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Exact Solutions of Two Nonlinear Space-time Fractional Differential Equations by Application of Exp-function Method

[1]Elahe Miri Eskandari and [2]Nasir Taghizadeh

Department of Mathematics
Faculty of Mathematical Sciences
University of Guilan
P.O.Box 41335-19141
Rasht, Iran
[1]elahe-eskandari@phd.guilan.ac.ir; [2]taghizadeh@guilan.ac.ir

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Abstract

In this paper, we discuss on the exact solutions of the nonlinear space-time fractional Burger-like equation and also the nonlinear fractional fifth-order Sawada-Kotera equation with the exp-function method. We use the functional derivatives in the sense of Riemann-Jumarie derivative and fractional convenient variable transformation in this study. Further, we obtain some exact analytical solutions including hyperbolic function.

Keywords: Exp-function method; Fractional differential equation; The Burger-like equation; The Sawada-Kotera Equation

MSC 2010 No.: 34A08, 35R11, 83C15

1. Introduction

Obvious, fractional differential equations (FDEs) have been the focus of many studies due to their frequent appearance in several applications in physics, biology, engineering, signal processing, systems identification, control theory, finance, fractional dynamics, and other different sciences (see Kilbas et al. (2006), Mille and Ross (1993), and Podlubny (1999)). The exact solution of the fractional differential equations have received tremendous attention of great, several investigators have successfully used reliable and algebraic methods for this purpose (see Taghizadeh et al. (2015, 2016)). However, there are several methods to obtain exact solutions of fractional differential Burger-like equation and fractional differential Sawada-Kotera equation; for example: the Sub-equation method, see Wang et al. (2017), the (G'/G) method, see Cerdid et al. (2019), the F-expansion method, see Inan et al.(2017), the $\exp(-\phi(\xi))$ method see Ali et al. (2016) and other methods, see Bulut et al. (2013), but we apply the exp-function method to nonlinear FDES's, space-time fractional Burger-like equation, see Cerdid et al. (2019) and Inan et al. (2017) and time-fractional fifth-order Sawada-Kotera equation, see Ali et al. (2016) and Guner et al. (2017).

The organization of the paper follows in the second and third sections of the paper we define the Riemann-Jumarie Derivative and we describe the exp-function method respectively. In Section 4 and Section 5 application of the exp-function method for Burger-like equation and Sawada-Kotera equation respectively. In the end, the general conclusion is given in Section 6.

2. The Riemann-Jumarie Derivative

Assume that $f : R \longrightarrow R, x \longrightarrow f(x)$ denotes a continuous (but not nesseciary differentiable) function, then Riemann-Jumarie derivative (see Jumarie(2006, 2009)),

$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, & \text{if} \qquad 0<\alpha\leq 1, \\ (f^{(n)}(t))^{(\alpha-n)}, & \text{if} \qquad n\leq\alpha<n+1, n\geq 1. \end{cases} \tag{1}$$

Some important properties for the Riemann-Jumarie derivative of order α are listed below,

$$D_t^{\alpha} t^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}, \tag{2}$$

$$D_t^{\alpha}(f(t)g(t)) = g(t)D_t^{\alpha} f(t) + f(t)D_t^{\alpha} g(t), \tag{3}$$

$$D_t^{\alpha} f[g(t)] = f_g^{'}[g(t)]D_t^{\alpha} g(t) = D_t^{\alpha} f[g(t)](g^{'}(t))^{\alpha}. \tag{4}$$

We consider the fractional partial differential equation, with independent variable $t, x_1, x_2, ..., x_n$ and dependent variable u :

$$P(u, D_t^{\alpha} u, D_{x_1}^{\alpha} u, ..., D_t^{2\alpha} U, ...) = 0 \qquad 0<\alpha\leq 1, t>0. \tag{5}$$

By using the fractional variable transformation

$$u(t, x_1, ..., x_n) = u(\xi) \quad \text{and} \quad \xi = \frac{ct^{\alpha}}{\Gamma(1+\alpha)} + \frac{k_1 x_1^{\alpha}}{\Gamma(1+\alpha)} + ... + \frac{k_n x_n^{\alpha}}{\Gamma(1+\alpha)}, \tag{6}$$

where c and $k_i, (i=1, ..., n)$ are nonzero arbitrary constants. The fractional differential equation (5) reduced to a nonlinear ordinary differential equation (ODE)

$$F(u, u', u'', ...) = 0, \tag{7}$$

where the prime denotes the derivation with respect to ξ and $u' = \frac{du}{d\xi}$.

3. The Exp-function method

In the exp-function method, we assume that the solution is a function,

$$u(\xi) = \frac{\sum_{p=-m_1}^{m_2} a_p \exp[p\xi]}{\sum_{q=-n_1}^{n_2} a_q \exp[q\xi]}, \tag{8}$$

where m_1, m_2, n_1 and n_2 are unknown positive integers that will be determined by using balance method, for constructing the relations between m_2, m_1, n_2 and n_1, first we consider the highest degree and the highest derivative order of function $u(\xi)$ in the nonlinear (ODE) (7). For relation between m_2 and n_2 by creating a balance between the first sentence of the highest degree and the first sentence of the highest derivative order. Similarly, the relation between m_1 and n_1 is obtained by balancing the last sentence of the highest degree and the last sentence of the highest derivative order. Ebaid (2012) proved in Theorem 1 and Theorem 2 that $m_2 = n_2$ and $m_1 = n_1$ are the only relations that can be obtained by applying the balancing method in the nonlinear (ODE) (7). In the exp-function method, additional calculations of balancing the highest order linear term with the highest order nonlinear term are not longer required in future because we only use their first and last sentences. Hence, the method becomes more straightforward.

4. The space-time fractional Burger-like equation

Burger's equation is related to applications in acoustic phenomena and has been utilized to model turbulence and certain steady-state viscous flows. We consider conformable space-time fractional Burger-like equation (see Cerdid et al. (2019) and Inan et al. (2017)),

$$D_t^{\alpha} u + D_x^{\alpha} u + u D_x^{\alpha} u + \frac{1}{2} D_x^{2\alpha} u = 0, \qquad t > 0, 0 < \alpha \leq 1, \tag{9}$$

where α is a parameter that describes the order of the fractional time derivative. By using variable transformations

$$u(x,t) = u(\xi), \qquad \text{and} \qquad \xi = \frac{ct^{\alpha}}{\Gamma(1+\alpha)} + \frac{kx^{\alpha}}{\Gamma(1+\alpha)}, \tag{10}$$

where c and k are constants, Equation (9) is reduced to an ordinary defferential equation

$$(c+k)u' + kuu' + \frac{k^2}{2} u'' = 0, \tag{11}$$

where $u' = \frac{du}{d\xi}$ .Integrating equation (11) with respect to ξ yields

$$(c+k)u + \frac{1}{2} k u^2 + \frac{k^2}{2} u' = 0. \tag{12}$$

By using Equation (8) and Equation (12), we have

$$u'(\xi) = \frac{A \exp[(n_2+m_2)\xi] + ... + B \exp[-(n_1+m_1)\xi]}{C \exp[2n_2\xi] + ... + D \exp[-2n_1\xi]}, \tag{13}$$

$$u^2(\xi) = \frac{E \exp[2m_2\xi] + ... + F \exp[-2m_1\xi]}{G \exp[2n_2\xi] + ... + H \exp[-2n_1\xi]}, \tag{14}$$

where A, B, C, D, E, F, G and H are coefficients determined by $a_{m_2}, a_{-m_1}, b_{n_2}$ and b_{-n_1}.

By balancing the highest order Exp-function in $u^{'}$ and u^2 in Equation (12), we have $n_2+m_2 = 2m_2$, which implies $m_2 = n_2$. Similarly, balancing the lowest order exp-function in $u^{'}$ and u^2, we have $n_1 + m_1 = 2m_1$, which implies $m_1 = n_1$. For simplicity, we set $m_2 = n_2 = 1$ and $m_1 = n_1 = 1$ so Equation (8) reduces to

$$u(\xi) = \frac{a_1 \exp[\xi] + a_0 + a_{-1} \exp[-\xi]}{b_1 \exp[\xi] + b_0 + b_{-1} \exp[-\xi]}. \tag{15}$$

Substitating Equation (15) into Equation (12) and the help of Maple, we have

$$\frac{1}{A}(R_1 \exp[2\xi] + R_2 \exp[\xi] + R_3 + R_4 \exp[-\xi] + R_5 \exp[-2\xi]) = 0, \tag{16}$$

where

$$\begin{aligned}
A &= (b_1 \exp[\xi] + b_0 + b_{-1} \exp[-\xi])^2,\\
R_1 &= (c+k)a_1b_1 + \frac{1}{2}ka_1^2,\\
R_2 &= (c+k)[a_0b_1 + a_1b_0] + ka_1a_0 + \frac{1}{2}k^2a_1b_0 - \frac{1}{2}k^2a_0b_1,\\
R_3 &= (c+k)[a_0b_0 + 2a_1b_{-1} + a_{-1}b_1] + k[a_1a_{-1} + a_{-1}b_1 + \frac{1}{2}a_0^2] + k^2[a_1b_{-1} - a_{-1}b_1],\\
R_4 &= (c+k)[a_0b_{-1} + a_{-1}b_0] + kb_{-1}a_0 - \frac{1}{2}k^2a_{-1}b_0 + \frac{1}{2}k^2a_0b_{-1},\\
R_5 &= (c+k)a_{-1}b_{-1} + \frac{1}{2}ka_{-1}^2.
\end{aligned} \tag{17}$$

Solving this system of algebraic equation by using symbolic computation, we obtain the following results.

If $a_1 = 0$, $a_0 = 0$, $a_{-1} = -2kb_{-1}$, $b_1 = b_1$, $b_0 = 0$, $b_{-1} = b_{-1}$, $c = k^2 - k$ and $k = k$.

If we set $b_1 = \frac{1}{2}$, $b_{-1} = -\frac{1}{2}$ and $k = 2$, then we have $a_{-1} = -2$ and $c = 2$, we obtain the following hyperbolic function solitary solution,

$$u(x,t) = \frac{2\exp[-2(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{t^\alpha}{\Gamma(1+\alpha)})]}{\sinh(2(\frac{-x^\alpha}{\Gamma(1+\alpha)} + \frac{t^\alpha}{\Gamma(1+\alpha)}))}. \tag{18}$$

5. The space-time fractional fifth-order Sawada-Kotera equation

We consider the space-time fractional fifth-order Sawada-Kotera equation (see Ali et al. (2016) and Guner et al. (2017)),

$$D_t^\alpha u + D_x^{5\alpha}u + 45U^2D_x^\alpha u + 15(D_x^\alpha uD_x^{2\alpha}u + uD_x^{3\alpha}u) = 0, \qquad t > 0, 0 < \alpha \leq 1, \tag{19}$$

where α is a parameter describing the order of the fractional time-derivative. The function $u(x,t)$ is assumed to be a causal function of time. As in the previous section, we use variable transformations

$$u(x,t) = u(\xi), \qquad \text{and} \qquad \xi = \frac{ct^{\alpha}}{\Gamma(1+\alpha)} + \frac{kx^{\alpha}}{\Gamma(1+\alpha)}, \tag{20}$$

where c and k are constants, Equation (19) is reduced to an ordinary defferential equation

$$cu' + k^5u^{(5)} + 45ku^2u' + 15(k^3u'u'' + k^3uu^{(3)}) = 0, \tag{21}$$

where $u' = \frac{du}{d\xi}$. Integrating Equation (21) with respect to ξ yields.

$$cu + k^5u^{(4)} + 15ku^3 + 15k^3(uu'') = 0. \tag{22}$$

We have

$$(u(\xi))^{(4)} = \frac{I\exp[(15n_2 + m_2)\xi] + ... + J\exp[-(15n_1 + m_1)\xi]}{K\exp[16n_2\xi] + ... + L\exp[-16n_1\xi]}, \tag{23}$$

$$uu''(\xi) = \frac{M\exp[(3n_2 + 2m_2)\xi] + ... + N\exp[-(3n_1 + 2m_1)\xi]}{P\exp[5n_2\xi] + ... + Q\exp[-5n_1\xi]}, \tag{24}$$

where I, J, K, L, M, N, P and Q are coefficients determined by $a_{m_2}, a_{-m_1}, b_{n_2}$ and b_{-n_1}.

By the same approach as in before this section, balancing the terms $u^{(4)}$ and $uu^{''}$ in Equation (22) we have $m_2 = n_2, m_1 = n_1$. To simplicity, we set $m_2 = n_2 = 1$ and $m_1 = n_1 = 1$ so equation (8) degrades to

$$u(\xi) = \frac{a_1\exp[\xi] + a_0 + a_{-1}\exp[-\xi]}{b_1\exp[\xi] + b_0 + b_{-1}\exp[-\xi]}. \tag{25}$$

Substiuting Equation (25) into Equation (22), and by the help of Maple, we have

$$\begin{aligned} &\frac{1}{A}(R_1\exp[5\xi] + R_2\exp[4\xi] + R_3\exp[3\xi] + R_4\exp[2\xi] + R_5\exp[\xi] + R_6 \\ &\quad + R_7\exp[-\xi] + R_8\exp[-2\xi] + R_9\exp[-3\xi] + R_{10}\exp[-4\xi] + R_{11}\exp[-5\xi]) = 0, \end{aligned} \tag{26}$$

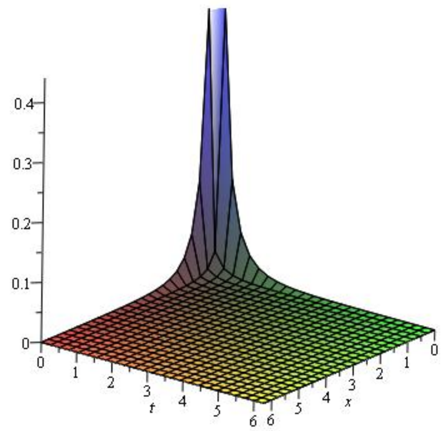
where

$$
\begin{aligned}
A &= (b_1 \exp[\xi] + b_0 + b_{-1} \exp[-\xi])^5, \\
R_1 &= ca_1 b_1^4 + 15ka_1^3 b_1^2, \\
R_2 &= 15k^3 a_1 a_0 b_1^3 + ca_0 b_1^4 + 4ca_1 b_1^3 b_0 - 15k^3 a_1^2 b_1^2 b_0 + 45ka_1^2 a_0 b_1^2 + k_5 a_0 b_1^4 - k^5 a_1 b_1^3 b_0 + 30ka_1^3 b_1 b_0, \\
R_3 &= 30ka_1^3 b_1 b_{-1} + 15ka_1^3 b_0^2 + ca_{-1} b_1^4 + 45ka_1 a_0^2 b_1^2 - 16k^5 a_1 b_1^3 b_{-1} + 15k^3 a_0^2 b_1^3 - 60k^3 a_1^2 b_1^2 b_{-1} \\
&\quad + 60k^3 a_1 a_{-1} b_1^3 + 16k^5 a_{-1} b_1^4 - 11k^5 a_0 b_1^3 b_0 + 4ca_0 b_1^3 b_0 + 90ka_1^2 a_0 b_1 b_0 + 6ca_1 b_1^2 b_0^2 + 45ka_1^2 a_{-1} b_1^2 \\
&\quad + 4ca_1 b_1^3 b_{-1} - 15k^3 a_1 a_0 b_1^2 b_0 + 11k^5 a_1 b_1^2 b_0^2, \\
R_4 &= -135k^3 a_1 a_0 b_1^2 b_{-1} + 77k^5 a_1 b_1^2 b_0 b_{-1} + 30ka_1^3 b_0 b_{-1} - 76k^5 a_0 b_1^3 b_{-1} + 4ca_1 b_1 b_0^3 + 45ka_1^2 a_0 b_0^2 \\
&\quad - 11k^5 a_1 b_1 b_0^3 + 15k^3 a_1^2 b_0^3 + 75k^3 a_0 a_{-1} b_1^3 + 6ca_0 b_1^2 b_0^2 - k^5 a_{-1} b_1^3 b_0 - 30k^3 a_1^2 b_1 b_{-1} b_0 \\
&\quad + 90k^3 a_1 a_{-1} b_1^2 b_0 + 12ca_1 b_1^2 b_0 b_{-1} + 11k^5 a_0 b_1^2 b_0^2 + 15ka_0^3 b_1^2 - 15k^3 a_1 a_0 b_1 b_0^2 + 4ca_{-1} b_1^3 b_0 \\
&\quad + 90ka_1^2 a_{-1} b_1 b_0 + 4ca_0 b_1^3 b_{-1} + 90ka_1^2 a_0 b_1 b_{-1} + 90ka_1 a_0 a_{-1} b_1^2 + 90ka_1 a_0^2 b_1 b_0, \\
R_5 &= 4ca_0 b_1 b_0^3 + k^5 a_1 b_0^4 - 60k^3 a_1 a_{-1} b_1^2 b_{-1} + 60k^3 a_1 a_{-1} b_1 b_0^2 + 105k^3 a_0 a_{-1} b_1^2 b_0 - 58k^5 a_1 b_1 b_0^2 b_{-1} \\
&\quad + 12ca_0 b_1^2 b_{-1} b_0 + 12ca_1 b_1 b_0^2 b_{-1} + 47k^5 a_0 b_1^2 b_{-1} b_0 + 90ka_1^2 a_0 b_0 b_{-1} - k^5 a_0 b_1 b_0^3 + 45ka_1 a_0^2 b_0^2 \\
&\quad + 30ka_0^3 b_1 b_0 - 15k^3 a_0^2 b_1 b_0^2 + 15k^3 a_0 a_1 b_0^3 + ca_1 b_0^4 + 15ka_1^3 b_{-1}^2 + 60k^3 a_{-1}^2 b_1^3 \\
&\quad + 180ka_1 a_0 a_{-1} b_1 b_0 - 150k^3 a_1 a_0 b_1 b_{-1} b_0 + 6ca_1 b_1^2 b_{-1}^2 + 4ca_{-1} b_1^3 b_{-1} + 6ca_{-1} b_1^2 b_0^2 \\
&\quad + 176k^5 a_1 b_1^2 b_{-1}^2 - 176k^5 a_{-1} b_1^3 b_{-1} + 11k^5 a_{-1} b_1^2 b_0^2 + 45ka_1^2 a_{-1} b_0^2 + 45ka_1 a_{-1}^2 b_1^2 + 45ka_0^2 a_{-1} b_1^2 \\
&\quad + 60k^3 a_1^2 b_0^2 b_{-1} - 75k^3 a_0^2 b_1^2 b_{-1} + 90ka_1^2 a_{-1} b_1 b_{-1} + 90ka_1 a_0^2 b_1 b_{-1}, \\
R_6 &= 30k^3 a_1 a_{-1} b_0^3 + 230k^5 a_0 b_1^2 b_{-1}^2 + 45ka_0 a_{-1}^2 b_1^2 + 105k^3 a_1^2 b_0 b_{-1}^2 + 30ka_0^3 b_1 b_{-1} + 90ka_1^2 a_{-1} b_0 b_{-1} \\
&\quad + 90ka_1 a_0^2 b_0 b_{-1} + 90ka_1 a_0 a_{-1} b_0^2 + 90ka_1 a_{-1}^2 b_1 b_0 + 90ka_0^2 a_{-1} b_1 b_0 - 75k^3 a_1 a_0 b_1 b_{-1}^2 \\
&\quad + 180ka_1 a_0 a_{-1} b_1 b_{-1} - 60k^3 a_1 a_{-1} b_1 b_{-1} b_0 - 115k^5 a_1 b_1 b_0 b_{-1}^2 - 115k^5 a_{-1} b_1^2 b_0 b_{-1} + 12ca_{-1} b_1^2 b_0 b_{-1} \\
&\quad + 12ca_1 b_1 b_0 b_{-1}^2 - 10k^5 a_0 b_1 b_{-1} b_0^2 + 4ca_1 b_0^3 b_{-1} - 120k^3 a_0^2 b_1 b_{-1} b_0 + 12ca_0 b_1 b_{-1} b_0^2 + 45ka_1^2 a_0 b_{-1}^2 \\
&\quad + 4ca_{-1} b_1 b_0^3 + 5k^5 a_{-1} b_1 b_0^3 + 5k^5 a_1 b_0^3 b_{-1} + 6ca_0 b_1^2 b_{-1}^2 + ca_0 b_0^4 + 15k^3 a_0^3 b_0^2 + 45k^3 a_1 a_0 b_{-1} b_0^2 \\
&\quad - 75k^3 a_0 a_{-1} b_1^2 b_{-1} + 45k^3 a_0 a_{-1} b_1 b_0^2 + 105k^3 a_{-1}^2 b_1^2 b_0, \\
R_7 &= -150k^3 a_0 a_{-1} b_1 b_{-1} b_0 + 180ka_1 a_0 a_{-1} b_0 b_{-1} + 4ca_0 b_{-1} b_0^3 - k^5 a_0 b_{-1} b_0^3 + 30ka_0^3 b_0 b_{-1} + 45ka_0^2 a_{-1} b_0^2 \\
&\quad + 15k^3 a_0 a_{-1} b_0^3 - 15k^3 a_0^2 b_{-1} b_0^2 + 4ca_1 b_1 b_{-1}^3 + 6ca_1 b_0^2 b_{-1}^2 + 6ca_{-1} b_1^2 b_{-1}^2 - 176k^5 a_1 b_1 b_{-1}^3 + 11k^5 a_1 b_0^2 b_{-1}^2 \\
&\quad + 176k^5 a_{-1} b_1^2 b_{-1}^2 + 45ka_1^2 a_{-1} b_{-1}^2 + 45ka_1 a_0^2 b_{-1}^2 + 45ka_1 a_{-1}^2 b_0^2 - 75k^3 a_0^2 b_1 b_{-1}^2 + 60k^3 a_{-1}^2 b_1 b_0^2 \\
&\quad + ca_{-1} b_0^4 + k^5 a_{-1} b_0^4 + 15ka_{-1}^3 b_1^2 + 60k^3 a_1^2 b_{-1}^3 - 58k^5 a_{-1} b_1 b_0^2 b_{-1} + 12ca_0 b_1 b_{-1}^2 b_0 + 12ca_{-1} b_1 b_0^2 b_{-1} \\
&\quad + 47k^5 a_0 b_1 b_{-1}^2 b_0 + 90ka_1 a_{-1}^2 b_1 b_{-1} + 90ka_0^2 a_{-1} b_1 b_{-1} + 90ka_0 a_{-1}^2 b_1 b_0 - 60k^3 a_1 a_{-1} b_1 b_{-1}^2 \\
&\quad + 105k^3 a_1 a_0 b_{-1}^2 b_0 + 60k^3 a_1 a_{-1} b_0^2 b_{-1}, \\
R_8 &= 4ca_0 b_1 b_{-1}^3 - 76k^5 a_0 b_1 b_{-1}^3 + 90k^3 a_1 a_{-1} b_0 b_{-1}^2 + 90ka_0 a_{-1}^2 b_1 b_{-1} - k^5 a_1 b_0 b_{-1}^3 + 12ca_{-1} b_1 b_0 b_{-1}^2 \\
&\quad + 90ka_0^2 a_{-1} b_0 b_{-1} + 77k^5 a_{-1} b_1 b_0 b_{-1}^2 - 15k^3 a_0 a_{-1} b_0^2 b_{-1} + 6ca_0 b_{-1}^2 b_0^2 + 30ka_{-1}^3 b_1 b_0 - 30k^3 a_{-1}^2 b_1 b_{-1} b_0 \\
&\quad + 45ka_0 a_{-1}^2 b_0^2 - 11k^5 a_{-1} b_0^3 b_{-1} + 15k^3 a_{-1}^2 b_0^3 + 4ca_1 b_0 b_{-1}^3 + 75k^3 a_1 a_0 b_{-1}^3 - 135k^3 a_0 a_{-1} b_1 b_{-1}^2 \\
&\quad + 90ka_1 a_0 a_{-1} b_{-1}^2 + 11k^5 a_0 b_{-1}^2 b_0^2 + 90ka_1 a_{-1}^2 b_0 b_{-1} + 15ka_0^3 b_{-1}^2 + 4ca_{-1} b_0^3 b_{-1}, \\
R_9 &= 4ca_{-1} b_1 b_{-1}^3 + 30ka_{-1}^3 b_1 b_{-1} + 6ca_{-1} b_0^2 b_{-1}^2 - 15k^3 a_0 a_{-1} b_0 b_{-1}^2 + 90ka_0 a_{-1}^2 b_0 b_{-1} + 15ka_{-1}^3 b_0^2 \\
&\quad + 60k^3 a_{-1} a_1 b_{-1}^3 + 45ka_0^2 a_{-1} b_{-1}^2 - 16k^5 a_{-1} b_1 b_{-1}^3 + 15k^3 a_0^2 b_{-1}^3 - 60k^3 a_{-1}^2 b_1 b_{-1}^2 + 11k^5 a_{-1} b_0^2 b_{-1}^2 \\
&\quad + ca_1 b_{-1}^4 + 4ca_0 b_{-1}^3 b_0 - 11k^5 a_0 b_{-1}^3 b_0 + 16k^5 a_1 b_{-1}^4 + 45ka_1 a_{-1}^2 b_{-1}^2, \\
R_{10} &= k^5 a_0 b_{-1}^4 + 30ka_{-1}^3 b_0 b_{-1} - k^5 a_{-1} b_0 b_{-1}^3 + ca_0 b_{-1}^4 + 45ka_0 a_{-1}^2 b_{-1}^2 + 4ca_{-1} b_0 b_{-1}^3, \\
&\quad + 15k^3 a_{-1} a_0 b_{-1}^3 - 15k^3 a_{-1}^2 b_0 b_{-1}^2 \\
R_{11} &= ca_{-1} b_{-1}^4 + 15ka_{-1}^3 b_{-1}^2.
\end{aligned} \tag{27}
$$

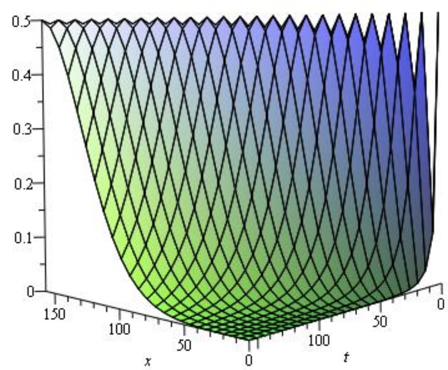
Solving the above algebraic equations by using Maple for a_1, a_0, a_{-1}, b_1, b_0, b_{-1}, c and k, we have

the solution if $a_1 = 0$, $a_0 = b_0$, $a_{-1} = 0$, $b_1 = b_1$, $b_0 = b_0$ and $b_{-1} = \frac{1}{4}\frac{b_0^2}{b_1}$, $k = -1$ and $c = -1$.
We take $b_1 = \frac{1}{2}$ and $b_0 = 1$,then $a_0 = 1$ and $b_{-1} = \frac{1}{2}$ so

$$u(x,t) = \frac{1}{2} sech^2 \left[\frac{1}{2} \left(\frac{-x^\alpha}{\Gamma(1+\alpha)} + \frac{t^\alpha}{\Gamma(1+\alpha)} \right) \right]. \tag{28}$$



(a) $u(x,t)$ of equation(18)



(b) $u(x,t)$ of equation(28)

Figure 1. a. solution of Burger-like equation for $\alpha = \frac{1}{2}$ b. solution of Sawada-Kotera equation for $\alpha = \frac{1}{2}$

6. Conclusion

In the present paper, the exact solution of some fractional differential equations is obtained by the exp-function method. Also, by balancing the first sentence of the highest degree and first sentence the highest derivative order of nonlinear (ODE) and similarly, balancing the last sentence of the highest degree and last sentence the highest derivative order of nonlinear (ODE), leads $m_2 = n_2$ and $m_1 = n_1$. Consequently, the simplest choice $m_2 = n_2 = 1$ and $m_1 = n_1 = 1$ has to be only considered. Hence, this method can be useful to solve other nonlinear FDEs in mathematical physics, because it is an easy, direct, concise, basic and powerful method to implement, also the exp-function method generalized solitary solutions and periodic solutions of nonlinear equations. The principal merits of this method over the other methods are that it provides more general solutions with some arbitrary parameters. It should be noticed that the solutions are accurate.

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