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Estimating Parameter of the Selected Uniform Population Under the Generalized Stein Loss Function

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Abstract

This paper deals with the problem of estimating scale parameter of the selected uniform population when sample sizes are unequal. The loss has been measured by the generalized Stein loss (GSL) function. The uniformly minimum risk unbiased (UMRU) estimator is derived, and the natural estimators are also constructed under the GSL function. One of the natural estimators is proved to be the generalized Bayes estimator with respect to a noninformative prior. For $k = 2$, we obtained a sufficient condition for an inadmissibility result and demonstrate that the natural estimator and UMRU estimator are inadmissible. A simulation investigation is also carried out for the performance of the risk functions of various competing estimators. Finally, this article represents a conclusion of our study.

Keywords: Generalized Stein loss (GSL) function; Uniform distributions; Inadmissibility; UMRU estimator; Natural estimators; Selection rule; Entropy loss function

MSC 2010 No.: 62E10, 62M90, 60G70, 60E05, 62E10, 62E15

1. Introduction

The problem of estimating parameters of a selected population commonly occurs in various practical applications in engineering, agricultural, medical experiments and social sciences. For example, a farmer not only wishes to select the type of fertilizer from $k(\geq 2)$ available fertilizers which provides the highest mean yield, but he also wants an estimate of mean of the selected fertilizer. Several types of medicines are used for a particular disease and a doctor is interested in selecting the most effective one among those. Naturally, he would be interested in an estimate of the effectiveness of the selected medicine. Such types of problems of estimation after selection have been widely investigated for various probability models due to its applications and perhaps the challenges involved in it. Some of the references in this area are due to Sackrowitz and Samuel-Cahn (1984), Kumar and Gangopadhyay (2005), Misra et al. (2006a, 2006b), Sill and Sampson (2007), Vellaisamy and Jain (2008), Vellaisamy and Al-Mosawi (2010), Al-Mosawi et al. (2012), Qomi et al. (2012), Arshad and Misra (2015, 2016), Nematollahi and Jozani (2016), Meena and Gangopadhyay (2017), Nematollahi (2017), Meena et al. (2018) and Arshad and Abdalghani (2020).

A good amount of the work relating to selection and estimation after selection problems summarized in the literature has been carried out over the last five decades under the assumption of equal nuisance parameters and/or sample sizes, and a limited amount of the research work has been conducted under the framework, where nuisance parameter and /or sample sizes may be unequal. For important papers in this direction, we refer the readers to Abughalous and Miescke (1989), Dhariyal et al. (1989), Gupta and Sobel (1958), Risko (1985). Recently, Pagheh and Nematollahi (2015) have examined the problem of estimation after selection concerning the uniform population based on the sample of equal sizes under the GSL function. In this article, we consider unequal sample sizes and a more general class of selection rules thereby extending the results of Pagheh and Nematollahi (2015). Arshad and Misra (2017) obtained the UMRU estimator and also established some inadmissible results for scale parameter of the selected population when used the entropy loss function. Vellaisamy et al. (1988) used the natural selection rule and investigated the problem of estimating the mean of the selected uniform population with respect to the squared error loss function and scale-invariant loss function. For the selected mean, they obtained UMVU estimator and a generalized Bayes estimator. Authors established that the natural estimator is inadmissible with respect to squared error loss function and provided a minimax estimator with respect to the scale-invariant loss function. They provided improvements on the UMVU estimator with respect to both loss functions. Afterwards, Nematollahi and Motamed-Shariati (2012) discussed the same problem with respect to the entropy loss function. They constructed the UMRU estimator of the scale parameter of the selected uniform population. For the case $k = 2$, they established that UMRU estimator is inadmissible and the generalized Bayes estimator is minimax. The problem of estimating the scale parameter of the selected uniform population using the asymmetric scale equivarient loss function has been studied by Arshad and Abdalghani (2019). To the best of our knowledge, this estimation problem has not been explored in literature before under the GSL function.

The manuscript is organized as follows. The formulation of the problem and selection process is discussed in Section 2. In Section 3, the UMRU estimator of μ_L is obtained using the UV procedure of Robbins (1988) and prove that the natural estimator $\xi_{N,2}(\boldsymbol{X})$ is a generalized Bayes estimator of μ_L under the GSL function. In Section 4, employing the procedure of Brewster and Zidek (1974), a sufficient condition for inadmissibility of scale parameter μ_L has been given under the GSL function. Moreover, it is shown that the natural estimator $\xi_{N,1}$ and the UMRU estimator are inadmissible, and improved estimators have been suggested for estimating μ_L. The simulation studies to compare various competing estimators are conducted in Section 5. Section 6 concludes our study.

2. Formulation of problem

Let $X_{i1}, X_{i2}, \ldots, X_{in_i}$ be independent random samples of size n_i from the population Π_i $(i = 1, 2 \ldots, k)$ which are individually uniformly distributed over the interval $(0, \mu_i)$ with unknown scale parameter $\mu_i > 0$. Let $X_i = \max\{X_{i1}, ..., X_{in_i}\}$, therefore $\mathbf{X} = (X_1, \ldots, X_k)$ is a complete and sufficient statistic for $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k) \in \mathbb{R}_+^k$; here $\mathbb{R}_+^k = \{(x_1, ..., x_k) \in \mathbb{R}^k : x_i > 0 \ \forall \ i = 1, 2, ..., k\}$ denotes a subset of $k-$ dimensional Euclidean space $\mathbb{R}^k$. Let $X_1, ..., X_k$ denote independent random variables and the density of X_i is

$$f_i(x|\mu_i) = \begin{cases} \frac{n_i x^{n_i - 1}}{\mu_i^{n_i}}, & \text{if} \quad 0 < x < \mu_i, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Here, $\mu_i > 0, (i = 1, \ldots, k)$ is an unknown scale parameter. The population Π_i is the "best" if $\mu_i > \mu_j$, for all $i, j = 1, \ldots, k, i \neq j$ i.e., the population associated with the largest scale parameter $\mu_{[k]} = \max\{\mu_1, \ldots, \mu_k\}$ to be the "best". If more than one of the μ_i are tied at the largest value, one of the population is assumed to be arbitrarily marked as "best" population. For selecting/identifying the "best" population, employ a nonrandomized selection procedure $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$, where $\delta_i(\boldsymbol{x})$ is the conditional probability of selecting population Π_i when $\boldsymbol{X} = \boldsymbol{x}$ is observed. Based on the maximum likelihood estimator (MLE) X_i of μ_i, we wish to construct a natural selection procedure for the goal of identifying the "best" population. Such a natural selection rule can be expressed as $\boldsymbol{\delta}^N(\boldsymbol{x}) = (\delta_1^N, \delta_2^N, ..., \delta_k^N)$, where

$$\delta_i^N(\boldsymbol{x}) = \begin{cases} 1, & \text{if} \quad \max\limits_{j \neq i} x_j < x_i, \\ 0, & \text{otherwise.} \end{cases}$$

For samples of equal sizes, i.e., $n_1 = n_2 = \cdots = n_k$, under the $0-1$ loss function, the natural selection rule $\boldsymbol{\delta}^N(\boldsymbol{x})$ is known to be minimax (Misra and Dhariyal (1994)). However, the natural selection rule $\boldsymbol{\delta}^N(\boldsymbol{x})$ is no longer minimax with respect to the $0-1$ loss function, when the sample sizes are unequal and it has many undesirable properties. For identifying (or selecting) the "best" uniform population, Arshad and Misra (2015b) introduced a class $\mathbb{C} = \{\boldsymbol{\delta}^{\boldsymbol{\nu}} : \boldsymbol{\delta}^{\boldsymbol{\nu}}(\boldsymbol{X}) = (\delta_1^{\boldsymbol{\nu}}, ..., \delta_k^{\boldsymbol{\nu}}), \boldsymbol{\nu} \in \mathbb{R}_+^k\}$ of selection rules, where

$$\delta_i^{\boldsymbol{\nu}}(\boldsymbol{X}) = \begin{cases} 1, & \text{if} \quad \nu_i X_i > \max\limits_{j \neq i} \nu_j X_j, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

and $\boldsymbol{\nu} = (\nu_1, ..., \nu_k) \in \mathbb{R}_+^k$. For $k = 2$ and $n_1 \neq n_2$, the class $\mathbb{C} = \{\boldsymbol{\delta}^\nu = (\delta_1^\nu, \delta_2^\nu), \nu > 0\}$, provides the selection procedures of the following forms:

$$\delta_1^\nu(\boldsymbol{X}) = \begin{cases} 1, & \text{if} \quad X_1 > \nu X_2, \\ 0, & \text{if} \quad X_1 \leq \nu X_2, \end{cases} ; \quad \delta_2^\nu(\boldsymbol{X}) = \begin{cases} 1, & \text{if} \quad X_1 \leq \nu X_2, \\ 0, & \text{if} \quad X_1 > \nu X_2. \end{cases}$$

The selection rule $\boldsymbol{\delta}^{\nu^*} = (\delta_1^{\nu^*}, \delta_2^{\nu^*})$ obtained by Arshad and Misra (2015a), where

$$\nu^* \equiv \nu^*(n_1, n_2) = \begin{cases} \left(\frac{n_1+n_2}{2n_2}\right)^{\frac{1}{n_1}}, & \text{if} \quad n_1 \leq n_2, \\ \left(\frac{2n_1}{n_1+n_2}\right)^{\frac{1}{n_2}}, & \text{if} \quad n_1 > n_2, \end{cases}$$

is admissible and minimax under the $0 - 1$ loss function and is a generalized Bayes rule with respect to non-informative prior.

The problem is to estimate the scale parameter μ_L associated with the population chosen by a selection rule $\boldsymbol{\delta^\nu}$ given in (2). Let $A_i = \{\boldsymbol{x} \in \chi : \nu_i x_i > \nu_j x_j \;\; \forall j \neq i, j = 1, 2, ..., k\}$ and let $I_A(.)$ be the partition of sample space χ. Then, scale parameter μ_L can be given by

$$\mu_L = \sum_{i=1}^{k} \mu_i I_{A_i}(\boldsymbol{X}). \tag{3}$$

Here, $I_A(.)$ denotes the indicator function of the set A.

For this research, we study the problem of estimation of scale parameter of the selected population using the GSL function. The GSL function has the form:

$$L(g(\boldsymbol{\mu}), \xi) = \left(\frac{\xi}{\boldsymbol{\mu}}\right)^q - q \ln\left(\frac{\xi}{\boldsymbol{\mu}}\right) - 1, \quad \boldsymbol{\mu} \in \Omega, \xi \in \mathbb{C}, \tag{4}$$

where $g(\boldsymbol{\mu})$ is real valued function of parameter $\boldsymbol{\mu}$ and $\mathbb{C}$ indicates the class of all estimators of $g(\boldsymbol{\mu})$. This loss function is asymmetric and convex when $\Delta = \frac{\xi}{g(\boldsymbol{\mu})}$ and quasi concave otherwise, but its risk function has unique minimum at $\Delta = 1$. The GSL function is a scale invariant loss function and is suitable for estimating the scale parameter. Therefore, the GSL function is useful in situations where under-estimation and over-estimation have not been assigned the same penalty. The GSL function with negative q values penalizes over-estimation more than under-estimation whereas it acts vice-versa with positive q values. In this article, under the GSL function, our aim is to estimate the parameter of the selected uniform population with sample sizes are unequal. Consider $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ and $g(\boldsymbol{\mu})$ be real valued function of $\boldsymbol{\mu}$. We would like to estimate a function of $\boldsymbol{\mu}$, i.e $\boldsymbol{\mu}$ by an estimator ξ with respect to the loss function $L(g(\boldsymbol{\mu}), \xi)$. Following Lehmann (1951), an estimator $\xi(\boldsymbol{X})$ is said to be risk-unbiased for the parameter $g(\boldsymbol{\mu})$ if it satisfies the inequality

$$E_{\boldsymbol{\mu}}\left[L\left(g(\boldsymbol{\mu}, \xi(\boldsymbol{X}))\right)\right] \leq E_{\boldsymbol{\mu}}[L(g(\boldsymbol{\mu}', \xi(\boldsymbol{X})))], \qquad \text{for all} \quad \boldsymbol{\mu}' \neq \boldsymbol{\mu}. \tag{5}$$

Using condition (5) and the GSL function (4), an estimator $\xi(\boldsymbol{X})$ is a risk-unbiased estimator of the parameter $g(\boldsymbol{\mu})$, if it satisfies the following condition

$$E_{\boldsymbol{\mu}}[\xi^q(\boldsymbol{X})] = g^q(\boldsymbol{\mu}), \;\; \text{for all } \boldsymbol{\mu}. \tag{6}$$

Since μ_L is dependent on $X_1, ..., X_k$. Thus, the condition for the risk-unbiased estimator of μ_L is defined as

$$E_{\boldsymbol{\mu}}\left[\xi^q(\boldsymbol{x})\right] = E_{\boldsymbol{\mu}}\left[\mu_L^q\right], \qquad \text{for all } \boldsymbol{\mu}.$$

Therefore, apply the $(U - V)$ procedure of Robbins (1988) to establish the risk unbiased and UMRU estimator of μ_L of selected uniform population.

We consider two natural estimators of μ_L based on the maximum likelihood estimator (MLE) and the UMRU estimator, under the GSL function. Therefore one may write the natural estimators of μ_L of the selected population as:

$$\xi_{N,1}(\boldsymbol{X}) = \sum_{i=1}^{k} X_i I_{A_i}(\boldsymbol{X}); \text{ and } \quad \xi_{N,2}(\boldsymbol{X}) = \sum_{i=1}^{k} \left(\frac{n_i + q}{n_i}\right)^{\frac{1}{q}} X_i I_{A_i}(\boldsymbol{X}). \tag{7}$$

3. UMRU Estimator and Generalized Bayes Estimator

We discuss the general form of uniformly minimum risk unbiased estimator and Generalized Bayes estimator of μ_L with respect to the GSL function (4) in this section. Utilizing the unbiased criterion (5), an estimate $\xi(\boldsymbol{X})$ is a risk unbiased estimator of the random parameter $g(\boldsymbol{\mu})$ with respect to the GSL function (4), if it satisfies

$$E_{\boldsymbol{\mu}}\left[\xi^q(\boldsymbol{x})\right] = E_{\boldsymbol{\mu}}\left[g^q(\boldsymbol{\mu})\right], \qquad \text{for all } \boldsymbol{\mu}.$$

To evaluate the UMRU estimator of μ_L, we adopt the (U-V) procedure of Robbins. The ensuing lemma is important in deriving the UMRU estimator.

Lemma 3.1.

Suppose $X_1, ..., X_k$ be k independent random variables, where X_i has a probability density function as given in (1). Let $U_1(\boldsymbol{X}), ..., U_k(\boldsymbol{X})$ be k real valued functions on $\mathbb{R}_+^k$ such that

(1) $E_{\boldsymbol{\mu}}\left[|X_i^q U_i(\boldsymbol{X})|\right] < \infty$, for all $\boldsymbol{\mu} \in \Omega, \quad i = 1, ..., k$.
(2) $\int_0^{x_i} x_i^q U_i(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_k) t^{n_i - 1} dt < \infty$, for all $\boldsymbol{x} \in \mathbb{R}_+^k, \quad i = 1, ..., k$.
(3) $\lim_{x_i \to 0}\left[x_i^q \int_0^{x_i} U_i(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_k) t^{n_i - 1} dt\right] = 0$, for all $\boldsymbol{x} \in \mathbb{R}_+^k, j \neq i, i = 1, ..., k$.

Then, the function $V_i(\boldsymbol{X})$ defined as

$$V_i(\boldsymbol{X}) = X_i^q U_i(\boldsymbol{X}) + q x_i^{q - n_i} \int_0^{x_i} U_i(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_k) t^{n_i - 1} dt,$$

satisfies

$$E_{\boldsymbol{\mu}}\left[\sum_{i=1}^{k} V_i(\boldsymbol{X})\right] = E_{\boldsymbol{\mu}}\left[\sum_{i=1}^{k} \mu_i^q U_i(\boldsymbol{X})\right].$$

Proof:

This lemma is a generalization of Theorem 3.1 of Nematollahi and Jozani (2016). Therefore, the proof of this Lemma follows from the Theorem 3.1 of Nematollahi and Jozani (2016). ∎

Theorem 3.1.

Consider the GSL function, as defined in (4), then the estimator

$$\xi_U(\boldsymbol{X}) = \sum_{i=1}^{k} X_i \left[1 + \frac{q}{n_i}\left\{1 - \left(\frac{\max\limits_{j\neq i} \nu_j X_j}{\nu_i X_i}\right)^{n_i}\right\}\right]^{\frac{1}{q}} I_{A_i(\boldsymbol{X})}, \tag{8}$$

is the UMRU estimator of scale parameter μ_L of the selected population.

Proof:

For $i = 1, ..., k$, let $V_i(\boldsymbol{X})$ be a function defined on the sample space χ such that $E\left[V_i(\boldsymbol{X})\right] = E\left[\mu_i^q I_{A_i}(\boldsymbol{X})\right]$.

Using Lemma 3.1, for $i = 1, ..., k$, we have

$$\begin{aligned}
V_i(\boldsymbol{X}) &= X_i^q I_{A_i}(\boldsymbol{X}) + qX_i^{q-n_i}\int_0^{x_i} I_{A_i}(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_k) t^{n_i - 1} dt \\
&= X_i^q I_{A_i}(\boldsymbol{X}) + qX_i^{q-n_i}\int_{\max\limits_{j\neq i}\frac{\nu_j X_j}{\nu_i}}^{X_i} t^{n_i-1} dt I_{A_i}(\boldsymbol{X}) \\
&= X_i^q I_{A_i}(\boldsymbol{X}) + \frac{qX_i^q}{n_i}\left[1 - \left(\frac{\max\limits_{j\neq i} \nu_j X_j}{\nu_i X_i}\right)^{n_i}\right] I_{A_i}(\boldsymbol{X}) \\
&= X_i^q \left[1 + \frac{q}{n_i}\left\{1 - \left(\frac{\max\limits_{j\neq i} \nu_j X_j}{\nu_i X_i}\right)^{n_i}\right\}\right] I_{A_i}(\boldsymbol{X}).
\end{aligned}$$

Clearly,

$$\xi_U^q(\boldsymbol{X}) = \sum_{i=1}^{k} V_i(\boldsymbol{X}).$$

It follows that

$$\begin{aligned}
E_{\boldsymbol{\mu}}\left[\xi_U^q(\boldsymbol{X})\right] &= E_{\boldsymbol{\mu}}\left[\sum_{i=1}^{k} V_i(\boldsymbol{X})\right] \\
&= \sum_{i=1}^{k} E_{\boldsymbol{\mu}}\left[\mu_i^q I_{A_i}(\boldsymbol{X})\right] \\
&= E_{\boldsymbol{\mu}}\left[\mu_L^q\right].
\end{aligned}$$

Since $\boldsymbol{X} = (X_1, ..., X_k)$ is a complete and sufficient statistics, the estimator $\xi_U(\boldsymbol{X})$ is a risk unbiased estimator of μ_L. ∎

In the following remarks we provide the UMRU estimator of the scale parameter μ_L obtained from the preceding theorem.

Remark 3.1.

Consider sample sizes are equal, i.e., $n_1 = n_2 = \cdots = n_k = n$ (say), and $\nu_1 = \nu_2 = \cdots = \nu_k = 1$. Then, the UMRU estimator of μ_L is

$$\xi_U(\boldsymbol{X}) = X_{[k]} \left[1 + \frac{q}{n} \left\{ 1 - \left(\frac{X_{[k-1]}}{X_{[k]}} \right)^n \right\} \right]^{\frac{1}{q}}.$$

The UMRU estimator depends only on two largest order statistics.

Proof:

In the equation (8) if we substitute $n_1 = n_2 = \cdots = n_k = n$ we get the above UMRU estimator. ■

Remark 3.2.

Using the entropy loss function, i.e., for $q = -1$, the UMRU estimator of μ_L is

$$\xi_U(\boldsymbol{X}) = \sum_{i=1}^{k} \frac{n_i X_i}{\left[(n_i - 1) + \left(\frac{\max_{j \neq i} \nu_j X_j}{\nu_i X_i} \right)^{n_i} \right]} I_{A_i}(\boldsymbol{X}). \tag{9}$$

Proof:

Substituting -1 in place of q in the equation (8) we get the result. ■

The next result leads to the generalized Bayes estimator of μ_L with respect to the GSL function as defined in (4).

Theorem 3.2.

Assume the noninformative prior distribution

$$\Pi_{\boldsymbol{\mu}}(\mu_1, ..., \mu_k) = \begin{cases} \frac{1}{\mu_1, ..., \mu_k}, & \text{if} \quad \boldsymbol{\mu} \in \Omega, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Then, the natural estimator $\xi_{N,2}(\boldsymbol{X})$ is the generalized Bayes estimator of μ_L under the GSL function (4).

Proof:

Consider the noninformative prior distribution (10) for $\boldsymbol{\mu} = (\mu_1, ..., \mu_k)$, then the posterior distribution of $\boldsymbol{\mu}$, given $\boldsymbol{X} = \boldsymbol{x}$ has the probability density function

$$\Pi^p_{\boldsymbol{\mu}}(\mu_1, ..., \mu_k | \boldsymbol{x}) = \begin{cases} \Pi_{i=1}^k \frac{n_i x_i^{n_i}}{\mu_i^{n_i+1}}, & \text{if} \quad x_i < \mu_i, \; i = 1, ..., k, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

The posterior risk of an estimator ξ under the GSL function (4), which can be written as

$$r^p(\xi, \boldsymbol{x}) = E_{\Pi^p}\left[\left\{\left(\frac{\xi}{\mu_L}\right)^q - q\ln\left(\frac{\xi}{\mu_L}\right) - 1\right\}\big|\boldsymbol{X} = \boldsymbol{x}\right]. \tag{12}$$

The generalized Bayes estimator of μ_L, denoted by $\xi^{GB}(\boldsymbol{X})$, which minimizes the posterior risk (12), is as follows

$$\xi^{GB}(\boldsymbol{x}) = \sum_{i=1}^{k}\left[E_{\Pi^p}\left(\frac{1}{\mu_i^q}\big|\boldsymbol{X} = \boldsymbol{x}\right)\right]^{-\frac{1}{q}} I_{A_i}(\boldsymbol{X}).$$

Using the posterior density (11), we obtained the generalized Bayes estimator of μ_L is given by

$$\xi^{GB}(\boldsymbol{x}) = \sum_{i=1}^{n}\left[\frac{(q+n_i)x_i^q}{n_i}\right]^{\frac{1}{q}} I_{A_i}(\boldsymbol{X}) = \xi_{N,2}(\boldsymbol{X}).$$

Hence, the result follows. ∎

4. Inadmissibility results

This section is devoted to the sufficient condition for inadmissability of a scale invariant estimator of scale parameter μ_L using the GSL function (4), for $k = 2$ uniform populations. It also gives dominated estimators in those cases, where the results satisfy the sufficient conditions. For this purpose, consider the class of scale invariant estimator of the form

$$\xi_\psi(X_1, X_2) = X_2\psi(Y),$$

where $Y = \frac{X_1}{X_2}$ and $\psi(.)$ is a non-negative real valued function defined on $\mathbb{R}_+$.

The following theorem is to study sufficient condition for inadmissibility of an estimator of μ_L using the application of Brewster and Zidek (1974) technique, under the GSL function (4).

Theorem 4.1.

Assume that $\xi_\psi(X_1, X_2) = X_2\psi(Y)$ provides a scale-invariant estimator of μ_L, where $Y = \frac{X_1}{X_2}$ and $\psi(.)$ is a non-negative real valued function defined on $\mathbb{R}_+$. Consider the function ψ_1 on $\mathbb{R}_+$ as:

$$\psi_1(Y) = \begin{cases} \left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}}, & \text{if } 0 < Y < \nu, \\ Y\left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}}, & \text{if } Y \geq \nu. \end{cases}$$

where $\nu = \frac{\nu_2}{\nu_1}$. Further, define the estimator ξ_{ψ_*} by $\xi_{\psi_*}(X_1, X_2) = X_2\psi_*(Y)$, where

$$\psi_*(Y) = \begin{cases} \psi_1(Y), & \text{if} \quad \psi(Y) \leq \psi_1(Y), \\ \psi(Y), & \text{if} \quad \psi(Y) > \psi_1(Y). \end{cases} \tag{13}$$

Then, the estimator ξ_ψ is inadmissible, and is dominated by ξ_{ψ_*} with respect to GSL function if $P_{\boldsymbol{\mu}}\left(\psi_1(Y) > \psi(Y)\right) \geq 0$ for all $\boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathbb{R}_+^2$, and strict inequality holds for some $\boldsymbol{\mu} \in \mathbb{R}_+^2$.

Proof:

For $\mu_1, \mu_2 \in \mathbb{R}_+^2$, consider the risk difference:

$$\begin{aligned}\Delta(\boldsymbol{\mu}) &= R(\boldsymbol{\mu}, \xi_\psi) - R(\boldsymbol{\mu}, \xi_{\psi_*}) \\ &= E_{\boldsymbol{\mu}}\left[\left(\frac{X_2\psi(Y)}{\mu_L}\right)^q - \left(\frac{X_2\psi_*(Y)}{\mu_L}\right)^q - q\ln\left(\frac{\psi(Y)}{\psi_*(Y)}\right)\right] \\ &= E_{\boldsymbol{\mu}}\left[\left(\frac{X_2}{\mu_L}\right)^q (\psi^q(Y) - \psi_*^q(Y)) - q\ln\left(\frac{\psi(Y)}{\psi_*(Y)}\right)\right] \\ &= E_{\boldsymbol{\mu}}\left[D_{\boldsymbol{\mu}}(Y)\right],\end{aligned}$$

where, for $y \in \mathbb{R}_+$ and $\mu \in \mathbb{R}_+^2$,

$$D_{\boldsymbol{\mu}}(y) = (\psi^q(y) - \psi_*^q(y))\, E_{\boldsymbol{\mu}}\left[\left(\frac{X_2}{\mu_L}\right)^q \big| Y = y\right] - q\ln\left(\frac{\psi(Y)}{\psi_*(Y)}\right). \tag{14}$$

The conditional p.d.f. of X_2, given $Y = y$, is

$$f_{X_1|Y}(x_1|y) = \begin{cases} \frac{(n_1+n_2)x_2^{n_1+n_2-1}}{\mu_2^{n_1+n_2}}, & \text{if} \quad 0 < x_2 < \mu_2,\ \ y < \frac{\mu_1}{\mu_2}, \\ \frac{(n_1+n_2)y^{n_1+n_2}x_2^{n_1+n_2-1}}{\mu_1^{n_1+n_2}}, & \text{if} \quad 0 < x_2 < \frac{\mu_1}{y}, y \geq \frac{\mu_1}{\mu_2}. \end{cases}$$

Let $\vartheta = \frac{\mu_1}{\mu_2}$, and let $\nu = \frac{\nu_2}{\nu_1}$. In derivation of $E_{\boldsymbol{\mu}}\left[\left(\frac{X_2}{\mu_L}\right)^q \big| Y = y\right]$, there are two cases which follows:
Case-I: when $y > \nu$

$$E\left(\left(\frac{X_2}{\mu_L}\right)^q \Big| Y = y\right) = \begin{cases} \frac{n_1+n_2}{n_1+n_2+q}\frac{1}{\vartheta^q}, & \text{if} \quad y < \vartheta, \\ \frac{n_1+n_2}{n_1+n_2+q}\frac{1}{y^q}, & \text{if} \quad y \geq \vartheta. \end{cases}$$

Case-II: when $y \leq \nu$

$$E\left(\left(\frac{X_2}{\mu_L}\right)^q \Big| Y = y\right) = \begin{cases} \frac{n_1+n_2}{n_1+n_2+q}, & \text{if} \quad y < \vartheta, \\ \frac{n_1+n_2}{n_1+n_2+q}\left(\frac{\vartheta}{y}\right)^q, & \text{if} \quad y \geq \vartheta. \end{cases}$$

It is observed from Case-I and Case-II that, for $\vartheta < \nu$

$$E\left(\left(\frac{X_2}{\mu_L}\right)^q \Big| Y = y\right) = \begin{cases} \frac{n_1+n_2}{n_1+n_2+q}, & \text{if} \quad 0 < y < \vartheta, \\ \frac{n_1+n_2}{n_1+n_2+q}\left(\frac{\vartheta}{y}\right)^q, & \text{if} \quad \vartheta \leq y < \nu, \\ \frac{n_1+n_2}{n_1+n_2+q}\frac{1}{y^q}, & \text{if} \quad 0 < \nu \leq y, \end{cases} \tag{15}$$

and, for $\vartheta \geq \nu$

$$E\left(\left(\frac{X_2}{\mu_L}\right) \Big| Y = y\right)^q = \begin{cases} \frac{n_1+n_2}{n_1+n_2+q}, & \text{if} \quad 0 < y < \nu, \\ \frac{n_1+n_2}{n_1+n_2+q}\frac{1}{\vartheta^q}, & \text{if} \quad \nu \leq y < \vartheta, \\ \frac{n_1+n_2}{n_1+n_2+q}\frac{1}{y^q}, & \text{if} \quad 0 < \vartheta \leq y. \end{cases} \tag{16}$$

In either cases, for $q < 0$ using (15) and (16), we get

$$\inf_{\vartheta\in(0,\infty)} E\left(\left(\frac{X_2}{\mu_L}\right)^q \Big| Y=y\right) = \begin{cases} \frac{n_1+n_2}{n_1+n_2+q}, & \text{if} \quad 0<y<\nu, \\ \frac{n_1+n_2}{n_1+n_2+q}\frac{1}{y^q}, & \text{if} \quad \nu \le y, \end{cases}$$

$$= \frac{1}{\psi_1^q(y)}, \tag{17}$$

and for $q > 0$, we get

$$\sup_{\vartheta\in(0,\infty)} E\left(\left(\frac{X_2}{\mu_L}\right)^q \Big| Y=y\right) = \begin{cases} \frac{n_1+n_2}{n_1+n_2+q}, & \text{if} \quad 0<y<\nu, \\ \frac{n_1+n_2}{n_1+n_2+q}\frac{1}{y^q}, & \text{if} \quad \nu \le y, \end{cases}$$

$$= \frac{1}{\psi_1^q(y)}. \tag{18}$$

It follows from (13), (14), (17) and (18) that, if $\psi_1(y) \ge \psi(y)$, then

$$\begin{aligned} D_{\boldsymbol{\mu}}(y) &= \left(\psi^q(y) - \psi_*^q(y)\right) E_{\boldsymbol{\mu}}\left[\left(\frac{X_2}{\mu_L}\right)^q \Big| Y\right] - q\ln\left(\frac{\psi(y)}{\psi_*(y)}\right) \\ D_{\boldsymbol{\mu}}(y) &= \left(\psi^q(y) - \psi_1^q(y)\right)\frac{1}{\psi_1^q(y)} - q\ln\left(\frac{\psi(y)}{\psi_1(y)}\right) \\ &\ge \left(\frac{\psi(y)}{\psi_1(y)}\right)^q - q\ln\left(\frac{\psi(y)}{\psi_1(y)}\right) - 1 \\ &\ge 0, \end{aligned}$$

and strict inequality holding for some $\boldsymbol{\mu} \in \mathbb{R}_+^2$. If $\psi_1(y) < \psi(y)$, then $D_{\boldsymbol{\mu}}(y)$=0. Therefore

$$R(\boldsymbol{\mu}, \xi_\psi) \ge R(\boldsymbol{\mu}, \xi_{\psi_*}), \quad \text{for all }, \quad \boldsymbol{\mu} \in \mathbb{R}_+^2,$$

and strict inequality holds for some μ. This completes the proof. ∎

Now we conclude the following dominance results for the proposed estimators which are the consequences of the preceding theorem.

Corollary 4.1.

Consider the case $k = 2$, the UMRU estimator $\xi_U(X)$ is inadmissible and is dominated by $\xi_U^D(X) = X_2 \max\{\psi^U(y), \psi_1(y)\}$, where

$$\psi^U(y) = \begin{cases} \left[1 + \frac{q}{n_2}\left(1 - \left(\frac{y}{\nu}\right)^{n_2}\right)\right]^{\frac{1}{q}}, & \text{if} \quad 0<y<\nu, \\ y\left[1 + \frac{q}{n_1}\left(1 - \left(\frac{\nu}{y}\right)^{n_1}\right)\right]^{\frac{1}{q}}, & \text{if} \quad y \ge \nu, \end{cases}$$

with respect to GSL function (4) and $\psi_1(y)$ is given in Theorem 4.1.

Proof:

The proof of this corollary follows from the Theorem 4.1 by replacing $\Psi_*(y)$ (given in Theorem 4.1) with $\psi^U(y)$. ■

Corollary 4.2.

Consider the case $k = 2$, under the GSL function (4), $\xi_{N,1}(\boldsymbol{X})$ is the natural estimator of the form defined in (7). Then, $\xi_{N,1}(\boldsymbol{X})$ is inadmissible and is dominated by

$$\xi_{N,1}^{ID}(\boldsymbol{X}) = \left(\frac{n_1 + n_2 + q}{n_1 + n_2}\right)^{\frac{1}{q}} \xi_{N,1}\left(\boldsymbol{X}\right).$$

Proof:

The proof of this corollary follows from the fact that $\left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}} \geq 1 \; \forall q, n_1, n_2$. ■

Corollary 4.3.

Consider the case $k = 2$ and $q < 0$, under the GSL function (4), $\xi_{N,2}(\boldsymbol{X})$ is the natural estimator of the form defined in (7). Then, $\xi_{N,2}(\boldsymbol{X})$ is inadmissible and is dominated by

$$\xi_{N,2}^{ID}(\boldsymbol{X}) = X_2 \max\{\xi_{N,2}(y), \psi_1(y)\}.$$

Proof:

This result follows from the fact that, for $q < 0$, $P\left(\psi_1(y) > \xi_{N,2}(y)\right) > 0 \; \forall \, \boldsymbol{\mu} \in \mathbb{R}_+^2$. ■

Remark 4.1.

From Corollary 4.1 that the UMRU estimator of μ_L is inadmissible and is dominated with respect to GSL function (4) for the case $k = 2, n_1 = n_2 = n$ and $\nu_1 = \nu_2 = 1$.

Proof:

Proof is obtained directly by Corollary 4.1 by substituting $n_1 = n_2 = n$ and $\nu_1 = \nu_2 = 1$. ■

Remark 4.2.

From Theorem 4.1 the UMRU estimator of μ_L is improved and dominated with respect to entropy loss function for the case $k = 2, n_1 = n_2 = n$ and $\nu_1 = \nu_2$.

Proof:

Proof is obtained directly by Theorem 4.1 by substituting $n_1 = n_2 = n$ and $\nu_1 = \nu_2$. ■

Remark 4.3.

From Corollary 4.2 the natural estimator $\xi_{N,1}$ corresponding to the MLE of μ_L is inadmissible with respect to GSL function (4) for the cases $k = 2$, $n_1 = n_2 = n$ and $\nu_1 = \nu_2$.

Proof:

The proof is obtained directly from Corollary 4.2 by substituting $n_1 = n_2 = n$ and $\nu_1 = \nu_2$. ∎

Now, we prove the following result.

Theorem 4.2.

Let $n_1 + n_2 + q > 0$. Let c_1 and c_2 be two possible real constants and let $\boldsymbol{c} = (c_1, c_2)$. Suppose that $c_i \in \left(0, \left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}}\right) \bigcup \left(\left(\frac{n_i+q}{n_i}\right)^{\frac{1}{q}}, \infty\right)$, for $i = 1, 2$. Define the natural-type estimators

$$\xi_{\boldsymbol{c}}(X_1, X_2) = \begin{cases} c_1 X_1, & \text{if} \quad \mathbf{X} \in A_1, \\ c_2 X_2, & \text{if} \quad \mathbf{X} \in A_2. \end{cases}$$

Then, the natural-type estimators ξ_c are inadmissible for estimating μ_L with respect to GSL function (4).

Proof:

It should be noted from Theorem 4.1 that the estimators ξ_c, for $c_i \in \left(0, \left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}}\right)$, $i = 1, 2$, are inadmissible and are dominated by

$$\xi^*_{\boldsymbol{c}}(\boldsymbol{X}) = \begin{cases} \left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}} X_1, & \text{if} \quad \mathbf{X} \in A_1, \\ \left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}} X_2, & \text{if} \quad \mathbf{X} \in A_2. \end{cases}$$

Further, assume that $c_i \in \left(\left(\frac{n_i+q}{n_i}\right)^{\frac{1}{q}}, \infty\right)$ for $i = 1, 2$. It is seen that the risk function of the estimator ξ_c is a function of $\vartheta = \frac{\mu_1}{\mu_2} \in (0, \infty)$. Therefore, consider the risk function of ξ_c as

$$\begin{aligned} R(\vartheta, \xi_c) &= E_{\boldsymbol{\mu}}\left[\left(\frac{\xi_c}{\mu_L}\right)^q - q \ln\left(\frac{\xi_c}{\mu_L}\right) - 1\right], q \neq 0 \\ &= \sum_{j=1}^{2} R_j(\vartheta, c_j) \quad \text{(say)}, \end{aligned}$$

where

$$R_j(\vartheta, c_j) = E_{\boldsymbol{\mu}}\left[\left\{\left(\frac{c_j X_j}{\mu_j}\right)^q - q \ln\left(\frac{c_j X_j}{\mu_j}\right) - 1\right\} I_{A_j(X)}\right].$$

The above risk function is a convex function of c, for a fixed $\vartheta \in (0, \infty)$ and fixed $j \in \{1, 2\}$, $R_j(\vartheta, c_j)$ achieves its minimum at $c_j^*(\vartheta) = M_j(\vartheta)$, where

$$M_j(\vartheta) = \left[\frac{E\left(I_{A_j}(\boldsymbol{X})\right)}{E\left(\left(\frac{X_1}{\mu_j}\right)^q I_{A_j}(\boldsymbol{X})\right)}\right]^{\frac{1}{q}}, \qquad j = 1, 2.$$

Now, using the p.d.f. of X_j, as defined in (1), obtain,

$$M_1(\vartheta)=\begin{cases}\left[\frac{1-\left(\frac{n_2}{n_1+n_2}\right)\left(\frac{\nu}{\vartheta}\right)^{n_1}}{\left(\frac{n_1}{n_1+q}\right)\left\{1-\left(\frac{n_2}{n_1+n_2+q}\right)\left(\frac{\nu}{\vartheta}\right)^{n_1+q}\right\}}\right]^{\frac{1}{q}}, & \text{if} \quad \vartheta>\nu,\\ \left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}}, & \text{if} \quad \vartheta\le\nu,\end{cases}$$

and

$$M_2(\vartheta)=\begin{cases}\left[\frac{1-\left(\frac{n_1}{n_1+n_2}\right)\left(\frac{\vartheta}{\nu}\right)^{n_2}}{\left(\frac{n_2}{n_2+q}\right)\left\{1-\left(\frac{n_1}{n_1+n_2+q}\right)\left(\frac{\vartheta}{\nu}\right)^{n_2+q}\right\}}\right]^{\frac{1}{q}}, & \text{if} \quad \vartheta\le\nu,\\ \left(\frac{n_1+n_2+q}{n_1+n_2}\right)^{\frac{1}{q}}, & \text{if} \quad \vartheta>\nu.\end{cases}$$

It is noticed that $M_1(\vartheta)$ and $M_2(\vartheta)$ are non-increasing and continuous function of $\vartheta\in(0,\infty)$. Therefore, $c_1^*(\vartheta)$ and $c_2^*(\vartheta)$ are non-increasing functions of ϑ, and $\sup_{\vartheta\in(0,\infty)} c_1^*(\vartheta)=\left(\frac{n_1+q}{n_1}\right)^{\frac{1}{q}}$ and $\sup_{\vartheta\in(0,\infty)} c_2^*(\vartheta)=\left(\frac{n_2+q}{n_2}\right)^{\frac{1}{q}}$. It is worth noting that, fixed $j=\{1,2\}$, and for any fixed $\vartheta\in(0,\infty)$, the risk function of $R_j(\vartheta,c)$ is a decreasing function of $c\in(0,c_j^*)$, and is an increasing function of $c\in[c_j^*,\infty)$ with $c_j^*\le\left(\frac{n_j+q}{n_j}\right)^{\frac{1}{q}}$. Therefore, for $c_j\ge\left(\frac{n_j+q}{n_j}\right)^{\frac{1}{q}}$,

$$R_j(\vartheta,c_j)>R_j\left(\vartheta,\left(\frac{n_j+q}{n_j}\right)^{\frac{1}{q}}\right)\quad\forall\ \vartheta\in(0,\infty).$$

This implies that

$$\begin{aligned}R(\vartheta,\xi_c)&=\sum_{j=1}^{2}R_j(\vartheta,c_j)\\&>\sum_{j=1}^{2}R_j\left(\vartheta,\left(\frac{n_j+q}{n_j}\right)^{\frac{1}{q}}\right)\\&=R(\vartheta,\xi_d)\ \ \forall\ \vartheta\in(0,\infty),\end{aligned}$$

where

$$\xi_{\boldsymbol{d}}(X_1,X_2)=\begin{cases}\left(\frac{n_1+q}{n_1}\right)^{\frac{1}{q}}X_1, & \text{if} \quad \mathbf{X}\in A_1,\\ \left(\frac{n_2+q}{n_2}\right)^{\frac{1}{q}}X_2, & \text{if} \quad \mathbf{X}\in A_2.\end{cases}$$

Hence, the proof of the theorem. ∎

Note: If we consider $q=1$, then GSL function becomes Stein loss function, and we can conclude the following results by using similar technique in this article.

(1)

$$\xi_U(\boldsymbol{X}) = \sum_{i=1}^{k} X_i \left[1 + \frac{1}{n_i} \left\{ 1 - \left(\frac{\max_{j \neq i} \nu_j X_j}{\nu_i X_i} \right)^{n_i} \right\} \right] I_{A_i(\boldsymbol{X})} \tag{19}$$

is the UMRU estimator of μ_L.

(2)

$$\xi^{GB}(\boldsymbol{x}) = \sum_{i=1}^{n} \left[\frac{(1+n_i)x_i}{n_i} \right] I_{A_i}(\boldsymbol{X})$$

is the generalized Bayes estimator and natural estimator $\xi_{N,2}(\boldsymbol{X})$.

(3) It should be noted here that we obtained the Theorem 4.2, Corollary 4.1 and Corollary 4.2 in this case.

(4) The natural estimators $\xi_{\boldsymbol{c}}(X_1, X_2)$ which is defined in Theorem 4.2, is inadmissible for estimating μ_L, if and only if $\frac{n_1+n_2+1}{n_1+n_2} \leq c \leq \frac{n_i+1}{n_i}$, for $i = 1, 2$.

Remark 4.4.

Consider equal sample sizes, i.e., $n_1 = n_2 = \cdots = n_k = n$ (say), and $\nu_1 = \nu_2 = \cdots = \nu_k = 1$. Then, from (19) that the UMRU estimator of μ_L is

$$\xi_U(\boldsymbol{X}) = \frac{X_{[k]}}{n} \left[n + 1 - \left(\frac{X_{[k-1]}}{X_{[k]}} \right)^n \right].$$

This UMRU estimator depends only on two largest order statistics.

Proof:

Consider $q = 1$. Then, the proof is obtained directly from equation (19) by the substitution of $k = 2$ and $n_1 = n_2 = n$. ■

5. Simulation Study

A simulation study is carried out using the MATLAB Software to evaluate the performance of the suggested estimators in previous sections under the GSL function. For $k = 2$ and $\vartheta = \frac{\mu_2}{\mu_1}$, it can be observed that the risk function of all the estimators depend on (μ_1, μ_2). The risks of the estimators $\xi_U(\boldsymbol{X})$, $\xi_U^D(\boldsymbol{X})$, $\xi_{N,1}$, $\xi_{N,1}^{ID}$ and $\xi_{N,2}$ of scale parameter μ_L are calculated. For simulation purpose, we take into account the minimax selection rule $\boldsymbol{\delta}^{\nu^*}$, as defined in Section 2 to choose the best population. It should be noted that the $\nu^* = \nu^*(n_1, n_2)$ is a function of n_1 and n_2. It is also noticed that ν^* depends on the different sample sizes n_1 and n_2, then seen that $\boldsymbol{\delta}^{\nu^*}$ is not same for various configurations of (n_1, n_2). For various sample sizes, we investigate the risk performances of the five competing estimators of μ_L for various values of ϑ. $R_1(\vartheta) = R(\vartheta, \xi_U(\boldsymbol{X}))$, $R_2(\vartheta) = R(\vartheta, \xi_U^D(\boldsymbol{X}))$, $R_3(\vartheta) = R(\vartheta, \xi_{N,1}(\boldsymbol{X}))$, $R_4(\vartheta) = R(\vartheta, \xi_{N,1}^{ID}(\boldsymbol{X}))$, and $R_5(\vartheta) = R(\vartheta, \xi_{N,2}(\boldsymbol{X}))$ represent the risk functions of the different estimators. The risk functions of proposed estimators are graphed for $(n_1, n_2) \in \{(3,4), (4,3), (5,8), (8,5)\}$. The following observations can be made from the Figures $1-8$ (Appendix $(1-2)$) as well as from Tables $1-8$ (Appendix $(3-5)$).

(1) For $q = 1$, the natural estimator $\xi_{N,1}$ is dominated by all the other estimators.
(2) For $q = -1$, the natural estimator $\xi_{N,1}$ is dominated by all the other estimators except $\xi_{N,2}$.
(3) The improved estimator ξ_U^D provides marginal improvement over the UMRU estimator ξ_U.
(4) The improved estimator $\xi_{N,1}^{ID}$ gives considerable improvement over the natural estimator $\xi_{N,1}$.
(5) For $0 < \vartheta < 0.8, 1.4 < \vartheta$ and $q = 1$, the estimator $\xi_{N,2}$ becomes better than all other estimators for all values of n_1, n_2.
(6) For $0 < \vartheta < 0.6$, $1.6 < \vartheta$ and $q = -1$, the estimator $\xi_{N,2}$ performs better than all other estimators when the values of n_1 and n_2 are $(3, 4)$ and $(4, 3)$.
(7) For $0 < \vartheta < 0.8$, $1.4 < \vartheta$ and $q = -1$, the estimator $\xi_{N,2}$ performs better than all other estimators when (n_1, n_2) is $(5, 8)$ and $(8, 5)$.
(8) The estimators ξ_U, ξ_U^D and $\xi_{N,1}^{ID}$ perform better for moderate values of ϑ.

Here, it is noted from the overall performance of all the estimators that the performance of $\xi_{N,1}^{ID}$ is satisfactory. Therefore, estimator $\xi_{N,1}^{ID}$ is recommended for use in practical applications.

6. Conclusion

This article focused on the problem of estimating scale parameter of the selected uniform population using the GSL function with unequal sample sizes has been addressed. We have derived the UMRU and generalized Bayes estimators for scale parameter of the selected uniform population with respect to the GSL function. We have demonstrated that the scale invariant estimators are inadmissible. Also, UMRU and natural estimators are inadmissible and dominated. Furthermore, the comparison among the estimators have been shown using simulation. Through simulation, we compared the estimators with respect to GSL function. It is observed that the risk function fourth performs very well and provides significant improvement over the risk function third for the case $q = 1$. In this article, we could not find general result giving a sufficient condition for the inadmissibility of scale-invariant estimators, it is an open problem for the general cases $k(> 2)$.

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Appendices

Appendix 1

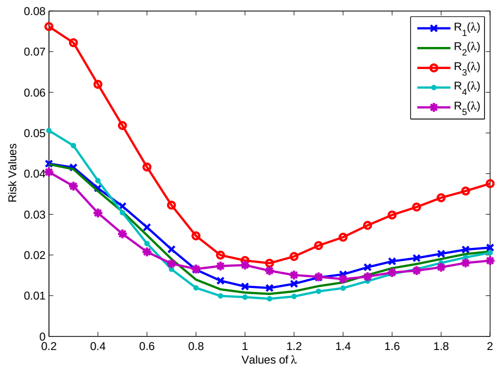

Figure 1. Risk performances of different estimators for $(n_1, n_2) = (3, 4)$ and $q = 1$

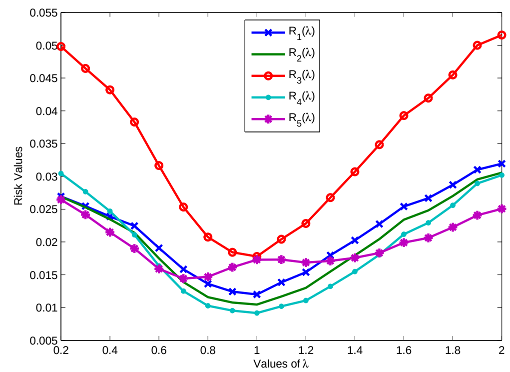

Figure 2. Risk performances of different estimators for $(n_1, n_2) = (4, 3)$ and $q = 1$

Appendix 2

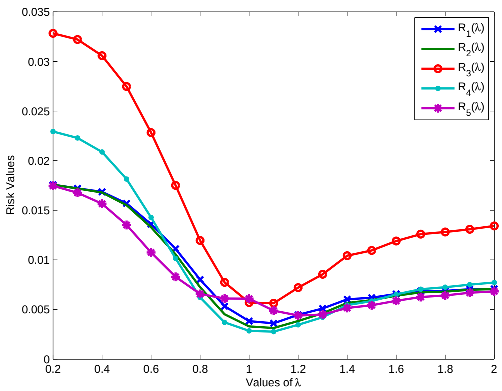

Figure 3. Risk performances of different estimators for $(n_1, n_2) = (5, 8)$ and $q = 1$

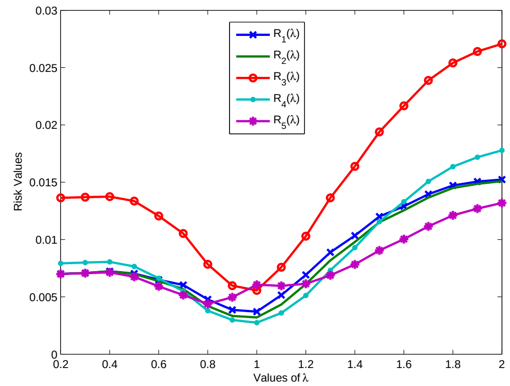

Figure 4. Risk performances of different estimators for $(n_1, n_2) = (8, 5)$ and $q = 1$

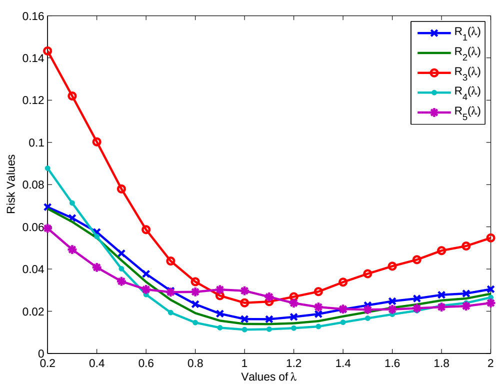

Figure 5. Risk performances of different estimators for $(n_1, n_2) = (3, 4)$ and $q = -1$

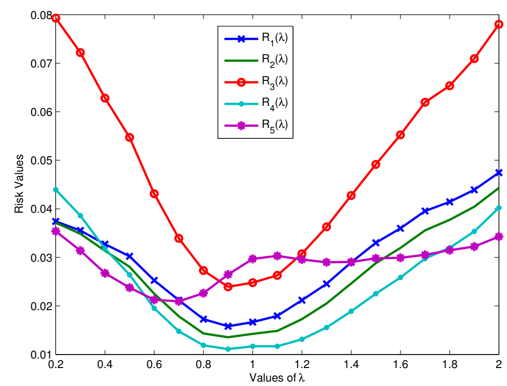

Figure 6. Risk performances of different estimators for $(n_1, n_2) = (4, 3)$ and $q = -1$

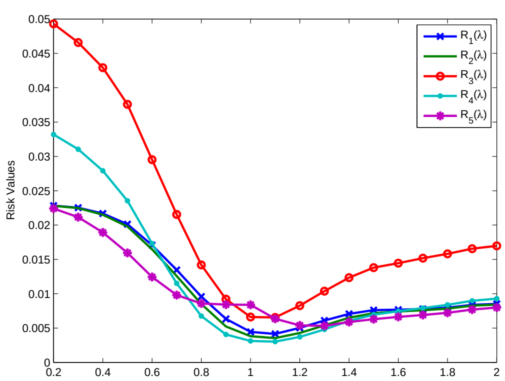

Figure 7. Risk performances of different estimators for $(n_1, n_2) = (5, 8)$ and $q = -1$

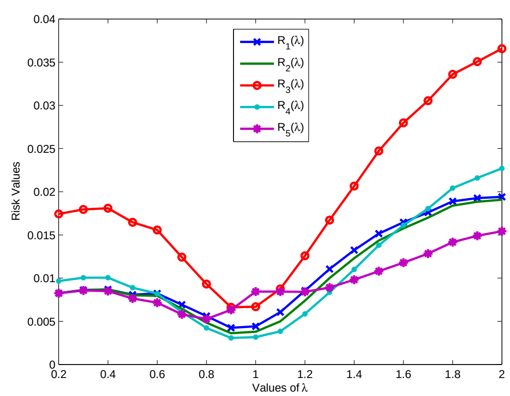

Figure 8. Risk performances of different estimators for $(n_1, n_2) = (8, 5)$ and $q = -1$

Appendix 3

Table 1. Risk performances of various estimators for $q = 1$ and different values of $\vartheta = \frac{\vartheta_2}{\vartheta_1}$

		$(n_1, n_2) = (3, 4); a^* = 0.9565$			
ϑ	$R(\vartheta, \xi_U)$	$R(\vartheta, \xi_U^D)$	$R(\vartheta, \xi_{N,1})$	$R(\vartheta, \xi_{N,1}^{ID})$	$R(\vartheta, \xi_{N,2})$
0.2	0.04416	0.04399	0.07833	0.05248	0.04189
0.4	0.03598	0.03520	0.06086	0.03748	0.02994
0.6	0.02600	0.02419	0.04094	0.02228	0.02040
0.8	0.01690	0.01445	0.02487	0.01227	0.01711
1.0	0.01188	0.01041	0.01769	0.00908	0.01726
1.2	0.01318	0.01128	0.01999	0.01011	0.01534
1.4	0.01572	0.01363	0.02471	0.01219	0.01442
1.6	0.01796	0.01634	0.02970	0.01505	0.01526
1.8	0.02022	0.01895	0.03356	0.01785	0.01701
2.0	0.02230	0.02124	0.03798	0.02086	0.01881

Table 2. Risk performances of various estimators for $q = 1$ and different values of $\vartheta = \frac{\vartheta_2}{\vartheta_1}$

		$(n_1, n_2) = (4, 3); a^* = 1.0455$			
ϑ	$R(\vartheta, \xi_U)$	$R(\vartheta, \xi_U^D)$	$R(\vartheta, \xi_{N,1})$	$R(\vartheta, \xi_{N,1}^{ID})$	$R(\vartheta, \xi_{N,2})$
0.2	0.02773	0.02769	0.05062	0.03123	0.02724
0.4	0.02479	0.02427	0.04360	0.02525	0.02212
0.6	0.01927	0.01774	0.03137	0.01639	0.01625
0.8	0.01347	0.01138	0.02086	0.01018	0.01447
1.0	0.01209	0.01063	0.01824	0.00939	0.01732
1.2	0.01585	0.01339	0.02359	0.01158	0.01703
1.4	0.02043	0.01810	0.03080	0.01555	0.01762
1.6	0.02426	0.02234	0.03792	0.02010	0.01923
1.8	0.02892	0.02724	0.04580	0.02583	0.02235
2.0	0.03185	0.03051	0.05172	0.03026	0.02500

Table 3. Risk performances of various estimators for $q = 1$ and different values of $\vartheta = \frac{\vartheta_2}{\vartheta_1}$

		$(n_1, n_2) = (5, 8); a^* = 0.9593$			
ϑ	$R(\vartheta, \xi_U)$	$R(\vartheta, \xi_U^D)$	$R(\vartheta, \xi_{N,1})$	$R(\vartheta, \xi_{N,1}^{ID})$	$R(\vartheta, \xi_{N,2})$
0.2	0.01753	0.01753	0.03305	0.02306	0.01743
0.4	0.01696	0.01690	0.03102	0.02117	0.01575
0.6	0.01388	0.01352	0.02335	0.01466	0.01091
0.8	0.00781	0.00708	0.01174	0.00606	0.00649
1.0	0.00384	0.00332	0.00576	0.00287	0.00607
1.2	0.00441	0.00380	0.00713	0.00343	0.00440
1.4	0.00579	0.00539	0.00994	0.00510	0.00490
1.6	0.00665	0.00649	0.01209	0.00665	0.00595
1.8	0.00705	0.00697	0.01308	0.00745	0.00660
2.0	0.00692	0.00688	0.01312	0.00751	0.00665

Appendix 4

Table 4. Risk performances of various estimators for $q = 1$ and different values of $\vartheta = \frac{\vartheta_2}{\vartheta_1}$

	$(n_1, n_2) = (8, 5); a^* = 1.0424$				
ϑ	$R(\vartheta, \xi_U)$	$R(\vartheta, \xi_U^D)$	$R(\vartheta, \xi_{N,1})$	$R(\vartheta, \xi_{N,1}^{ID})$	$R(\vartheta, \xi_{N,2})$
0.2	0.00730	0.00730	0.01401	0.00825	0.00729
0.4	0.00710	0.00709	0.01365	0.00794	0.00702
0.6	0.00674	0.00659	0.01221	0.00680	0.00610
0.8	0.00475	0.00417	0.00773	0.00374	0.00436
1.0	0.00382	0.00331	0.00566	0.00285	0.00613
1.2	0.00704	0.00622	0.01038	0.00520	0.00624
1.4	0.01077	0.01015	0.01702	0.00974	0.00796
1.6	0.01333	0.01295	0.02217	0.01371	0.01030
1.8	0.01447	0.01422	0.02493	0.01600	0.01195
2.0	0.01549	0.01534	0.02761	0.01819	0.01340

Table 5. Risk performances of various estimators for $q = -1$ and different values of $\vartheta = \frac{\vartheta_2}{\vartheta_1}$

	$(n_1, n_2) = (3, 4); a^* = 0.9565$				
ϑ	$R(\vartheta, \xi_U)$	$R(\vartheta, \xi_U^D)$	$R(\vartheta, \xi_{N,1})$	$R(\vartheta, \xi_{N,1}^{ID})$	$R(\vartheta, \xi_{N,2})$
0.2	0.06931	0.06828	0.14297	0.08734	0.05885
0.4	0.05672	0.05410	0.10120	0.05575	0.04033
0.6	0.03794	0.03388	0.05882	0.02787	0.03015
0.8	0.02378	0.01952	0.03437	0.01493	0.02961
1.0	0.01644	0.01412	0.02420	0.01141	0.02989
1.2	0.01752	0.01472	0.02680	0.01225	0.02425
1.4	0.02091	0.01738	0.03342	0.01448	0.02087
1.6	0.02451	0.02141	0.04130	0.01841	0.02069
1.8	0.02726	0.02486	0.04800	0.02236	0.02208
2.0	0.03001	0.02793	0.05467	0.02637	0.02373

Table 6. Risk performances of various estimators for $q = -1$ and different values of $\vartheta = \frac{\vartheta_2}{\vartheta_1}$

	$(n_1, n_2) = (4, 3); a^* = 1.0455$				
ϑ	$R(\vartheta, \xi_U)$	$R(\vartheta, \xi_U^D)$	$R(\vartheta, \xi_{N,1})$	$R(\vartheta, \xi_{N,1}^{ID})$	$R(\vartheta, \xi_{N,2})$
0.2	0.03704	0.03691	0.07942	0.04393	0.03524
0.4	0.03339	0.03194	0.06345	0.03242	0.02735
0.6	0.02546	0.02262	0.04328	0.01961	0.02119
0.8	0.01794	0.01474	0.02800	0.01230	0.02259
1.0	0.01608	0.01390	0.02382	0.01122	0.02987
1.2	0.02157	0.01764	0.03102	0.01351	0.03004
1.4	0.02931	0.02505	0.04336	0.01938	0.02954
1.6	0.03642	0.03224	0.05578	0.02612	0.02993
1.8	0.04241	0.03875	0.06792	0.03344	0.03153
2.0	0.04534	0.04243	0.07561	0.03846	0.03338

Appendix 5

Table 7. Risk performances of various estimators for $q = -1$ and different values of $\vartheta = \frac{\vartheta_2}{\vartheta_1}$

$(n_1, n_2) = (5, 8); a^* = 0.9593$					
ϑ	$R(\vartheta, \xi_U)$	$R(\vartheta, \xi_U^D)$	$R(\vartheta, \xi_{N,1})$	$R(\vartheta, \xi_{N,1}^{ID})$	$R(\vartheta, \xi_{N,2})$
0.2	0.02231	0.02231	0.04886	0.03275	0.02201
0.4	0.02175	0.02157	0.04328	0.02814	0.01892
0.6	0.01716	0.01656	0.02927	0.01719	0.01255
0.8	0.00976	0.00868	0.01460	0.00706	0.00863
1.0	0.00435	0.00374	0.00649	0.00306	0.00844
1.2	0.00517	0.00436	0.00838	0.00380	0.00543
1.4	0.00687	0.00634	0.01217	0.00589	0.00573
1.6	0.00769	0.00740	0.01443	0.00742	0.00660
1.8	0.00802	0.00788	0.01586	0.00841	0.00726
2.0	0.00830	0.00823	0.01666	0.00906	0.00779

Table 8. Risk performances of various estimators for $q = -1$ and different values of $\vartheta = \frac{\vartheta_2}{\vartheta_1}$

$(n_1, n_2) = (8, 5); a^* = 1.0424$					
ϑ	$R(\vartheta, \xi_U)$	$R(\vartheta, \xi_U^D)$	$R(\vartheta, \xi_{N,1})$	$R(\vartheta, \xi_{N,1}^{ID})$	$R(\vartheta, \xi_{N,2})$
0.2	0.00846	0.00846	0.01783	0.00993	0.00846
0.4	0.00868	0.00864	0.01777	0.00991	0.00846
0.6	0.00790	0.00766	0.01535	0.00796	0.00688
0.8	0.00555	0.00481	0.00930	0.00422	0.00524
1.0	0.00437	0.00377	0.00655	0.00310	0.00845
1.2	0.00830	0.00724	0.01231	0.00567	0.00841
1.4	0.01279	0.01193	0.02033	0.01072	0.00956
1.6	0.01669	0.01597	0.02824	0.01631	0.01184
1.8	0.01895	0.01840	0.03340	0.02030	0.01414
2.0	0.02062	0.02020	0.03794	0.02381	0.01609