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# Introduce Gâteaux and Frêchet Derivatives in Riesz Spaces

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## Abstract

In this paper, the Gâteaux and Frêchet differentiations of functions on Riesz space are introduced without topological structure. Thus, we aim to study Gâteaux and Frêchet differentiability functions in vector lattice by developing topology-free techniques, and also, we give some relations with other kinds of operators.

**Keywords:** Gâteaux differentiable; Frêchet differentiable; Riesz space; Order bounded operator

**MSC 2010 No.:** 49J50, 58C20, 46A40, 46B20

## 1. Introduction

The concept of Gâteaux and Frêchet differentiability provides natural and efficient tools in the theory of applied mathematics and physics. The Gâteaux differentiable are defined on Banach spaces and used norm convergences to calculate derivatives of functions. Gâteaux derivative is a generalization of the concept of directional derivative in differential calculus. Also, Frêchet derivative is a special case of Gâteaux differentiability. They are often used to formalize the functional derivative commonly used in Physics, particularly Quantum field theory. Riesz space is another concept of functional analysis, and it was introduced by F. Riesz (see Riesz (1928)). Riesz space is an ordered vector space, and it has many applications in measure theory, operator theory and optimization.

This current paper aims to combine the notions of Gâteaux, Frêchet differentiabilities, and vector lattices. It is known that the order convergence in vector lattices is not topological in general. Nevertheless, via order convergence, the order continuous operator can be defined in vector lattices without using any topological structure. Thus, our purpose is to introduce and study Gâteaux and Frêchet differentiabilities functions in vector lattices by developing topology-free techniques. Moreover, we show the linear properties of the order differentiabilities (see Theorem 3.1), and also, we introduce a relation between the order convergence and the order differentiabilities (see Theorem 3.2).

## 2. Basic Properties of Vector Lattices

In this section, we present simple and basic notions from elementary Riesz spaces. First of all, we recall some of the basic concepts related to vector lattice. We refer the reader for more information on vector lattices (Abramovich and Aliprantis (2002), Abramovich and Aliprantis (2006), Luxemburg and Zaanen (1971), Zaanen (1983)), and for some applications (Aydın (2018), Aydın (2020a), Aydın (2020b), Aydın et al. (2018), Aydın et al. (2019), Hussain et al. (2019), Zada et al. (2018)).

**Definition 2.1.**

Let "$\leq$" be an order relation (i.e., it is antisymmetric, reflexive and transitive) on a real-valued vector space $E$. Then, $E$ is called an *ordered vector space* if it satisfies the following properties:
(1) if $x \leq y$ for any $x, y \in E$ then, $x + z \leq y + z$ for every $z \in E$,
(2) if $x \leq y$ for any $x, y \in E$ then, $\lambda x \leq \lambda y$ for every $\lambda \in \mathbb{R}_+$.

If, for any two vectors $x, y \in E$, the infimum and the supremum

$$x \wedge y = \inf\{x, y\} \text{ and } x \vee y = \sup\{x, y\},$$

exist in $E$ then, the ordered vector space $E$ is called *vector lattice* or *Riesz space*. A vector lattice $E$ is called *Archimedean* whenever $\frac{1}{n}x \downarrow 0$ holds in $E$ for each $x \in E_+$. In this article, unless otherwise, all vector lattices are assumed to be real and Archimedean. For an element $x$ in a vector lattice $E$, the *positive part*, *the negative*, and the *absolute value* of $x$ are following, respectively,

$$x^+ := x \vee 0, \ \ x^- := (-x) \vee 0 \text{ and } |x| := x \vee (-x).$$

A vector lattice is called *order complete* if every nonempty bounded above subset has a supremum (or, every nonempty bounded below subset has an infimum). A given partially ordered set $I$ is called *directed upward* (or, *directed downward*) if, for each $a_1, a_2 \in I$, there is another $a \in I$ such that $a \geq a_1$ and $a \geq a_2$ (or, $a \leq a_1$ and $a \leq a_2$). A function from a directed set $I$ into a set $E$ is called *net* in $E$. The meaning of $(x_\alpha)_{\alpha \in I} \downarrow x$ in a vector lattice is both $(x_\alpha)_{\alpha \in I} \downarrow$ and $\inf(x_\alpha)_{\alpha \in I} = x$. Also, a net $(x_\alpha)_{\alpha \in A}$ in a vector lattice $E$ is called *order convergent* to $x \in E$, if

(1) there exists another net $(y_\beta)_{\beta \in B}$ satisfying $y_\beta \downarrow 0$,
(2) for any $\beta \in B$, there exists $\alpha_\beta \in A$ such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_\beta$.

In this case, we write $x_\alpha \xrightarrow{o} x$. On the other hand, the net $(x_\alpha)_{\alpha \in A}$ is called *relatively uniform convergent* to $x \in E$ ($x_\alpha \xrightarrow{ru} x$, for short) if there exists $u \in E_+$, such that, for any $n \in \mathbb{N}$, there exists $\alpha_n$ such that $|x_\alpha - x| \leq \frac{1}{n}u$ for all $\alpha \geq \alpha_n$. For Archimedean Riesz spaces, the $ru$-convergence implies the order convergence; see, for example, Theorem 16.2$(i)$ (Luxemburg and Zaanen (1971)).

Recall that a subset $B$ in a topological vector space is called *topological bounded* if, for every zero neighborhood $U$, there is a positive scalar $\lambda > 0$ such that $B \subseteq \lambda U$. On the other hand, an *order interval* defined by $[-a, a] = \{x \in E : -a \leq x \leq a\}$ for a positive element $a$ (i.e., $0 \leq a$) in a vector lattice $E$. Moreover, if any subset in $E$ is included in an order interval then, it is called *order bounded*. Moreover, a map $T$ between two vector spaces is called *operator* if it is linear. An operator $T : E \to F$ between two ordered vector spaces is called *positive* if $T(x) \geq 0$ for all $x \geq 0$. An *order bounded operator* between vector lattices sends order bounded subsets to order bounded subsets. Moreover, an operator between vector lattices is called *order continuous* if $x_\alpha \xrightarrow{o} 0$ implies $T(x_\alpha) \xrightarrow{o} 0$.

Recall that a norm $\|\cdot\|$ on a vector lattice is called a *lattice norm* if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. In addition, a lattice norm on a vector lattice is said to be *order continuous* whenever $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \downarrow 0$. A vector lattice with a lattice norm is called *normed vector lattice* or *normed Riesz space*. Moreover, a normed vector lattice referred to as a *Banach lattice* whenever it is norm complete. Examples of Banach lattices with order continuous norms are provided by the classical $c_0$, $\ell_p$ and $L_p(\mu)$-spaces, where $1 \leq p < \infty$.

## 3. The Order Differentiability

We begin with standard definitions of Gâteaux and Frêchet derivative functions from an open subset in a Banach space into another Banach space. A function $f : U \subseteq X \to Y$ is said to be *Gâteaux differentiable* at $x_0 \in U$ if there is a bounded linear operator $T : X \to Y$ such that

$$T_{x_0}(u) = \lim_{t \to 0} \frac{\big(f(x_0 + tu) - f(x_0)\big)}{t} \tag{1}$$

for every $u \in X$. Then, $T$ is said to be *the Gâteaux derivative* of $f$ at $x_0$, and also, it is denoted by $D_f(x_0)$; for much more detail, see Benyamini and Lindenstrauss (2000), Hale (2009), and Lasalle (1968). If, for any fixed $u \in X$, the limit

$$f'(u, x_0) = \lim_{t \to 0} \frac{\big(f(x_0 + tu) - f(x_0)\big)}{t},$$

exists, then, it said that $f$ has a directional derivative at $x_0$ in the direction $u$. Hence, $f$ is Gâteaux differentiable at $x_0$ if and only if all directional derivatives exist and they form a bounded linear operator on $u$. Moreover, if the limit $(1)$ is *uniform* then, the operator $T$ is called the Frêchet derivative of $f$ at $x_0$. The following notion is motivated by the above definitions, and it is developed by topology-free techniques.

**Definition 3.1.**

Let $E$ and $F$ be two vector lattices and $[-a, a]$ be an order interval for a positive element $a \in E_+$. A function $f$ from the interval $[-a, a]$ to $F$ is said to be *order Gâteaux differentiable function* at a vector $e \in [-a, a]$ if there exists an order bounded operator $T : E \to F$ such that

$$z_\alpha \bigg( f(e + t_\alpha u) - f(e) \bigg) \xrightarrow{o} T(u), \tag{2}$$

for all $u \in E$ whenever $t_\alpha \to 0$ in $\mathbb{R}$, $z_\alpha = 1/t_\alpha$, and $t_\alpha \neq 0$ for all scalar $\alpha$.

Therefore, $T$ is said to be *the order Gâteaux derivative* of $f$ at $e$ and it is abbreviated as $OD_{G_f}(e)$. If the order Gâteaux derivative exists then, it is unique because the order limit is unique whenever it exists.

If we take the convergence in $(2)$ as a relatively uniform converge instead of the order convergence then, the function $f$ is said to be *order Frĉchet differentiable function* at a vector $e \in [-a, a]$. Thus, $T$ is said to be *the order Frêchet derivative* of $f$ at $e$, and also, it is abbreviated as $OD_{F_f}(e)$.

Since the $ru$-convergence implies the order convergence in Archimedean Riesz spaces, the order Gâteaux differentiable implies the order Frêchet differentiable whenever $F$ is taken as an Archimedean Riesz spaces in Definition 3.1.

**Remark 3.1.**

The norm convergence does not necessarily imply the order convergence, and similarly, it also does not imply relatively uniform convergence. By the way, the Gâteaux or Frêchet differentiables do not imply the order Gâteaux or the order Frêchet differentiable in general; see Example 100.3. (Zaanen (1983)).

To show that the Gâteaux or Frêchet differentiable implies the order version of them is not easy. Consider an atomic order continuous Banach lattice. Then, the norm convergence implies order convergence; see, for example, Lemma 5.1. (Deng at el. (2017)). However, the norm boundedness of an operator does not imply the order boundedness of it, and so, the existence of Gâteaux or Frêchet derivative of a function does not imply the existence of the order version of it, yet.

For the partial converse case of Remark 3.1, we give the following observations.

**Example 3.1.**

On order continuous Banach lattices, the order Gâteaux differentiability implies the Gâteaux differentiability because every order bounded operator is norm bounded; see for example Theorem 1.31 (Abramovich and Aliprantis (2002)).

**Example 3.2.**

Take $E = F = \mathbb{R}$. If we choose $u = 1$ then, the order Gâteaux differentiability of a function implies the classical differentiability on $\mathbb{R}$ because the order convergence on $\mathbb{R}$ is classical topological convergence on it.

In the following work, we prove that the order Gâteaux differentiable is linear.

**Theorem 3.1.**

Let $E$ and $F$ be vector lattices, and $f, g : [-a, a] \subseteq E \to F$ be order Gâteaux differentiable functions. Then, $\lambda f + \beta g$ is also an order Gâteaux differentiable for all $\lambda, \beta \in \mathbb{R}$.

***Proof:***

Assume $f$ and $g$ are order Gâteaux differentiable functions at $e \in [-a, a]$. Then, there exist order bounded operators $T_1,\ T_2$ from $E$ to $F$ such that

$$z_\alpha\big(f(e + t_\alpha u) - f(e)\big) \xrightarrow{o} T_1(u),$$

and also,

$$z_\alpha\big(g(e + t_\alpha u) - g(e)\big) \xrightarrow{o} T_2(u),$$

holds for each $u \in E$ whenever $t_\alpha \to 0$, $z_\alpha = 1/t_\alpha$, and $t_\alpha \neq 0$ for all scalar $\alpha$. Fix $u \in E$ and $\alpha, \beta \in \mathbb{R}$. Take arbitrary net $(t_\alpha)$ that satisfies the above conditions.

Now, it follows from the following inequality

$$\begin{aligned} \big|z_\alpha\big[(\lambda f + \beta g)\ (e + t_\alpha u) - (\lambda f + \beta g)(e)\big] - (T_1 - T_2)(u)\big| \\ \leq \big|z_\alpha\big[\lambda f(e + t_\alpha u) - \lambda f(e)\big] - T_1(u)\big| \\ + \big|z_\alpha\big[\beta g(e + t_\alpha u) - \beta g(e)\big] - T_2(u)\big|, \end{aligned}$$

that $\lambda f + \beta g$ is order Gâteaux differentiable, and $OD_{G_{\lambda f + \beta g}}(e) = OD_{G_{\lambda f}}(e) + OD_{G_{\beta g}}(e)$. ■

Similar to Theorem 3.1, one can show the linearity for the order Frêchet differentiability.

**Remark 3.2.**

We can observe that if a function $g$ is order Gâteaux differentiable at $e$, and another function $f$ is order continuous then, $f \circ g$ is order Gâteaux differentiable at $e$. Similarly, if a function $g$ is order Frêchet differentiable at $e$, and $f$ is relatively uniform continuous then, $f \circ g$ is order Frêchet differentiable at $e$.

**Definition 3.2.**

Let $f : E \to F$ be a map between Riesz spaces. Then, $f$ is called *order Lipschits* if there exists a positive scalar $\lambda$ such that $|f(x) - f(y)| \leq \lambda|x - y|$ for all $x, y \in E$.

Moreover, $L_o(f) := \inf\{\lambda : \lambda \geq 0\ and\ |f(x) - f(y)| \leq \lambda|x - y|\ for\ all\ x, y \in E\}$ is called the *order Lipschitz constant* of $f$. On the other hand, it can be observed that the order Lipschitzness implies the Lipschitz on normed vector lattices. It is well known that the differentiability of $f$ at $g(e)$ is not sufficient for the differentiability of $f \circ g$, but it is sufficient if $f$ is Lipschitz; see page 84 (Benyamini and Lindenstrauss (2000)). Thus, we can obtain that the order Lipschitzness of $f$ is enough for the differentiability of $f \circ g$.

**Theorem 3.2.**

If a function $f : [-a, a] \subseteq E \to F$ is order Gâteaux differentiable at $e$ in an order interval $[-a, a]$ then, $f$ is also order Frêchet differentiable at $e$ whenever there exists a sequence $0 \leq \lambda \uparrow < \infty$ in $\mathbb{R}$ such that $\lambda_n x_n \xrightarrow{o} 0$ for every sequence $(x_n) \xrightarrow{o} 0$ in $F$.

***Proof:***

Suppose that $f$ is order Gâteaux differentiable function at $e$. Thus, there is an order bounded operator $T : E \to F$ such that $z_\alpha\big(f(e + t_\alpha u) - f(e)\big) \xrightarrow{ru} T(u)$ for all $u \in E$ whenever $t_\alpha \to 0$ in $\mathbb{R}$, $z_\alpha = 1/t_\alpha$, and $t_\alpha \neq 0$ for all scalar $\alpha$. Now, by applying Theorem 16.3 (Luxemburg and Zaanen (1971)), we can obtain $z_\alpha\big(f(e + t_\alpha u) - f(e)\big) \xrightarrow{o} T(u)$ for all $u \in E$. Therefore, we obtain the desired result. ∎

## 4. Conclusion

In general, the Gâteaux and Frêchet differentiability are defined on Banach spaces with respect to the norm topology. However, in this paper, we use the order relation and order convergence on vector lattice to introduce both Gâteaux and Frêchet differentiability functions in vector lattice by developing topology-free techniques under the names order Gâteaux and order Frêchet differentiabilities. Also, we get some relationt with order and standard differentiabilities.

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