ell}_{N,s},t). \tag{30}$$

Then,

$$\phi_N(x,t) = \sum_{s=0}^{N}\Big(\sum_{r=0}^{N}\frac{1}{h^{\ell}_r}P_{\ell,r}(x^{\ell}_{N,s})P_{\ell,r}(x)\varpi^{\ell}_{N,s}\Big)\phi(x^{\ell}_{N,s},t). \tag{31}$$

On the other hand, the spatial partial derivative $\frac{\partial^{\nu(x)}\phi(x,t)}{\partial x^{\nu(x)}}$ is approximated as

$$\frac{\partial^{\nu(x)}\phi_N(x,t)}{\partial x^{\nu(x)}} = \sum_{s=0}^{N}\Big(\sum_{r=0}^{N}\frac{1}{h_r^\ell}P_{\ell,r}(x_{N,s}^\ell)\frac{\partial^{\nu(x)}P_{\ell,r}(x)}{\partial x^{\nu(x)}}\varpi_{N,s}^\ell\Big)\phi(x_{N,s}^\ell,t). \tag{32}$$

For $0<\nu(x)<1$, the variable order R-LFD (Lin et al. (2009)) is defined as

$$\begin{aligned}\frac{\partial^{\nu(x)}x^k}{\partial x^{\nu(x)}} &= \frac{}{\Gamma(1-\nu(x))}\Big(\frac{\partial}{\partial\eta}\int_0^\eta\frac{\chi^k}{(\eta-\chi)^{\nu(x)}}d\chi\Big)_{\eta=x}\\ &= \frac{x^{k-\nu(x)}\Gamma(1+k)}{\Gamma(1+k-\nu(x))}.\end{aligned} \tag{33}$$

Thus,

$$\frac{\partial^{\nu(x)}P_{\ell,i}(x)}{\partial x^{\nu(x)}} = P_{\ell,i}^{(\nu(x))}(x) = \sum_{k=1}^{i}(-1)^{i+k}\frac{(i+k)!\Gamma(1+k)}{(i-k)!\,(k!)^2\Gamma(1+k-\nu(x))\,\ell^k}\,x^{k-\nu(x)}. \tag{34}$$

Consequently,

$$\frac{\partial^{\nu(x)}\phi_N(x,t)}{\partial x^{\nu(x)}} = \sum_{s=0}^{N}\Big(\sum_{r=0}^{N}\frac{1}{h_r^\ell}P_{\ell,r}(x_{N,s}^\ell)P_{\ell,s}^{(\nu(x))}(x)\varpi_{N,s}^\ell\Big)\phi(x_{N,s}^\ell,t). \tag{35}$$

Similarly, we can obtain $\frac{\partial}{\partial x}\Big(\frac{\partial^{\nu(x)}\phi_N(x,t)}{\partial x^{\nu(x)}}\Big)$ as

$$\frac{\partial}{\partial x}\Big(\frac{\partial^{\nu(x)}\phi_N(x,t)}{\partial x^{\nu(x)}}\Big) = \sum_{s=0}^{N}\Big(\sum_{r=0}^{N}\frac{1}{h_r^\ell}P_{\ell,r}(x_{N,s}^\ell)\frac{\partial}{\partial x}(P_{\ell,s}^{(\nu(x))}(x))\varpi_{N,s}^\ell\Big)\phi(x_{N,s}^\ell,t), \tag{36}$$

the previous fractional derivatives can be computed at the shifted Legendre Gauss-Lobatto interpolation nodes as

$$\Big(\frac{\partial^{\nu(x)}\phi_N(x,t)}{\partial x^{\nu(x)}}\Big)_{x=x_{N,n}^\ell} = \sum_{s=0}^{N}\varsigma_{n,s}\phi_s(t), \quad n=0,1,\cdots,N, \tag{37}$$

$$\Big(\frac{\partial}{\partial x}\big(\frac{\partial^{\nu(x)}\phi_N(x,t)}{\partial x^{\nu(x)}}\big)\Big)_{x=x_{N,n}^\ell} = \sum_{s=0}^{N}\varrho_{n,s}\phi_s(t), \quad n=0,1,\cdots,N, \tag{38}$$

where,

$$\phi_s(t) = \phi(x^{\ell}_{N,s}, t),$$

$$\varsigma_{n,s} = \sum_{r=0}^{N} \frac{\varpi^{\ell}_{N,s}}{h^{\ell}_r} P_{\ell,r}(x^{\ell}_{N,s}) \big(\frac{\partial^{\nu(x)} P_{\ell,r}(x)}{\partial x^{\nu(x)}}\big)_{x=x^{\ell}_{N,n}},$$

$$\varrho_{n,s} = \sum_{r=0}^{N} \frac{\varpi^{\ell}_{N,s}}{h^{\ell}_r} P_{\ell,r}(x^{\ell}_{N,s}) \big(\frac{\partial}{\partial x}(P^{(\nu(x))}_{\ell,s}(x))\big)_{x=x^{\ell}_{N,n}}.$$

In the novel SL-GL-C method, the residual of (25) is set to zero at $N-1$ of SL-GL points. Furthermore, the boundary conditions (27) are satisfied exactly at the two collocation points 0 and ℓ.

Based on (28)-(38), Eq. (25) is written as

$$\begin{aligned} \dot{\phi_n}(t) =& g_5(x^{\ell}_{N,n})\Big(\sum_{s=1}^{N-1} \varsigma_{n,s}\phi_s(t) + \varsigma_{n,0}\phi_0(t) + \varsigma_{n,N}\phi_N(t)\Big) + f(x^{\ell}_{N,n}, t) + \\ & g_1(x)\Big(\sum_{s=1}^{N-1} \varrho_{n,s}\phi_s(t) + \varrho_{n,0}\phi_0(t) + \varrho_{n,N}\phi_N(t)\Big). \end{aligned} \tag{39}$$

More precisely, the VO-FPEs (25)-(27) is reduced into the following SODEs

$$\begin{aligned} \dot{\phi_n}(t) =& g_5(x^{\ell}_{N,n})\Big(\sum_{s=1}^{N-1} \varsigma_{n,s}\phi_s(t) + \varsigma_{n,0}g_3(t) + \varsigma_{n,N}g_4(t)\Big) + \chi(x^{\ell}_{N,n}, t) + \\ & g_1(x)\Big(\sum_{s=1}^{N-1} \varrho_{n,s}\phi_s(t) + \varrho_{n,0}g_3(t) + \varrho_{n,N}g_4(t)\Big), \quad n = 1, \cdots, N-1, \end{aligned} \tag{40}$$

subject to the initial values

$$\phi_n(0) = g_2(x^{\ell}_{N,n}), \quad n = 1, \cdots, N-1. \tag{41}$$

The previous equations can be rearranged in a matrix form as:

$$\begin{pmatrix} \dot{\phi_1}(t) \\ \dot{\phi_2}(t) \\ \cdots \\ \cdots \\ \cdots \\ \dot{\phi_{N-2}}(t) \\ \dot{\phi_{N-1}}(t) \end{pmatrix} = \begin{pmatrix} F_1(t, \phi_1(t), \phi_2(t), \cdots, \phi_{N-1}(t)) \\ F_2(t, \phi_1(t), \phi_2(t), \cdots, \phi_{N-1}(t)) \\ \cdots \\ \cdots \\ \cdots \\ F_{N-2}(t, \phi_1(t), \phi_2(t), \cdots, \phi_{N-1}(t)) \\ F_{N-1}(t, \phi_1(t), \phi_2(t), \cdots, \phi_{N-1}(t)) \end{pmatrix}, \tag{42}$$

$$\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \cdots \\ \cdots \\ \cdots \\ \phi_{N-2}(0) \\ \phi_{N-1}(0) \end{pmatrix} = \begin{pmatrix} g_2(x^{\ell}_{N,1}) \\ g_2(x^{\ell}_{N,2}) \\ \cdots \\ \cdots \\ \cdots \\ g_2(x^{\ell}_{N,N-2}) \\ g_2(x^{\ell}_{N,N-1}) \end{pmatrix}, \tag{43}$$

where,

$$\begin{aligned} &F_n(t, \phi_1(t), \phi_2(t), \cdots, \phi_{N-1}(t)) \\ &\quad = g_5(x^{\ell}_{N,n})\Big(\sum_{s=1}^{N-1} \varsigma_{n,s}\phi_s(t) + \varsigma_{n,0}g_3(t) + \varsigma_{n,N}g_4(t)\Big) + \chi(x^{\ell}_{N,n}, t) + \\ &\qquad g_1(x^{\ell}_{N,n})\Big(\sum_{s=1}^{N-1} \varrho_{n,s}\phi_s(t) + \varrho_{n,0}g_3(t) + \varrho_{n,N}g_4(t)\Big). \end{aligned} \tag{44}$$

3.2. Initial, boundary and non-local integral conditions

An integral condition is added in this subsection. We present the following VO-FPEs:-

$$\frac{\partial \phi(x,t)}{\partial t} = \frac{\partial}{\partial x}\Big(g_1(x)\frac{\partial^{\nu(x)}\phi(x,t)}{\partial x^{\nu(x)}}\Big) + \chi(x,t), \quad (x,t) \in [0,\ell] \times [0,\tau], \tag{45}$$

subject to the initial condition

$$\phi(x,0) = g_2(x), \quad x \in [0,\ell], \tag{46}$$

the boundary condition

$$\phi(0,t) = g_3(t), \quad t \in [0,\tau], \tag{47}$$

and the non-local integral condition

$$\int_{b_1}^{b_2} \phi(x,t)dt = g_4(t), \quad 0 \le b_1 < b_2 \le \ell, \quad t \in [0,\tau]. \tag{48}$$

where, $\chi(x,t)$, $g_1(x)$, $g_2(x)$, $g_3(t)$ and $g_4(t)$ are given valued functions. Here, we take care of the treatment of the non-local condition (48). In order to do this, we reconstructed the integral condition (48) as

$$\int_{b_1}^{b_2}\sum_{s=0}^{N}\Big(\sum_{r=0}^{N}\frac{1}{h_r^{\ell}}P_{\ell,r}(x_{N,s}^{\ell})P_{\ell,r}(x)\varpi_{N,s}^{\ell}\Big)\phi_s(t)dx=g_2(t),\quad 0\le b_1<b_2\le\ell. \tag{49}$$

We rearrange the above formula to give

$$\sum_{s=0}^{N}\Big(\sum_{r=0}^{N}\frac{1}{h_r^{\ell}}P_{\ell,r}(x_{N,s}^{\ell})\varpi_{N,s}^{\ell}(\int_{b_1}^{b_2}P_{\ell,r}(x)dx)\Big)\phi_s(t)=g_2(t),\quad 0\le b_1<b_2\le\ell, \tag{50}$$

or briefly,

$$\sum_{s=0}^{N}I_s\phi_s(t)=g_2(t),\quad 0\le b_1<b_2\le\ell. \tag{51}$$

where,

$$I_s=\sum_{r=0}^{N}\frac{1}{h_r^{\ell}}P_{\ell,r}(x_{N,s}^{\ell})\varpi_{N,s}^{\ell}(\int_{b_1}^{b_2}P_{\ell,r}(x)dx).$$

Consequently, we get

$$\begin{aligned}\phi_N(t)=&\frac{1}{I_N}\Big(g_4(t)-I_0\phi_0(t)-\sum_{s=1}^{N-1}I_s\phi_s(t)\Big),\quad 0\le b_1<b_2\le\ell,\\ =&\frac{1}{I_N}\Big(g_4(t)-I_0g_3(t)-\sum_{s=1}^{N-1}I_s\phi_s(t)\Big).\end{aligned} \tag{52}$$

By means of the information mentioned above, we get the SODEs

$$\dot{\phi_n}(t)=g_5(x_{N,n}^{\ell})\Delta_1(t)+g_1(x_{N,n}^{\ell})\Delta_2(t)+\chi(x_{N,n}^{\ell},t),\quad n=1,\cdots,N-1, \tag{53}$$

where,

$$\Delta_1(t) = \Big(\sum_{s=1}^{N-1} \varsigma_{n,s}\phi_s(t) + \varsigma_{n,0} g_3(t) + \frac{\varsigma_{n,N}}{I_N}\Big(g_4(t) - I_0 g_3(t) - \sum_{s=1}^{N-1} I_s \phi_s(t)\Big)\Big),$$

$$\Delta_2(t) = \Big(\sum_{s=1}^{N-1} \varrho_{n,s}\phi_s(t) + \varrho_{n,0} g_3(t) + \frac{\varrho_{n,N}}{I_N}\Big(g_4(t) - I_0 g_3(t) - \sum_{s=1}^{N-1} I_s \phi_s(t)\Big)\Big).$$

This provides a $(N-1)$ SODEs in $\phi_s(t)$. In other words, the problem (45)-(48) is reduced to the following SODEs

$$\dot{\phi_n}(t) = g_5(x^{\ell}_{N,n})\Delta_1(t) + g_1(x^{\ell}_{N,n})\Delta_2(t) + \chi(x^{\ell}_{N,n}, t), \quad n = 1, \cdots, N-1, \tag{54}$$

subject to the initial values

$$\phi_n(0) = g_2(x^{\ell}_{N,n}), \quad n = 1, \cdots, N-1. \tag{55}$$

Or in matrix notation as:

$$\begin{pmatrix} \dot{\phi_1}(t) \\ \dot{\phi_2}(t) \\ \cdots \\ \cdots \\ \cdots \\ \dot{\phi_{N-2}}(t) \\ \dot{\phi_{N-1}}(t) \end{pmatrix} = \begin{pmatrix} F_1(t, \phi_1(t), \phi_2(t), \cdots, \phi_{N-1}(t)) \\ F_2(t, \phi_1(t), \phi_2(t), \cdots, \phi_{N-1}(t)) \\ \cdots \\ \cdots \\ \cdots \\ F_{N-2}(t, \phi_1(t), \phi_2(t), \cdots, \phi_{N-1}(t)) \\ F_{N-1}(t, \phi_1(t), \phi_2(t), \cdots, \phi_{N-1}(t)) \end{pmatrix}, \tag{56}$$

$$\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \cdots \\ \cdots \\ \cdots \\ \phi_{N-2}(0) \\ \phi_{N-1}(0) \end{pmatrix} = \begin{pmatrix} g_2(x^{\ell}_{N,1}) \\ g_2(x^{\ell}_{N,2}) \\ \cdots \\ \cdots \\ \cdots \\ g_2(x^{\ell}_{N,N-2}) \\ g_2(x^{\ell}_{N,N-1}) \end{pmatrix}, \tag{57}$$

where

$$F_n(t, \boldsymbol{\phi}(t)) = g_5(x^{\ell}_{N,n})\Delta_1(t) + g_1(x^{\ell}_{N,n})\Delta_2(t) + \chi(x^{\ell}_{N,n}, t). \tag{58}$$

3.3. SC-GR-C scheme for time variable

Here, we extend the collocation strategy discussed in the previous subsection to approximate the SODEs

$$\begin{pmatrix} \dot{\phi_1}(t) \\ \dot{\phi_2}(t) \\ \cdots \\ \cdots \\ \cdots \\ \dot{\phi_{N-2}}(t) \\ \dot{\phi_{N-1}}(t) \end{pmatrix} = \begin{pmatrix} F_1(t, \phi_1(t), \phi_2(t), \cdots, \phi_{N-1}(t)) \\ F_2(t, \phi_1(t), \phi_2(t), \cdots, \phi_{N-1}(t)) \\ \cdots \\ \cdots \\ \cdots \\ F_{N-2}(t, \phi_1(t), \phi_2(t), \cdots, \phi_{N-1}(t)) \\ F_{N-1}(t, \phi_1(t), \phi_2(t), \cdots, \phi_{N-1}(t)) \end{pmatrix}, \tag{59}$$

subject to the initial values

$$\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \cdots \\ \cdots \\ \cdots \\ \phi_{N-2}(0) \\ \phi_{N-1}(0) \end{pmatrix} = \begin{pmatrix} g_2(x^{\ell}_{N,1}) \\ g_2(x^{\ell}_{N,2}) \\ \cdots \\ \cdots \\ \cdots \\ g_2(x^{\ell}_{N,N-2}) \\ g_2(x^{\ell}_{N,N-1}) \end{pmatrix}. \tag{60}$$

We use the SC-GR-C method to deal with the temporal variable t. We approximate $\phi_n(t)$, $n = 1, \cdots, N-1$, as

$$\phi_{n,M}(t) = \sum_{\varsigma=0}^{M} a_{n,\varsigma} T_{\tau,\varsigma}(t), \quad n = 1, \cdots, N-1. \tag{61}$$

Furthermore, the time derivative can be computed as

$$\frac{d}{dt}(\phi_{n,M}(t)) = \sum_{\varsigma=0}^{M} a_{n,\varsigma} \frac{d}{dt}(T_{\tau,\varsigma}(t)) = \sum_{\varsigma=0}^{M} a_{n,\varsigma} T^{(1)}_{\tau,\varsigma}(t), \quad n = 1, \cdots, N-1, \tag{62}$$

here $T^{(1)}_{\tau,k}(t)$ represents the first time derivative of shifted Chebyshev polynomials which can be easily evaluated at any point $t^{\tau}_{K,s}$ (shifted Chebyshev Gauss Radau points).

Adopting (61)-(62), Equations (59)-(60) can be rewritten in the form:

$$\begin{pmatrix} \sum_{\varsigma=0}^{M} a_{1,\varsigma} T_{\tau,\varsigma}^{(1)}(t) \\ \sum_{\varsigma=0}^{M} a_{2,\varsigma} T_{\tau,\varsigma}^{(1)}(t) \\ \cdots \\ \cdots \\ \cdots \\ \sum_{\varsigma=0}^{M} a_{N-2,\varsigma} T_{\tau,\varsigma}^{(1)}(t) \\ \sum_{\varsigma=0}^{M} a_{N-1,\varsigma} T_{\tau,\varsigma}^{(1)}(t) \end{pmatrix} = \begin{pmatrix} F_1(t, \sum_{\varsigma=0}^{M} a_{1,\varsigma} T_{\tau,\varsigma}(t), \cdots, \sum_{\varsigma=0}^{M} a_{N-1,\varsigma} T_{\tau,\varsigma}(t)) \\ F_2(t, \sum_{\varsigma=0}^{M} a_{1,\varsigma} T_{\tau,\varsigma}(t), \cdots, \sum_{\varsigma=0}^{M} a_{N-1,\varsigma} T_{\tau,\varsigma}(t)) \\ \cdots \\ \cdots \\ \cdots \\ F_{N-2}(t, \sum_{\varsigma=0}^{M} a_{1,\varsigma} T_{\tau,\varsigma}(t), \cdots, \sum_{\varsigma=0}^{M} a_{N-1,\varsigma} T_{\tau,\varsigma}(t)) \\ F_{N-2}(t, \sum_{\varsigma=0}^{M} a_{1,\varsigma} T_{\tau,\varsigma}(t), \cdots, \sum_{\varsigma=0}^{M} a_{N-1,\varsigma} T_{\tau,\varsigma}(t)) \end{pmatrix}, \tag{63}$$

$$\begin{pmatrix} \sum_{\varsigma=0}^{M} a_{1,\varsigma} T_{\tau,\varsigma}(0) \\ \sum_{\varsigma=0}^{M} a_{2,\varsigma} T_{\tau,\varsigma}(0) \\ \cdots \\ \cdots \\ \cdots \\ \sum_{\varsigma=0}^{M} a_{N-2,\varsigma} T_{\tau,\varsigma}(0) \\ \sum_{\varsigma=0}^{M} a_{N-1,\varsigma} T_{\tau,\varsigma}(0) \end{pmatrix} = \begin{pmatrix} g_2(x_{N,1}^{\ell}) \\ g_2(x_{N,2}^{\ell}) \\ \cdots \\ \cdots \\ \cdots \\ g_2(x_{N,N-2}^{\ell}) \\ g_2(x_{N,N-1}^{\ell}) \end{pmatrix}. \tag{64}$$

In the proposed technique, the residual of (63) has to be enforced to zero at $(M(N-1))$ collocation points. Otherwise, we collocate Equation (63) at the $(M(N-1))$ shifted Chebyshev collocation points, which immediately yields

$$\begin{aligned} \sum_{\varsigma=0}^{M} a_{n,\varsigma} T_{T,\varsigma}^{(1)}(t_{M,s}^{\tau})) =& F_n(t_{M,s}^{\tau}, \sum_{\varsigma=0}^{M} a_{1,\varsigma} T_{\tau,\varsigma}(t_{M,s}^{\tau}), \cdots, \sum_{\varsigma=0}^{M} a_{N-1,\varsigma} T_{\tau,\varsigma}(t_{M,s}^{\tau})), \\ & n = 1, \cdots, N-1, \quad s = 1, \cdots, M, \end{aligned} \tag{65}$$

by virtue of (64), we get

$$\sum_{\varsigma=0}^{M} a_{n,\varsigma} T_{\tau,\varsigma}(0) = \tau_n, \quad n = 1, \cdots, N-1. \tag{66}$$

The set of previous equations is equivalent to a system of $(N-1)(M+1)$ algebraic equations in the unknowns $a_{i,j}$, $i = 1, \cdots, N-1$; $j = 0, \cdots, M$,

$$\begin{pmatrix} \kappa_{1,0} & \cdots & \kappa_{1,M} \\ \kappa_{2,0} & \cdots & \kappa_{2,M} \\ \cdots & \ddots & \cdots \\ \cdots & \ddots & \cdots \\ \cdots & \ddots & \cdots \\ \kappa_{N-2,0} & \cdots & \kappa_{N-2,M} \\ \kappa_{N-1,0} & \cdots & \kappa_{N-1,M} \end{pmatrix} = \begin{pmatrix} \xi_{1,0} & \cdots & \xi_{1,M} \\ \xi_{2,0} & \cdots & \xi_{2,M} \\ \cdots & \ddots & \cdots \\ \cdots & \ddots & \cdots \\ \cdots & \ddots & \cdots \\ \xi_{N-2,0} & \cdots & \xi_{N-2,M} \\ \xi_{N-1,0} & \cdots & \xi_{N-1,M} \end{pmatrix}, \tag{67}$$

where,

$$\kappa_{n,s} = \begin{cases} \sum\limits_{\varsigma=0}^{M} a_{n,\varsigma} T_{\tau,\varsigma}(0), \qquad s=0, \quad n=1,\cdots,N-1, \\ \sum\limits_{\varsigma=0}^{M} a_{n,\varsigma} T^{(1)}_{T,j}(t^{\tau}_{M,s})), \, n=1,\cdots,N-1, \quad s=1,\cdots,M, \end{cases} \tag{68}$$

and

$$\xi_{l,m} = \begin{cases} \tau_n, & s=0, \quad n=1,\cdots,N-1, \\ F_n(t^{\tau}_{M,s}, \sum\limits_{\varsigma=0}^{M} a_{1,\varsigma} T_{\tau,\varsigma}(t^{\tau}_{M,s}), \cdots, \sum\limits_{\varsigma=0}^{M} a_{N-1,\varsigma} T_{\tau,\varsigma}(t^{\tau}_{M,s})), \, n=1,\cdots,N-1, & s=1,\cdots,M. \end{cases} \tag{69}$$

After the coefficents $a_{i,j}$ are determined, it is very easy to compute the approximate solution $u_{N,M}(x,t)$ at any value of (x,t) in the given domain from the following equation

$$\phi_{N,M}(x,t) = \sum_{k=0}^{M} \sum_{s=0}^{N} \sum_{r=0}^{N} a_{s,k} \Big(\frac{P_{\ell,r}(x^{\ell}_{N,s}) \varpi^{\ell}_{N,s}}{h^{\ell}_{r}} \Big) P_{\ell,r}(x) T_{\tau,k}(t). \tag{70}$$

4. Numerical results

To show the effectiveness of our scheme and the accuracy of the results, we adapt the above analysis into some numerical examples. We also give a comparison between the obtained results with others of those which used novel implicit finite difference method (Chen et al. (2011)).

Example 4.1.

We list the one-dimensional VO-FPEs

$$\frac{\partial \phi(x,t)}{\partial t} = \frac{\partial}{\partial x} \Big(\frac{\Gamma(3-\nu(x))}{2} \frac{\partial^{\nu(x)} \phi(x,t)}{\partial x^{\nu(x)}} \Big) - e^{-t} \left(x^2 - x^{1-\nu(x)} \left(\nu(x) - 2 + x \ln(x) \nu'(x) \right) \right), \quad (x,t) \in [0,1] \times [0,1], \tag{71}$$

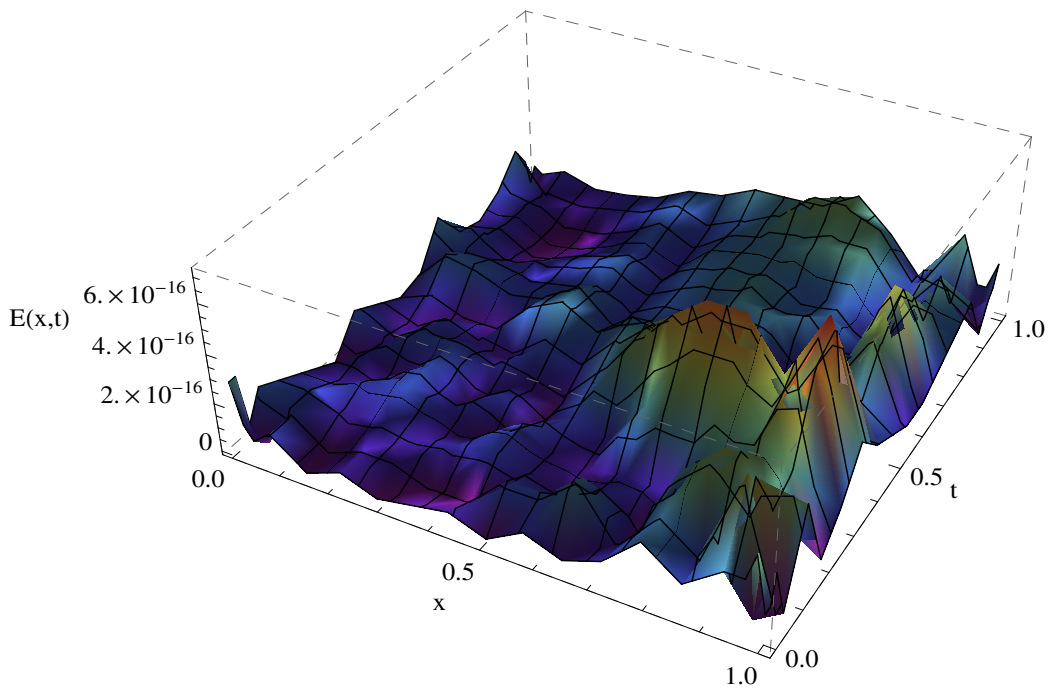

Figure 1. The absolute error of the problem (71), where $N = M = 16$

related to the initial-boundary conditions

$$\begin{aligned} \phi(x,0) =& x^2, & x \in [0,1], \\ \phi(0,t) =& 0, \quad \phi(1,t) = e^{-t}, & t \in [0,1]. \end{aligned} \tag{72}$$

The exact solution of (71) is $\phi(x,t) = e^{-t}x^2$. The maximum absolute error (M_E) of equation (71) are listed in Table 1 with several values of N and M.

Table 1. M_E for problem (71)

$N = M$	4	8	12	16
M_E	6.59×10^{-6}	1.50×10^{-11}	1.05×10^{-15}	5.27×10^{-16}

Three-dimensional graph of the absolute error (AE) of problem (71) at $N = M = 16$ is displayed in Figure 1. In addition, the curve of the AE of problem (71) at $x = 0.0$ is displayed in Figure 2. Moreover in Figure 3, we sketch the logarithmic graphs of M_E ($log_{10}M_E$) at different values of $N(N = M = 4, 6, \cdots, 16)$, which ensure an accurate approximation and reasonable convergence rates of the proposed method.

Example 4.2.

Here, we present the one-dimensional VO-FPEs (Chen et al. (2011))

$$\frac{\partial \phi(x,t)}{\partial t} = \frac{\partial}{\partial x}\Big((30 - x^2)\frac{\partial^{0.5}\phi(x,t)}{\partial x^{0.5}}\Big) - e^{-t}x^2 - \frac{e^{-t}\Gamma(3)}{\Gamma(2.5)}(45\,x^{0.5} - 3.5\,x^{2.5}), \quad x \in [0,1], \quad t > 0, \tag{73}$$

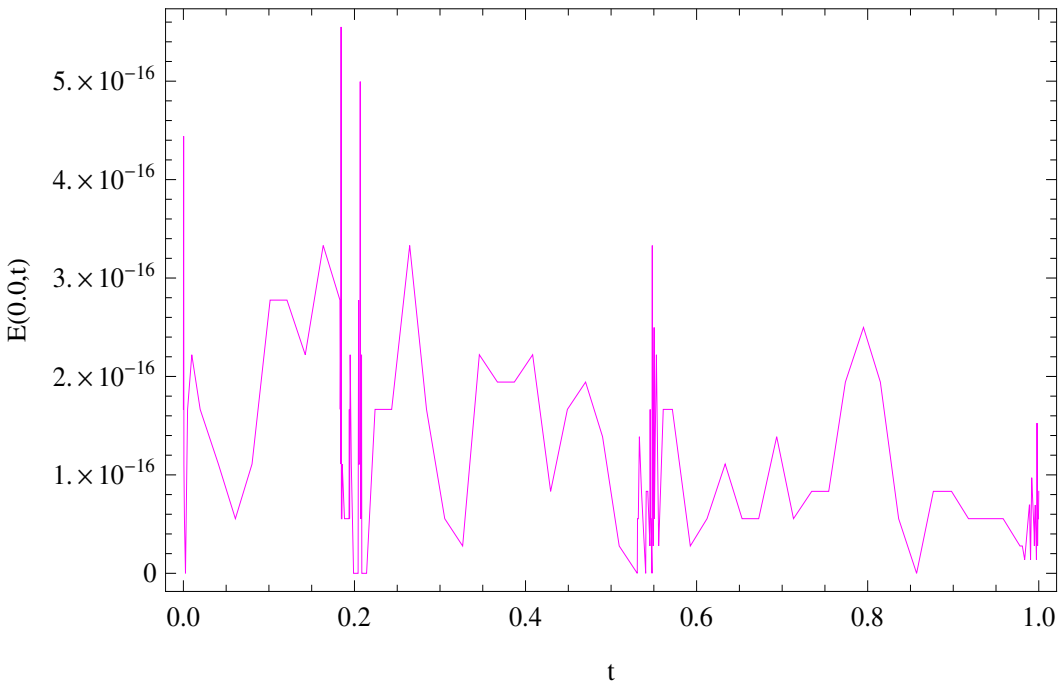

Figure 2. t-direction curve of the AE related to problem (71), where $N = M = 16$

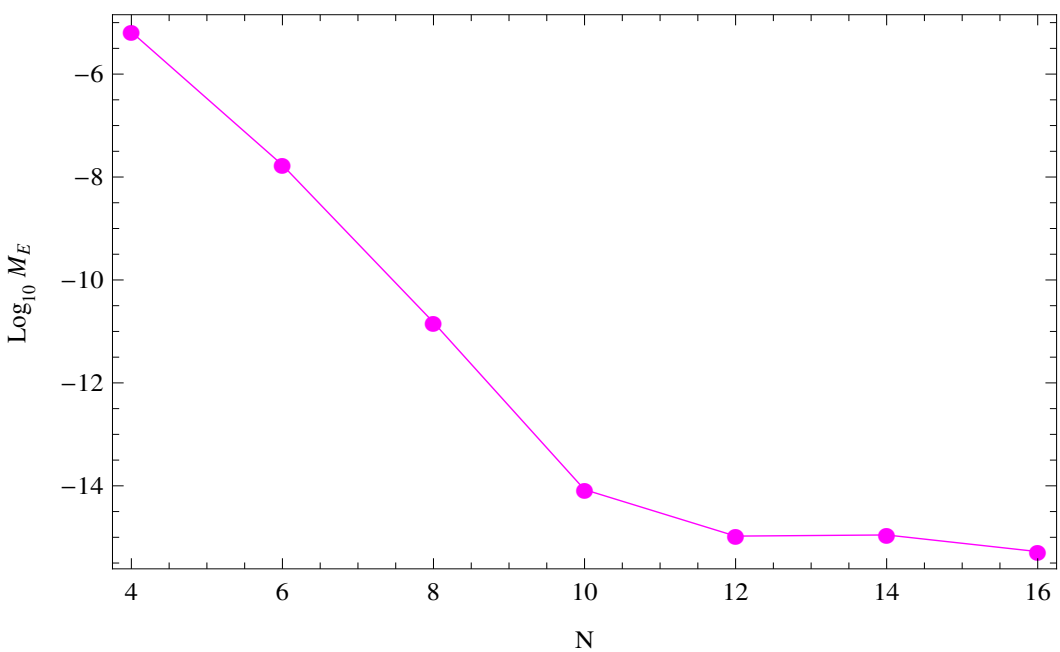

Figure 3. Convergence of problem (71)

related to the initial-boundary conditions

$$\begin{aligned} \phi(x,0) =& x^2, & x \in [0,1], \\ \phi(0,t) =& 0, \quad \phi(1,t) = e^{-t}, & t > 0. \end{aligned} \tag{74}$$

The exact solution of (73) is $\phi(x,t) = e^{-t}x^2$.

A comparison between the M_E achieved by using the proposed method and novel implicit finite difference method (Chen et al. (2011)) is introduced in Table 2.

Table 2. M_E for problem (73) for $T_{end} = 10$

Our Method at $N = M =$				Method (Chen et al. (2011)) at $\Delta t = \Delta x =$	
8	12	16	20	$\frac{1}{40}$	$\frac{1}{80}$
3.56297×10^{-4}	6.71811×10^{-7}	4.36289×10^{-10}	1.2151×10^{-13}	1.5187×10^{-7}	7.6818×10^{-8}

Example 4.3.

Now, we deal with the one-dimensional VO-FPEs subject to non-local condition

$$\frac{\partial \phi(x,t)}{\partial t} = \frac{\partial}{\partial x}\Big(\frac{\Gamma(3-\nu(x))}{2}\frac{\partial^{\nu(x)}\phi(x,t)}{\partial x^{\nu(x)}}\Big) - e^{-t}\left(x^2 - x^{1-\nu(x)}\left(\nu(x) - 2 + x\ln(x)\nu'(x)\right)\right), \quad (x,t) \in [0,1]\times[0,1], \tag{75}$$

subject to the initial-boundary conditions

$$\begin{aligned} \phi(x,0) =& x^2, \quad x \in [0,1], \\ \phi(0,t) =& 0, \quad t \in [0,1], \end{aligned} \tag{76}$$

the non-local integral condition

$$\int_0^1 \phi(x,t)dx = \frac{e^{-t}}{3}, \quad t \in [0,1]. \tag{77}$$

The exact solution of (75) is $\phi(x,t) = e^{-t}x^2$.

Using different choices of nodes, M_E of the problem (75) are declared in Table 3. Figure 4 compares the graphs of numerical and exact solutions of problem (75) for three values of t. In addition, the curve of the AE of problem (75) at $x = 0.0$ is plotted in Figure 5

Table 3. M_E for problem (75)

N	4	6	8	10	12
M_E	8.0274×10^{-5}	3.0037×10^{-7}	5.9460×10^{-10}	4.6385×10^{-13}	9.1038×10^{-15}

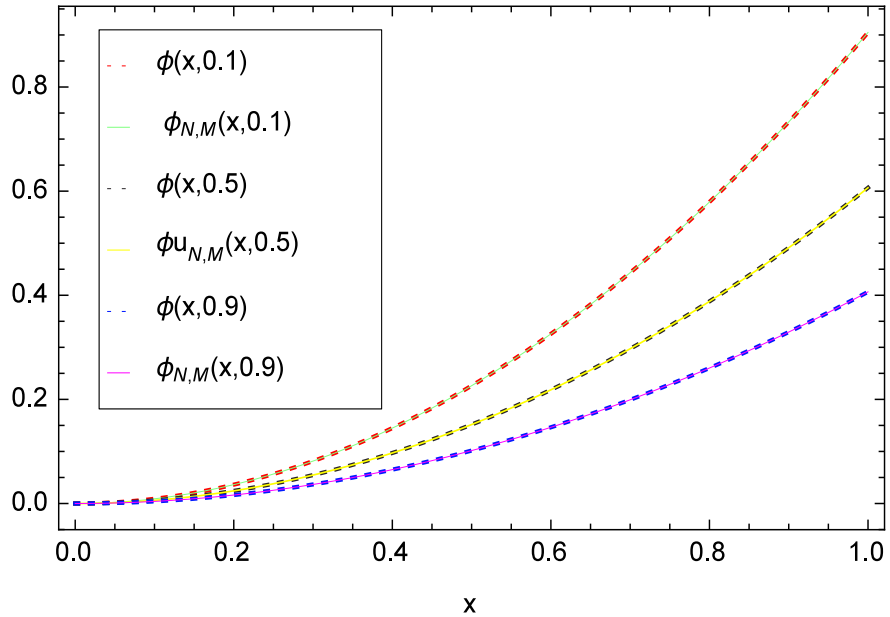

Figure 4. x-direction curves of exact and approximate solutions of problem (75), where $N = M = 12$

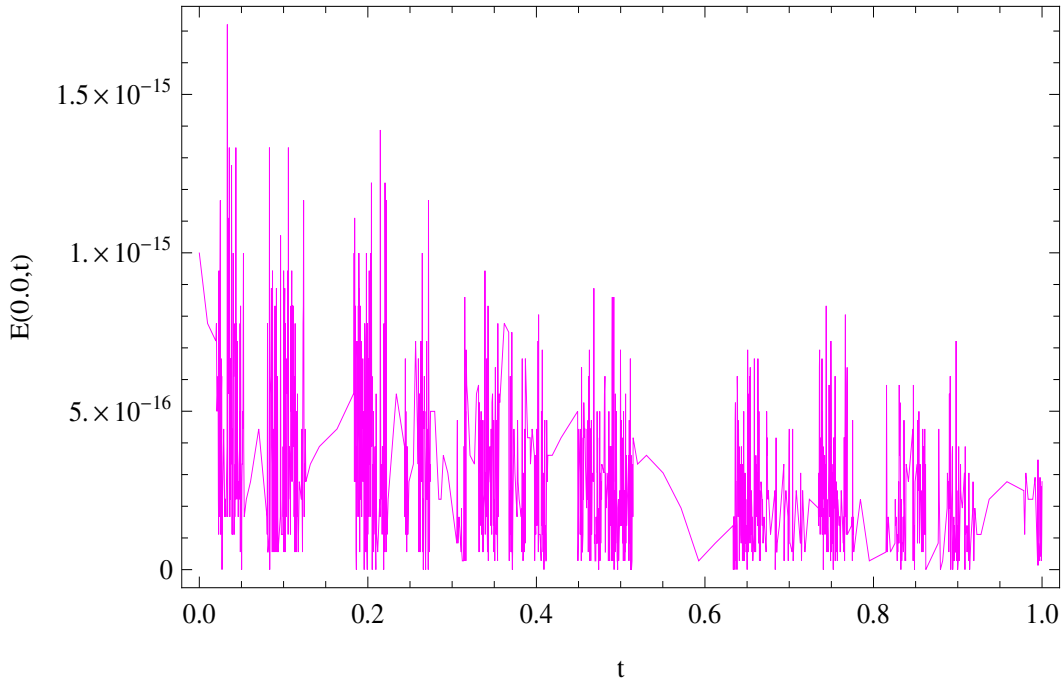

Figure 5. t-direction curve of AE related to the problem (75), where $N = M = 12$

5. Conclusion

For one-dimensional space, we introduce an accurate and efficient numerical algorithm related to SL-GL-C and SC-GR-C methods to get the numerical solutions for VO-FPEs. According to the

numerical results obtained above, we can conclude the high accuracy of our technique. The underline problem subject to the Dirichlet or non-local boundary condition is presented and compared with the results in literature, which reveals wonderful results.

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