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Analysis of $\mathrm{MAP}/\mathrm{PH}/1$ Queueing Model with Breakdown, Instantaneous Feedback and Server Vacation

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Abstract

In this article, we analyze a single server queueing model with feedback, a single vacation under Bernoulli schedule, breakdown and repair. The arriving customers follow the Markovian Arrival Process (MAP) and service follow the phase-type distribution. When the server returns from vacation, if there is no one present in the system, the server will wait until the customer's arrival. When the service completion epoch if the customer is not satisfied then that customer will get the service immediately. Under the steady-state probability vector that the total number of customers are present in the system is probed by the Matrix-analytic method. In our model, the stability condition, some system performance measures are discussed and we have examined the analysis of the busy period. Numerical results and some graphical representation are discussed for the proposed model.

Keywords: Markovian arrival process; Instantaneous feedback; Phase-type distributions; Breakdown; Repair; Single vacation; Matrix-analytic method

MSC (2010) No.: 60K25, 68M20, 90B22

1. Introduction

The Markovian Arrival Process is a rich class of point processes, a special class of tractable Markov renewal process that includes many well-known process such as Poisson, Markov-Modulated Poisson process and PH-renewal processes. One of the most significant features of the MAP is the underlying Markov structure and fits ideally in the context of matrix-analytic methods.

The Matrix-analytic methods (MAM) had been first introduced and examined by Neuts (1984). Chakravarthy (2010) has described the various types of arrivals in which the customer's arrival follows the Markovian Arrival Process (MAP) with representation (D_0, D_1) and the service times with representation (α, T) which follows phase type distributions whose matrix order is n. Let the generator Q be defined by Q = $D_0 + D_1$, an irreducible stochastic matrix. Here D_0 and D_1 are square matrices of order m, D_0 has non-negative off-diagonal elements and negative diagonal elements and it replicates there is no arrival, D_1 has both diagonal and off-diagonal elements are non-negative and it replicates there is an arrival in the system.

If π_1 is the unique probability vector of the Markov process described by the irreducible generator Q satisfying $\pi_1 Q = 0$ and $\pi_1 e = 1$. The constant $\lambda = \pi_1 D_1 e$ indicates the fundamental arrival rate of customers arrives per unit time based on the Markovian arrival process. The Versatile Markovian Point Process (VMPP) has been first introduced by Neuts (1979). The VMPP forms the devolution of MAP and Batch Markovian Arrival Process (BMAP).

In recent years, queueing models with server breakdown appears as one of the major areas of queueing theory. Ayyappan and Karpagam (2018) have investigated non-Markovian queue with a breakdown, second optional repair, standby server and multiple vacations. Chan et al. (1993) have analyzed the queueing model with breakdowns, and at the time of breakdown, the customer may become discouraged which leads the customer to leave the system with constant probability. Shoukry et al. (2018) have analyzed the matrix geometric method for the Markovian queueing system that the server may or may not struck with breakdown using the transition structure of the Markov chain.

Ayyappan and Deepa (2018) have studied the batch arrival and bulk service queueing model with multiple vacations and optional repair. Haghighi and Mishev (2016a) have studied external arrival tasks from outside to the system and internal arrival through immediate feedback or through a splitting process both have occurred with the concept of delay. After service completion, it may leave the system forever; otherwise, it seeks feedback or else it may go to a unit called a splitter. The splitter splits into two tasks. One task reveals a return to the service station and seek feedback with some probability, otherwise they leave out of the system with some other probability. Ayyappan and Nirmala (2018) have examined the non-Markovian queueing model and the server serves the customers based on general bulk service with the breakdown and two-phase repair under multiple vacations. Haghighi and Mishev (2013) have investigated the three-stage hiring queueing model with bulk arrival and Erlang phase-type selection.

The concept of instantaneous feedback explains after getting service from the service station, the customers may leave the system with satisfaction or otherwise, they will get service immediately once again. Haghighi and Mishev (2016b) have examined a single server service queueing system with delayed feedback and splitting. Badamchi Zadeh (2015) has studied a bulk arrival of multi-phase service queueing system with single vacation and random feedback in such a case the customers have enough to get service in the first phase itself otherwise if they would like to get feedback then they have to join the endpoint of the original queue and will procure service in the second phase. Similarly, they will get feedback service up to k phases and also examined a single vacation policy.

Krishna Kumar et al. (2008) have analyzed the Markovian arrival process and the phase-type service is provided by two servers with multiple vacations under the Bernoulli schedule. A vacation mechanism in a queueing system describes customers based on their classes. Such class of queues which extensively appears in a number of situations of everyday life in major production management and many real-life situations. Sreenivasan et al. (2013) have examined the $MAP/PH/1$ queueing model with N-Policy, vacation interruption and working vacations.

Wang et al. (2007) have investigated the batch Markovian queueing model with multiple vacations and the server struck with breakdown. A single server queue with service interruptions, arrivals based on Markovian arrival process (MAP) and the server is not available at certain times and the service times are considered to be general distribution have been analyzed by Sengupta and Takine (1996). Wortman et al. (1991) have studied $M/GI/1$ queueing model with feedback and server vacations. The study of queueing model with vacations and breakdown have been analyzed by Gray et al. (2000). A Multiphase non-Markovian queueing model with Bernoulli feedback and multi-server vacation studied by Maragathasundari and Srinivasan (2012). However, they have incorporated that after received service from the server, the customers who wish to get feedback have to rejoin at the endpoint of the queue, otherwise the satisfied customer leaves the system.

Madan (2003) has incorporated a single server queueing model with the concepts of breakdown and repair in which whenever the server struck with breakdown immediately go for repair proces. The server would affect by breakdown not only in the service time, but the breakdown also occurs while the server in the idle state and he has found the probability generating function that the number of customers in the queueing system.

Kulkarni and Choi (1990) have incorporated the two types of retrial queue with the concept of server breakdown in which the first model replicates that when the server struck with breakdown, the customer who currently rendering service from the server go to the orbit that is joins the retrial queue and the second model replicates that when the server struck with breakdown who rendering service from the server will be waiting in the system itself up to server repair completion. Renisagaya Raj and Chandrasekar (2015) have analyzed the matrix geometric method of queueing model with server breakdown, multiple vacations under N-policy and the service would start immediately after the vacation completion. Jain and Agrawal (2009) have studied the bulk arrival queueing system with multiple breakdowns during the busy period of the system under N-policy. The repair process has a minimum number of stages and they using a matrix geometric method to compute the characteristics of system performance measures.

In this paper, the main motivation come from the scenarios of manufacturing unit. For instance, in Manufacturing units, Computer systems, Communication units, Production units and many other types of units which might suddenly breakdown. In Manufacturing unit, the server can breakdown at any moment during busy period in that instant, the server go to repair for rejuvenation immediately. In general, during the repair period, the customers have choice that they may freeze in the system up to repair completion of the server. Furthermore, after completing the working process in the Manufacturing unit, some amount of time the machines would be in the vacation period. After received service from the server, if a person who aspires to get feedback from the manufacturing unit, obviously they will get service once again immediately. The Queueing models with phase-type service have been investigated by resolve the crowded congestion in many real-life situations and industrial frameworks.

Similarly, in the case of the banking sectors, if they are doing any of the works in the computers or printers suddenly the server may be struck with breakdown. Then after completion of the repair process, they will do the same job from the starting point. For instance, while preparing Demand Draft (DD) the server (computer or printer) breakdown may happen then after repair completion of any small adjustment in the system then they will start the work freshly. This sort of incident is suitable for our model.

The rest of the structure of this article is as follows. In Section 2, we formulate the mathematical model description. In Section 3, the notations and matrix formation for our model are given. In Section 4, we discuss the stability condition and the steady-state probability vector. In Section 5, we analyze the busy period for our model. In Section 6, measures of system performance are described. In Section 7, we discuss some special cases and in Section 8, we illustrate some numerical and graphical representations. Then the conclusion is given in Section 9.

2. Mathematical Model Description

In this model, arrival follows the Markovian arrival process(MAP) with the representation (D_0, D_1) whose square matrix of order is 'm'. The matrix D_0 governs transitions for no arrival and the matrix D_1 governs transitions for arrival and the service process follows phase-type distribution with representation (α, T) whose matrix order is 'n' with $T^0 + Te = 0$ implies that $T^0 = -Te$. Let us denote λ be the arrival rate and is defined by $\lambda = \pi_1 D_1 e_m$ where π_1 is the stationary probability vector of the generator matrix $D = D_0 + D_1$. Then the normal service rate is denoted by $\delta = [\alpha(-T)^{-1}e_n]^{-1}$.

We describe a single server queueing model with instantaneous feedback, breakdown, repair and server vacation. After received service from the server, the customer has satisfaction with probability d_1 otherwise the customer with dissatisfaction who is aspiring for getting service one more time immediately and they will get service from the server with probability c_1 such that $c_1 + d_1 = 1$. The server may affect by the breakdown at any time while offering service to the customers. In that situation, the breakdown server immediately goes for the repair process. In the event of a server breakdown, the customer who currently rendering service from the server will remain in the frozen state until the server would get rid of it from repair. When the server completes the repair, the server will start a new service for the frozen customer.

Breakdown times and repair times follow an exponential distribution with parameter σ and Ψ, respectively. Whenever the service completion epoch the server can go for vacation with probability c_2 otherwise the server will be in the system itself and start the service to those who are

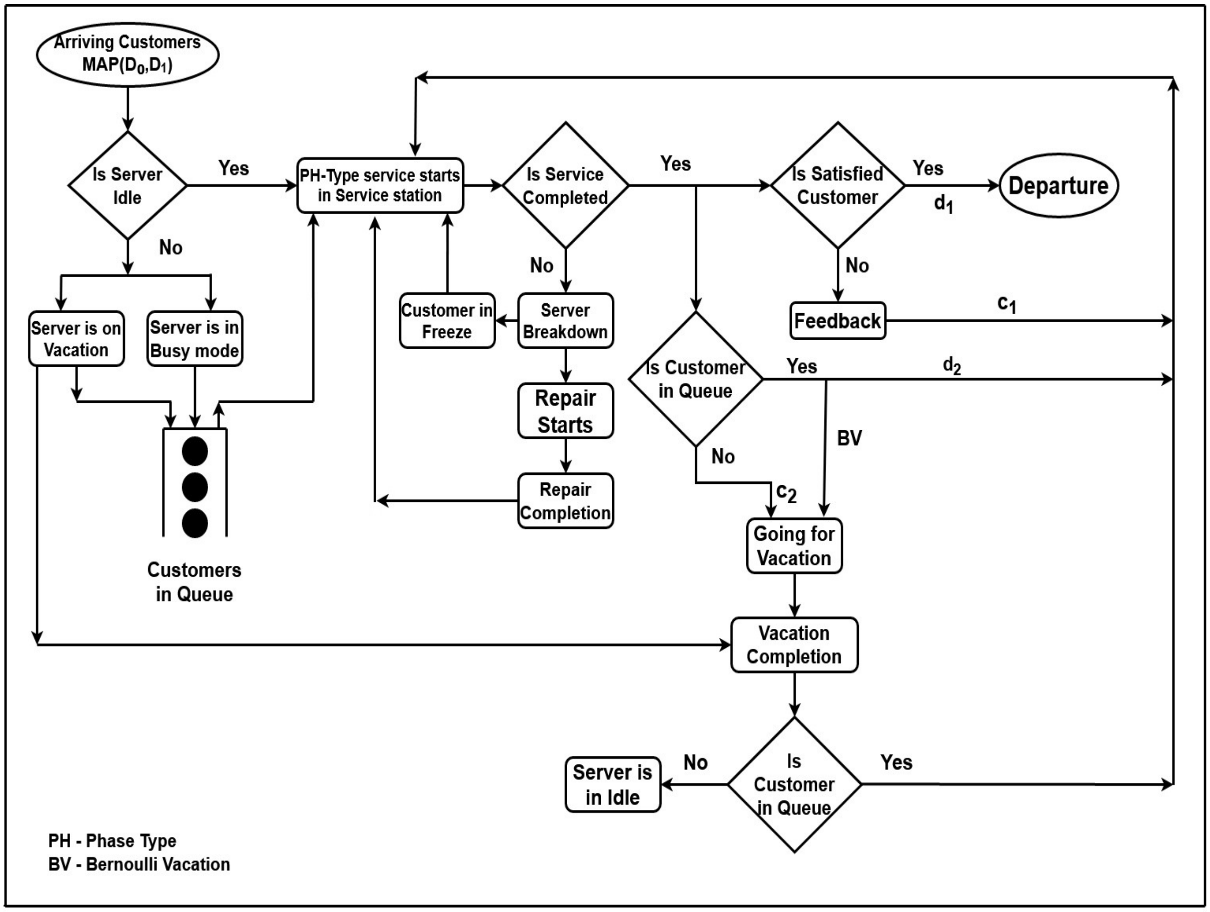

Figure 1. Pictorial representation of our proposed model

standing next to in the queue. Suppose, if there is no customer in the system, the server will be in an idle state up to customer's arrival with probability d_2 such that $c_2 + d_2 = 1$. When the vacation period is over, the server will start the service at the service station whether the customer available in the queue or else the server will be in the idle state until the customer arrive at the system. The vacation period follows an exponential distribution with parameter η, respectively (see Figure 1).

The overall motivation and goal of this paper comes from the manufacturing unit in industries in which the machines in manufacturing unit is considered as server and analyse the scenarios of how the machines has affected by breakdown and it has rejuvenated from repair process. However, we discussed the concepts of instantaneous feedback in manufacturing unit. We derived the stability condition and steady-state probability vector whereas the system required to be stable and we described the busy period of the system. Finally, we derived the performance characteristics of the system and the computation of numerical and graphical representations in an elegant manner.

3. The Matrix Generation - QBD process

In this section, we describe the following notation needed to describe our model for the purpose of generating QBD Process.

Notations for our model

- $\otimes$ - Kronecker product of two different dimensions of matrices using this symbol.
- $\oplus$ - Kronecker sum of two different dimensions of matrices using this symbol.
- I_m - It indicates an $(m \times m)$ Identity matrix.
- e - Column vector of suitable dimension each of its entry is 1.
- e_n - Column vector whose dimension $(n \times 1)$ each of its entries is 1.
- $N(t)$ indicates the number of customers in the system at time t.
- $V(t)$ indicates the status of the server at time t, where

$$V(t) = \begin{cases} 0, & \text{if the server is idle.} \\ 1, & \text{if the server is on vacation.} \\ 2, & \text{if the server is in busy mode.} \\ 3, & \text{if the server is in the breakdown.} \end{cases}$$

- $J(t)$ indicates the service process considered by phases.
- $I(t)$ indicates the Markovian arrival process considered by phases.
- Let $\{(N(t), V(t), J(t), I(t)) \; : \; t \; \geqslant \; 0\}$ is continuous time Markov chain with state level independent Quasi-Birth-and-Death process whose state space is as follows,

$$\Phi = l(0) \cup l(p), \tag{1}$$

where

$$l(0) = \{(0,0,s) : 1 \leq s \leq m\} \cup \{(0,1,s) : 1 \leq s \leq m\},$$

and for $p \geq 1$,

$$l(p) = \{(p,1,s): 1 \leq s \leq m\} \cup \{(p,2,r,s): 1 \leq r \leq n; 1 \leq s \leq m\} \cup \{(p,3,s) : 1 \leq s \leq m\}.$$

The QBD process of infinitesimal generator matrix is given by:

$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots\cdots \\ B_{10} & A_1 & A_0 & 0 & 0 & 0 & \cdots\cdots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \cdots\cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots\cdots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2}$$

The block matrices of Q are defined as follows:

$$B_{00} = \begin{pmatrix} D_0 & 0 \\ \eta I_m & D_0 - \eta I_m \end{pmatrix}; \quad B_{01} = \begin{pmatrix} 0 & \alpha \otimes D_1 & 0 \\ D_1 & 0 & 0 \end{pmatrix};$$

$$B_{10} = \begin{pmatrix} 0 & 0 \\ d_2 d_1 T^0 \otimes I_m & c_2 d_1 T^0 \otimes I_m \\ 0 & 0 \end{pmatrix};$$

$$A_1 = \begin{pmatrix} D_0 - \eta I_m & \alpha \otimes \eta I_m & 0 \\ c_2 c_1 T^0 \otimes I_m & (T + d_2 c_1 T^0 \alpha) \oplus D_0 - \sigma I_{nm} & e_n \otimes \sigma I_m \\ 0 & \alpha \otimes \Psi I_m & D_0 - \Psi I_m \end{pmatrix};$$

$$A_0 = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & I_n \otimes D_1 & 0 \\ 0 & 0 & D_1 \end{pmatrix}; \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ c_2 d_1 T^0 \otimes I_m & d_2 d_1 T^0 \alpha \otimes I_m & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

4. Stability Condition

We analyze our model under the condition that the system is stable.

4.1. Analysis of Stability condition

Let us define the matrix A as $A = A_0 + A_1 + A_2$. It is an arrangement of the square matrix A of the order is $(2m + nm)$. Let us define $\boldsymbol{\xi}$ be the steady-state probability vector of A and it satisfies the condition $\boldsymbol{\xi}$A = 0 and $\boldsymbol{\xi}$e = 1. The vector $\boldsymbol{\xi}$ is partitioned by $\boldsymbol{\xi}$ = (ξ_0, ξ_1, ξ_2), where ξ_0 and ξ_1 are of dimension m and ξ_2 is of dimension nm.

The condition $\boldsymbol{\xi} A_0 e < \boldsymbol{\xi} A_2 e$ is the necessary and sufficient condition for stability of the queueing system. The vector ξ is obtained by solving the following equations

$$\xi_0[D - \eta I_m] + \xi_1[c_2 T^0 \otimes I_m] = 0, \tag{3}$$

$$\xi_0[\alpha \otimes \eta I_m] + \xi_1[(T + d_2 T^0 \alpha) \oplus (D_0) - \sigma I_{nm}] + \xi_2[\alpha \otimes \Psi I_m] = 0, \tag{4}$$

$$\xi_1[e_n \otimes \sigma I_m] + \xi_2[D - \Psi I_m] = 0, \tag{5}$$

subject to the normalizing condition

$$\xi_0 e_m + \xi_1 e_{nm} + \xi_2 e_m = 1. \tag{6}$$

After some algebraical manipulation, the stability condition which is turned to be

$$\{(\xi_0+\xi_2)[D_1 e_m]+\xi_1[e_n\otimes D_1 e_m]\}<\{\xi_1[d_1 T^0\otimes e_m]\}. \tag{7}$$

4.2. Analysis of the Steady-state probability vector

Let us take the variable **x** is the probability vector and it is subdivided as **x** = ($\mathbf{x}_0$,$\mathbf{x}_1$,$\mathbf{x}_2$, ...). Thus, **x** be the steady-state probability vector of the QBD. Here we mention $\mathbf{x}_0$ is of dimension 2m and $\mathbf{x}_1$,$\mathbf{x}_2$,$\mathbf{x}_3$,... are of dimension $(2m+mn)$. Then the probability vector **x** satisfies the condition **x**Q = 0 and **x**e = 1. Moreover, when the stability condition has satisfied with the subvectors of **x** except for $\mathbf{x}_0$ and $\mathbf{x}_1$, commensurate to the different level states are given by the equation as:

$$\mathbf{x}_j=\mathbf{x}_1 R^{j-1},\quad j\geqslant 2, \tag{8}$$

where the matrix R represents the least non-negative solution of the matrix quadratic equation '$R^2A_2+RA_1+A_0=0$' and satisfies the relation $RA_2e = A_0e$. The subvectors $\mathbf{x}_0$ and $\mathbf{x}_1$ are obtained by solving the following equations:

$$\mathbf{x}_0 B_{00}+\mathbf{x}_1 B_{10}=0, \tag{9}$$

$$\mathbf{x}_0 B_{01}+\mathbf{x}_1(A_1+RA_2)=0, \tag{10}$$

subject to the normalizing condition

$$\mathbf{x}_0 e_{2m}+\mathbf{x}_1(I-R)^{-1}e_{2m+nm}=1. \tag{11}$$

Therefore, we can mathematically compute the R matrix using the essential steps of the logarithmic reduction algorithm.

Logarithmic Reduction Algorithm

Step 0:

$H\leftarrow(-A_1)^{-1}A_0, L\leftarrow(-A_1)^{-1}A_2, G=L$, and $T=H$.

Step 1:
$U=HL+LH$
$M=H^2$
$H\leftarrow(I-U)^{-1}M$
$M\leftarrow L^2$
$L\leftarrow(I-U)^{-1}M$
$G\leftarrow G+TL$
$T\leftarrow TH$
Continue Step 1 until $||e-Ge||_\infty<\epsilon$.

Step 2:
$R = -A_0(A_1 + A_0G)^{-1}$.

Theorem 4.1.

The structure of the rate matrix R is

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}. \tag{12}$$

Proof:

In the computation of the R matrix, it is clearly shown that the R matrix must have the structure for our model as given in (12) in which the server may be struck with breakdown while offering service immediately go for a repair process which leads to the customer who renders service from the server would be in a frozen state up to the server return to the service station. Furthermore, here we will give proof of the construction of R. We can rewrite the matrix quadratic equation $R^2A_2 + RA_1 + A_0 = 0$ is given by

$$R = (R^2A_2 + A_0)(-A_1)^{-1}.$$

It can easily verify that the structure of the matrix $(-A_1)^{-1}$ as follows:

$$(-A_1)^{-1} = \frac{1}{V}\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}, \tag{13}$$

where the elements of $(-A_1)^{-1}$ are

$$f_{11} = [\alpha \otimes \Psi I_m][e_n \otimes \sigma I_m] - [(T + d_2c_1T^0\alpha) \oplus D_0 - \sigma I_{nm}][D_0 - \Psi I_m],$$

$$f_{12} = [\alpha \otimes \eta I_m][D_0 - \Psi I_m], \quad f_{13} = -[\alpha \otimes \eta I_m][e_n \otimes \sigma I_m], \quad f_{21} = [c_2c_1T^0 \otimes I_m][D_0 - \Psi I_m],$$

$$f_{22} = -[D_0 - \eta I_m][D_0 - \Psi I_m], \qquad f_{23} = [D_0 - \eta I_m][e_n \otimes \sigma I_m],$$

$$f_{31} = -[c_2c_1T^0 \otimes I_m][\alpha \otimes \Psi I_m], \qquad f_{32} = [D_0 - \eta I_m][\alpha \otimes \Psi I_m],$$

$$f_{33} = [\alpha \otimes \eta I_m][c_2c_1T^0 \otimes I_m] - [D_0 - \eta I_m][(T + d_2c_1T^0\alpha) \oplus D_0 - \sigma I_{nm}].$$

In the same way, pre-multiplying a diagonal block matrix with $(-A_1)^{-1}$ matrix won't change the structure as seen in (13). Hence, the structure of matrix $A_0(-A_1)^{-1}$ is given by

$$A_0(-A_1)^{-1} = \frac{1}{V}\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}, \tag{14}$$

where the elements of $A_0(-A_1)^{-1}$ are

$$g_{11} = D_1 f_{11}, \quad g_{12} = D_1 f_{12}, \quad g_{13} = D_1 f_{13}, \quad g_{21} = [I_n \otimes D_1] f_{21}, \quad g_{22} = [I_n \otimes D_1] f_{22},$$

$$g_{23} = [I_n \otimes D_1] f_{23}, \quad g_{31} = D_1 f_{31}, \quad g_{32} = D_1 f_{32}, \quad g_{33} = D_1 f_{33}.$$

Here, pre-multiply a block matrix A_2 with $(-A_1)^{-1}$ matrix. Therefore, the structure of matrix $A_2(-A_1)^{-1}$ is given by

$$A_2(-A_1)^{-1} = \frac{1}{V}\begin{bmatrix} 0 & 0 & 0 \\ h_{21} & h_{22} & h_{23} \\ 0 & 0 & 0 \end{bmatrix}, \tag{15}$$

where the elements of $A_2(-A_1)^{-1}$ are

$$h_{21} = [c_2 d_1 T^0 \otimes I_m] f_{11} + [d_2 d_1 T^0 \alpha \otimes I_m] f_{21},$$

$$h_{22} = [c_2 d_1 T^0 \otimes I_m] f_{12} + [d_2 d_1 T^0 \alpha \otimes I_m] f_{22},$$

$$h_{23} = [c_2 d_1 T^0 \otimes I_m] f_{13} + [d_2 d_1 T^0 \alpha \otimes I_m] f_{23}.$$

The sequence of $\{R^{(n)}, n = 0, 1, 2, 3, ...\}$ is defined by

$$R^{(n+1)} = \big[(R^{(n)})^2 A_2 + A_0\big](-A_1)^{-1}, \quad n = 0, 1, 2, 3, ...$$

The $R^{(0)} = 0$ monotonically converges to the minimal non-negative solution to the quadratic equation $R^2 A_2 + R A_1 + A_0 = 0$. Hence, the structure of $\{A_0(-A_1)^{-1}\}$ and $\{(R^{(n)})^2 A_2(-A_1)^{-1}$, where $n = 1, 2, 3, ...\}$ will remain the same as that of $(-A_1)^{-1}$. Using $R^{(0)} = 0$, we can compute the first iteration of R matrix, i.e., $R^{(1)}$. Then using the first iteration of R matrix we can compute the second iteration of R matrix, i.e., $R^{(2)}$. Similarly, we can compute the further iterations of the R matrix. Therefore, each iteration of the R matrix, i.e., $R^{(n)}$. It retains the same structure. ∎

5. Analysis of the Busy Period

- The busy period means that the interval between the customers arrives into the empty system and afterward the first interval once again the system becomes empty. So, it is the first passage from level 1 to 0. The busy cycle describes the first return time to level 0 with at least one visit to a state at any other level.

- Before we examine the busy period, we introduced an overview of the fundamental period. Under consideration of the QBD process, this is the first passage time from level j to level $j-1, j \geqslant 2$.
- The cases $j = 0, 1$ corresponding to the boundary states have to be discussed separately. Note that for each and every level j, $j \geqslant 1$ there corresponds $(2m + nm)$ states. Thus, by the state (j, k) of level j we mention that the k^{th} state of level j when the states are arrange in lexicographic order.
- Let $G_{kk'}(u, x)$ be the conditional probability that the QBD process visits the level $j-1$ to make changes to u transitions to the left and also enter the state (j, k') with the constraint that it start up in the state (j, k) at time t = 0.

Let us introduce the concept of joint transform

$$\overline{G}_{kk'}(z, s) = \sum_{u=1}^{\infty} z^u \int_0^{\infty} e^{-sx} dG_{kk'}(u, x) \quad ; \mid z \mid \leq 1, Re(s) \geq 0, \tag{16}$$

and the matrix is denoted as $\overline{G}(z, s) = \overline{G}_{kk'}(z, s)$. Then, the above-defined matrix $\overline{G}(z, s)$ satisfies the equation,

$$\overline{G}(z, s) = z(SI - A_1)^{-1} A_2 + (SI - A_1)^{-1} A_0 \overline{G}^2(z, s). \tag{17}$$

The matrix $G = G_{kk'} = \overline{G}(1, 0)$ is taken for the first passage times, exclude for the boundary states. If we already know the matrix R, we can find the matrix G using the result as given below

$$G = -(A_1 + RA_2)^{-1} A_2. \tag{18}$$

Otherwise, we may use the concept of a logarithmic reduction algorithm method to find the values of the G matrix.

Notations of boundary level states for busy Period

- $G_{kk'}^{(1,0)}(u, x)$ denotes the conditional probability discussed for the first passage time level 1 to level 0 at time $t = 0$.
- $G_{kk'}^{(0,0)}(u, x)$ denotes the conditional probability discussed for the first return time to level 0.
- $\mathscr{F}_{1j}$ denotes the average first passage time from the level j to level $j-1$, given that the process is in the state (j, k) at time $t = 0$.
- $\vec{\mathscr{F}}_1$ denotes the column vector with entries $\mathscr{F}_{1j}$.
- $\mathscr{F}_{2j}$ denotes the average number of customers required to serve during the first passage time from level j to level $j-1$, given that the first passage time begins in the state (j, k).
- $\vec{\mathscr{F}}_2$ denotes the column vector with entries $\mathscr{F}_{2j}$.
- $\vec{\mathscr{F}}_1^{(1,0)}$ denotes the average first passage time from level 1 to level 0.
- $\vec{\mathscr{F}}_2^{(1,0)}$ denotes the average number of service completed during the first passage time from level 1 to level 0.

- $\vec{\mathscr{F}}_1^{(0,0)}$ denotes the first return time to level 0.
- $\vec{\mathscr{F}}_2^{(0,0)}$ denotes the average number of service completes between first return time to level 0.

The following equations, which are given by $\overline{G}^{(1,0)}(z,s)$ and $\overline{G}^{(0,0)}(z,s)$, are for the boundary levels 1 and 0, respectively,

$$\overline{G}^{(1,0)}(z,s) = z(SI - A_1)^{-1}B_{10} + (SI - A_1)^{-1}A_0\overline{G}(z,s)\overline{G}^{(1,0)}(z,s), \tag{19}$$

$$\overline{G}^{(0,0)}(z,s) = (SI - B_{00})^{-1}B_{01}\overline{G}^{(1,0)}(z,s). \tag{20}$$

Thus, the following instances are calculated using the matrices as G, $\overline{G}^{(0,0)}(1,0)$ and $\overline{G}^{(1,0)}(1,0)$ are stochastic in nature. We can compute the instants as follows:

$$\vec{\mathscr{F}}_1 = -\frac{\partial}{\partial s}\overline{G}(z,s)\bigg|_{z=1,s=0} e = -[A_1 + A_0(I+G)]^{-1}e, \tag{21}$$

$$\vec{\mathscr{F}}_2 = \frac{\partial}{\partial z}\overline{G}(z,s)\bigg|_{z=1,s=0} e = -[A_1 + A_0(I+G)]^{-1}A_2 e, \tag{22}$$

$$\vec{\mathscr{F}}_1^{(1,0)} = -\frac{\partial}{\partial s}\overline{G}^{(1,0)}(z,s)\bigg|_{z=1,s=0} e = -[A_1 + A_0 G]^{-1}(A_0\vec{\mathscr{F}}_1 + e), \tag{23}$$

$$\vec{\mathscr{F}}_2^{(1,0)} = \frac{\partial}{\partial z}\overline{G}^{(1,0)}(z,s)\bigg|_{z=1,s=0} e = -[A_1 + A_0 G]^{-1}(A_0\vec{\mathscr{F}}_2 + B_{10}e), \tag{24}$$

$$\vec{\mathscr{F}}_1^{(0,0)} = -\frac{\partial}{\partial s}\overline{G}^{0,0}(z,s)\bigg|_{z=1,s=0} e = -B_{00}^{-1}[B_{01}\vec{\mathscr{F}}_1^{(1,0)} + e], \tag{25}$$

$$\vec{\mathscr{F}}_2^{(0,0)} = \frac{\partial}{\partial z}\overline{G}^{(0,0)}(z,s)\bigg|_{z=1,s=0} e = -B_{00}^{-1}[B_{01}\vec{\mathscr{F}}_2^{(1,0)}]. \tag{26}$$

6. Performance measures of the system

We investigate the qualitative behavior of our model under steady-state. In this section, we itemized a few system performance measures along with their expressions.

- The probability of the server is idle
  $P_{IDLE} = \sum_{s=1}^{m}\mathbf{x}_{00s}$
- The probability of the server is on vacation
  $P_{VAC} = \sum_{p=0}^{\infty}\sum_{s=1}^{m}\mathbf{x}_{p1s}$
- The probability of the server is busy in the system
  $P_B = \sum_{p=1}^{\infty}\sum_{r=1}^{n}\sum_{s=1}^{m}\mathbf{x}_{p2rs}$
- The probability of the server is in the breakdown

$P_{BD} = \sum_{p=1}^{\infty}\sum_{s=1}^{m}\mathbf{x}_{p3s}$

- Expected system size

  $\mu_{system} = \sum_{p=1}^{\infty} p\mathbf{x}_p e_{2m+nm} = \mathbf{x}_1 (I-R)^{-2} e_{2m+nm}$
- Expected Queue size during busy Period

  $\mu_{QSB} = \sum_{p=1}^{\infty}(p-1)\mathbf{x}_p e_{2m+nm} = \mathbf{x}_1 R(I-R)^{-2} e_{2m+nm}$
- Expected Queue size during the breakdown

  $\mu_{BD} = \sum_{p=1}^{\infty}\sum_{s=1}^{m} p\mathbf{x}_{p3s}$
- Expected queue size during vacation

  $\mu_{VAC} = \sum_{p=0}^{\infty}\sum_{s=1}^{m} p\mathbf{x}_{p1s}$
- Average queue size

  $\mu_{queue} = \mu_{BD} + \mu_{VAC} + \mu_{QSB}$

7. Particular Cases

(i) M/PH/1

The arrival follows the Poisson process. The inter-arrival times follow an exponential distribution with rate λ and the service follows phase type distribution with the representation (α, T) of order n. The service rate is given by $\delta = [\alpha(-T)^{-1}e]^{-1}$. Similarly, the vacation times, breakdown and repair times are exponentially distributed. In this type, the infinitesimal generator is thus reduced as

$$Q_1 = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots\cdots \\ B_{10} & A_1 & A_0 & 0 & 0 & 0 & \cdots\cdots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \cdots\cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots\cdots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{27}$$

The block matrices of Q_1 are defined as follows:

$$B_{00} = \begin{pmatrix} -\lambda & 0 \\ \eta & -(\lambda+\eta) \end{pmatrix}; \quad B_{01} = \begin{pmatrix} 0 & \alpha\lambda & 0 \\ \lambda & 0 & 0 \end{pmatrix}; \quad B_{10} = \begin{pmatrix} 0 & 0 \\ d_2 d_1 T^0 & c_2 d_1 T^0 \\ 0 & 0 \end{pmatrix};$$

$$A_1 = \begin{pmatrix} -(\lambda+\eta) & \alpha\eta & 0 \\ c_2 c_1 T^0 & T + d_2 c_1 T^0 \alpha - (\lambda+\sigma) I_n & \sigma \\ 0 & \alpha\Psi & -(\lambda+\Psi) \end{pmatrix}; \quad A_0 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda I_n & 0 \\ 0 & 0 & \lambda \end{pmatrix};$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ c_2 d_1 T^0 & d_2 d_1 T^0 \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Subsequently, the generator matrix of A is

$$A = \begin{pmatrix} -\eta & \alpha\eta & 0 \\ c_2 T^0 & T + d_2 T^0 \alpha - \sigma I_n & \sigma \\ 0 & \alpha\Psi & -\Psi \end{pmatrix}. \tag{28}$$

We consider the stationary probability vector $\boldsymbol{\xi}$ of A for equation (28) and it obeys $\boldsymbol{\xi}$A = 0 and $\boldsymbol{\xi}$e = 1. The vector $\boldsymbol{\xi}$ has partitioned as $\boldsymbol{\xi}$=(ξ_0, ξ_1, ξ_2) and one can easily find the values of $\boldsymbol{\xi}$ by solving the following equations:

$$\xi_0[-\eta] + \xi_1[c_2 T^0] = 0, \tag{29}$$

$$\xi_0[\alpha\eta] + \xi_1[T + d_2 T^0 \alpha - \sigma I_n] + \xi_2[\alpha\Psi] = 0, \tag{30}$$

$$\xi_1[\sigma] + \xi_2[-\Psi] = 0, \tag{31}$$

subject to the normalizing condition

$$\xi_0 + \xi_1 e_n + \xi_2 = 1. \tag{32}$$

After algebraical manipulation, $\boldsymbol{\xi} A_0 e < \boldsymbol{\xi} A_2 e$ is the necessary and sufficient condition to attain the stability condition which is turning to be

$$\{(\xi_0 + \xi_2)[\lambda] + \xi_1[\lambda e_n]\} < \{\xi_1[d_1 T^0]\}. \tag{33}$$

(ii) PH/M/1

The arrival follows phase-type distribution with the representation (α, T) of order m. The arrival rate is given by λ = $[\alpha(-T)^{-1}e]^{-1}$. The service times are assumed to be exponentially distributed with rate δ. Similarly, the vacation times, breakdown and repair times are exponentially distributed. In this type, the infinitesimal generator is thus reduced as

$$Q_2 = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots\cdots \\ B_{10} & A_1 & A_0 & 0 & 0 & 0 & \cdots\cdots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \cdots\cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots\cdots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{34}$$

The block matrices of Q_2 are defined as follows:

$$B_{00} = \begin{pmatrix} T & 0 \\ \eta I_m & T - \eta I_m \end{pmatrix}; \quad B_{01} = \begin{pmatrix} 0 & T^0\alpha & 0 \\ T^0\alpha & 0 & 0 \end{pmatrix}; \quad B_{10} = \begin{pmatrix} 0 & 0 \\ d_2 d_1 \delta I_m & c_2 d_1 \delta I_m \\ 0 & 0 \end{pmatrix};$$

$$A_1 = \begin{pmatrix} T - \eta I_m & \eta I_m & 0 \\ c_2 c_1 \delta I_m & T - (\sigma + \delta) I_m + d_2 c_1 \delta I_m & \sigma I_m \\ 0 & \Psi I_m & T - \Psi I_m \end{pmatrix}; \quad A_0 = \begin{pmatrix} T^0\alpha & 0 & 0 \\ 0 & T^0\alpha & 0 \\ 0 & 0 & T^0\alpha \end{pmatrix};$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ c_2 d_1 \delta I_m & d_2 d_1 \delta I_m & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Subsequently, the generator matrix of A is

$$A = \begin{pmatrix} (T + T^0\alpha) - \eta I_m & \eta I_m & 0 \\ c_2 \delta I_m & (T + T^0\alpha) - (\delta + \sigma) I_m + d_2 \delta I_m & \sigma I_m \\ 0 & \Psi I_m & (T + T^0\alpha) - \Psi I_m \end{pmatrix}. \tag{35}$$

We consider the stationary probability vector $\boldsymbol{\xi}$ of A for Equation (35) and it obeys $\boldsymbol{\xi}$A = 0 and $\boldsymbol{\xi}$e = 1. The vector $\boldsymbol{\xi}$ has partitioned as $\boldsymbol{\xi}$=(ξ_0, ξ_1, ξ_2) and one can easily find the values of $\boldsymbol{\xi}$ by solving the following equations:

$$\xi_0[(T + T_0\alpha) - \eta I_m] + \xi_1[c_2 \delta I_m] = 0, \tag{36}$$

$$\xi_0[\eta I_m] + \xi_1[(T + T^0\alpha) - (\delta + \sigma) I_m + d_2 \delta I_m] + \xi_2[\Psi I_m] = 0, \tag{37}$$

$$\xi_1[\sigma I_m] + \xi_2[(T + T^0\alpha) - \Psi I_m] = 0, \tag{38}$$

subject to the normalizing condition

$$\xi_0 e_m + \xi_1 e_m + \xi_2 e_m = 1. \tag{39}$$

After algebraical manipulation, $\boldsymbol{\xi} A_0 e < \boldsymbol{\xi} A_2 e$ is the necessary and sufficient condition to attain the stability condition which is turning to be

$$\{(\xi_0 + \xi_1 + \xi_2)[T^0 \alpha e_m]\} < \{\xi_1[d_1 \delta e_m]\}. \tag{40}$$

(iii) M/M/1

The arrival follows the Poisson process. The inter-arrival times follow an exponential distribution with rate λ and the service times are assumed to be exponentially distributed with rate δ. Similarly, vacation times, breakdown and repair times are also exponentially distributed. In this type, the infinitesimal generator is thus reduced as

$$Q_3 = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots\cdots \\ B_{10} & A_1 & A_0 & 0 & 0 & 0 & \cdots\cdots \\ 0 & A_2 & A_1 & A_0 & 0 & 0 & \cdots\cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots\cdots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots\ \ddots \end{pmatrix}. \tag{41}$$

The block matrices of Q_3 are defined are

$$B_{00} = \begin{pmatrix} -\lambda & 0 \\ \eta & -(\lambda+\eta) \end{pmatrix}; \quad B_{01} = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \end{pmatrix}; \quad B_{10} = \begin{pmatrix} 0 & 0 \\ d_2 d_1 \delta & c_2 d_1 \delta \\ 0 & 0 \end{pmatrix};$$

$$A_1 = \begin{pmatrix} -(\lambda+\eta) & \eta & 0 \\ c_2 c_1 \delta & -(\lambda+\sigma+\delta) + d_2 c_1 \delta & \sigma \\ 0 & \Psi & -(\lambda+\Psi) \end{pmatrix}; \quad A_0 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix};$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ c_2 d_1 \delta & d_2 d_1 \delta & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Subsequently, the generator matrix of A is

$$A = \begin{pmatrix} -\eta & \eta & 0 \\ c_2 \delta & -(\delta+\sigma) + d_2 \delta & \sigma \\ 0 & \Psi & -\Psi \end{pmatrix}. \tag{42}$$

We consider the stationary probability vector $\boldsymbol{\xi}$ of A for Equation (42) which satisfies $\boldsymbol{\xi}$A = 0 and $\boldsymbol{\xi}$e = 1 such that it has partitioned as $\boldsymbol{\xi}$=(ξ_0, ξ_1, ξ_2) and after computation we will get the following results:

$$\xi_1 = \frac{\Psi c_2 \delta}{\eta\sigma + \Psi(c_2\delta + \eta)}; \quad \xi_2 = \frac{\Psi \eta}{\eta\sigma + \Psi(c_2\delta + \eta)}; \quad \xi_3 = \frac{\sigma \eta}{\eta\sigma + \Psi(c_2\delta + \eta)}. \tag{43}$$

After algebraical manipulation, $\boldsymbol{\xi} A_0 e < \boldsymbol{\xi} A_2 e$ is the necessary and sufficient condition to attain the stability condition which is turning to be

$$\lambda < \frac{\Psi \eta [d_1 \delta]}{\eta\sigma + \Psi(c_2\delta + \eta)}. \tag{44}$$

8. Numerical Results

In this part, we analyze the model behavior in numerical and graphical form. The following five different MAP representations, their representation mean values are 1 for the arrival process. These five sets of arrival values are literally incurred from Chakravarthy (2010).

Arrival in Erlang (ERLA) :

$$D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix},$$

Arrival in Exponential (EXPA) :

$$D_0 = \begin{bmatrix} -1 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 1 \end{bmatrix},$$

Arrival in Hyper-exponential (HEXA) :

$$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix},$$

Arrival in MAP-Negative Correlation (MNCA) :

$$D_0 = \begin{bmatrix} -1.00243 & 1.00243 & 0 \\ 0 & -1.00243 & 0 \\ 0 & 0 & -225.797 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.01002 & 0 & 0.99241 \\ 223.539 & 0 & 2.258 \end{bmatrix},$$

Arrival in MAP-Positive Correlation (MPCA) :

$$D_0 = \begin{bmatrix} -1.00243 & 1.00243 & 0 \\ 0 & -1.00243 & 0 \\ 0 & 0 & -225.797 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.99241 & 0 & 0.01002 \\ 2.258 & 0 & 223.539 \end{bmatrix}.$$

Let us consider three phase type distributions for the service process. Normalization of these three representations has been made to obtain service rate δ. These sets of service values are literally incurred from Chakravarthy (2010).

Service in Erlang (ERLS) :

$$\alpha = \big(1, 0\big), \qquad T = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix},$$

Service in Exponential (EXPS) :

$$\alpha = \big(1\big), \qquad T = \big[-1\big],$$

Service in Hyper-exponential (HEXS) :

$$\alpha = \big(0.8, 0.2\big), \qquad T = \begin{bmatrix} -2.80 & 0 \\ 0 & -0.28 \end{bmatrix}.$$

Illustrative Example 8.1.

We have examined the consequence of the breakdown rate σ against the expected system size in the following table 1 through 3. We fix $\lambda = 1$; $\delta = 20$; $c_1 = 0.5$; $d_1 = 0.5$; $\Psi = 10$; $c_2 = 0.4$; $\eta = 15$; $d_2 = 0.6$.

We have observed from Table 1 through 3 that when we are maximizing the breakdown rate, the variety of arrangements of service and arrival times then the expected system size also maximizes. Furthermore, comparing to hyper-exponential and exponential arrival increases than Erlang arrival. While increasing the breakdown occurs more often in which the queueing line will increases so that the expected system size increases. Nonetheless, the expected system size increases more slowly in exponential arrival.

Illustrative Example 8.2.

We have examined the impact of the vacation rate η against the expected system size in the following table 4 through 6. We fix $\lambda = 1$; $\delta = 20$; $c_1 = 0.5$; $d_1 = 0.5$; $\Psi = 10$; $c_2 = 0.4$; $\sigma = 2$; $d_2 = 0.6$.

We have observed from Table 4 through 6 replicates that when we are maximizing the vacation rate, the variety of arrangements of service and arrival times then the expected system size minimizes. Due to the vacation under the Bernoulli schedule server has an option to go vacation when completing service of a customer so it explicates the increase in expected system size. When comparing to Erlang and exponential arrival decreases slowly than hyper-exponential arrival. Nevertheless, the expected system size decreases more slowly in the Erlang arrival.

Illustrative Example 8.3.

We have analyzed the consequence of the repair rate Ψ against the expected system size in the following table 7 through 9. We fix $\lambda = 1$; $\delta = 20$; $c_1 = 0.5$; $d_1 = 0.5$; $\eta = 15$; $c_2 = 0.4$; $\sigma = 2$; $d_2 = 0.6$.

From Table 7 through 9, we observed that when we are maximizing the repair rate, the variety of arrangements of service and arrival times expected system size minimizes. While we increase the repair rate which means the server rejuvenated from repair much faster leads availability of the server in the system reduces the expected system size. Subsequently, comparing Erlang and exponential arrival decreases slower than the hyper-exponential arrival. Nonetheless, the expected system size decreases gradually in the Erlang arrival.

Illustrative Example 8.4.

We have examined the impact of the service rate δ against the expected system size as shown in the following table 10 through 12. We fix $\lambda = 1$; $c_1 = 0.5$; $d_1 = 0.5$; $\eta = 15$; $\Psi = 10$; $c_2 = 0.4$; $\sigma = 2$; $d_2 = 0.6$.

Our observation from Table 10 through 12 that when we are maximizing the service rate in the variety of arrangements of service and arrival times then the expected system size minimizes. While we increase the service rate to the customers automatically expected system size decrease will happen. When comparing to Erlang and Exponential arrival decreases slower than the hyper-exponential arrival. Nevertheless, the expected system size decreases more slowly in the Erlang arrival.

Illustrative Example 8.5.

We have analyzed both the repair rate Ψ and service rate δ against the probability of the server being idle. We fix $\lambda = 1$; $c_1 = 0.5$; $d_1 = 0.5$; $\eta = 15$; $c_2 = 0.4$; $\sigma = 2$; $d_2 = 0.6$.

We noticed from Figure 2 through 16 explains that the probability that the server is idle increases when we maximize both the repair and service rate for all the possible combinations with arrival and service times. When the repair completion epoch service would start for the frozen customer afterward if the server decides to stay in the system in idle state is increases. However, Erlang service increases slowly and gradually increase in hyper-exponential service with all types of arrival times.

Illustrative Example 8.6.

We have examined both the breakdown rate σ and repair rate Ψ against the expected system size. We fix $\lambda = 1$; $c_1 = 0.5$; $d_1 = 0.5$; $\eta = 15$; $c_2 = 0.4$; $\delta = 20$; $d_2 = 0.6$.

From viewpoint of Figure 17 through 31, observe that the expected system size decreases when we maximize both the breakdown and repair rate for all the possible groupings of arrival and service times. While we increase both the repair rate and breakdown which leads whenever breakdown occurs server would go for the repair process and when the server comes back from repair will start offering service to the customers, i.e., it will enhance the utility of the server so that the average system size decreases. Therefore, the expected system size increases slowly in Erlang arrival and fastly in MAP-Positive correlation arrival.

Table 1. Breakdown rate vs. μ_{system} - **ERLS**

σ	ERLA	EXPA	HEXA
1.1	0.1507279212	0.1708058912	0.2003442734
1.2	0.1521677952	0.1725360417	0.2026598388
1.3	0.1536141594	0.1742748213	0.2049918267
1.4	0.1550670529	0.1760222818	0.2073403813
1.5	0.1565265152	0.1777784754	0.2097056480
1.6	0.1579925863	0.1795434547	0.2120877740
1.7	0.1594653062	0.1813172728	0.2144869081
1.8	0.1609447157	0.1830999831	0.2169032005
1.9	0.1624308555	0.1848916398	0.2193368033

Table 2. Breakdown rate vs. μ_{system} - **EXPS**

σ	ERLA	EXPA	HEXA
1.1	0.1499058805	0.1703097992	0.2006076378
1.2	0.1511720332	0.1718317359	0.2026536200
1.3	0.1524407557	0.1733573237	0.2047081286
1.4	0.1537120590	0.1748865759	0.2067712087
1.5	0.1549859542	0.1764195057	0.2088429056
1.6	0.1562624522	0.1779561263	0.2109232650
1.7	0.1575415643	0.1794964511	0.2130123326
1.8	0.1588233016	0.1810404934	0.2151101545
1.9	0.1601076753	0.1825882669	0.2172167771

Table 3. Breakdown rate vs. μ_{system} - **HEXS**

σ	ERLA	EXPA	HEXA
1.1	0.1452616282	0.1662205280	0.1994024042
1.2	0.1455231551	0.1665288722	0.1997882840
1.3	0.1457924733	0.1668465766	0.2001876397
1.4	0.1460692897	0.1671732948	0.2005999847
1.5	0.1463533263	0.1675086991	0.2010248583
1.6	0.1466443198	0.1678524781	0.2014618244
1.7	0.1469420202	0.1682043367	0.2019104691
1.8	0.1472461902	0.1685639945	0.2023704002
1.9	0.1475566045	0.1689311851	0.2028412453

Table 4. Vacation rate vs. μ_{system} - **ERLS**

η	ERLA	EXPA	HEXA
10	0.1851160320	0.2152446675	0.2667891049
12	0.1741573932	0.2004593446	0.2431456728
14	0.1667772378	0.1905247305	0.2276657859
16	0.1614714635	0.1834038694	0.2167877908
18	0.1574737671	0.1780558151	0.2087448063
20	0.1543534596	0.1738948626	0.2025660129
22	0.1518501030	0.1705668669	0.1976757605
24	0.1497970281	0.1678453839	0.1937119539
26	0.1480826438	0.1655790033	0.1904358387

Table 5. Vacation rate vs. μ_{system} - **EXPS**

η	ERLA	EXPA	HEXA
10	0.1825094294	0.2125000000	0.2638373400
12	0.1715933908	0.1978142077	0.2404554607
14	0.1642389785	0.1879464286	0.2251457522
16	0.1589499063	0.1808734940	0.2143868746
18	0.1549636220	0.1755614657	0.2064317363
20	0.1518514263	0.1714285714	0.2003203107
22	0.1493540180	0.1681230408	0.1954833131
24	0.1473054163	0.1654199475	0.1915626361
26	0.1455944663	0.1631688963	0.1883221466

Table 6. Vacation rate vs. μ_{system} - **HEXS**

η	ERLA	EXPA	HEXA
10	0.1684911939	0.1966470147	0.2452402478
12	0.1578415770	0.1824882097	0.2232181090
14	0.1506552808	0.1729751983	0.2087981830
16	0.1454800970	0.1661571421	0.1986648955
18	0.1415750692	0.1610369835	0.1911727635
20	0.1385231904	0.1570536966	0.1854174488
22	0.1360719796	0.1538680641	0.1808626460
24	0.1340596718	0.1512631963	0.1771709821
26	0.1323778420	0.1490940791	0.1741199961

Table 7. Repair rate vs. μ_{system} - **ERLS**

Ψ	ERLA	EXPA	HYPA
11	0.1613821135	0.1836098899	0.2174937795
14	0.1560828120	0.1771889180	0.2086469733
17	0.1527569016	0.1731666522	0.2031783962
20	0.1504749646	0.1704123780	0.1994690650
23	0.1488119756	0.1684088399	0.1967898645
26	0.1475460761	0.1668862054	0.1947649383
29	0.1465501429	0.1656900241	0.1931811536
32	0.1457460966	0.1647255466	0.1919087491
35	0.1450833280	0.1639314353	0.1908642467

Table 8. Repair rate vs. μ_{system} - **EXPS**

Ψ	ERLA	EXPA	HEXA
11	0.1589022416	0.1811221210	0.2151338663
14	0.1537040183	0.1748338237	0.2064785253
17	0.1504405357	0.1708930825	0.2011241127
20	0.1482010003	0.1681938911	0.1974902327
23	0.1465687019	0.1662300292	0.1948644673
26	0.1453260517	0.1647373205	0.1928793080
29	0.1443483440	0.1635645085	0.1913262417
32	0.1435589714	0.1626187820	0.1900782643
35	0.1429082723	0.1618400475	0.1890536421

Table 9. Repair rate vs. μ_{system} - **HEXS**

Ψ	ERLA	EXPA	HEXA
11	0.1456544762	0.1666422826	0.1996515464
14	0.1410206162	0.1610832589	0.1920611484
17	0.1381067334	0.1575930234	0.1873493741
20	0.1361050381	0.1551993595	0.1841439734
23	0.1346450526	0.1534561805	0.1818237248
26	0.1335330090	0.1521302787	0.1800671612
29	0.1326577215	0.1510879499	0.1786914517
32	0.1319508260	0.1502470668	0.1775850210
35	0.1313679742	0.1495544081	0.1766759513

Table 10. Service rate vs. μ_{system} - **ERLS**

δ	ERLA	EXPA	HEXA
20	0.1639237671	0.1866922973	0.2217878700
21	0.1567572632	0.1781677053	0.2105311242
22	0.1503166845	0.1705339667	0.2005570412
23	0.1444962601	0.1636582756	0.1916587746
24	0.1392099199	0.1574329880	0.1836715712
25	0.1343868947	0.1517699300	0.1764626087
26	0.1299684488	0.1465961844	0.1699236342
27	0.1259054179	0.1418509261	0.1639655455
28	0.1221563289	0.1374830147	0.1585143432

Table 11. Service rate vs. μ_{system} - **EXPS**

δ	ERLA	EXPA	HEXA
20	0.1613946967	0.1841397849	0.2193322471
21	0.1544733466	0.1758962624	0.2084066836
22	0.1482428339	0.1684997588	0.1987034446
23	0.1426039185	0.1618261069	0.1900291591
24	0.1374755878	0.1557742782	0.1822290478
25	0.1327910426	0.1502612330	0.1751777853
26	0.1284947013	0.1452180872	0.1687728331
27	0.1245399326	0.1405872193	0.1629295008
28	0.1208873183	0.1363200590	0.1575772347

Table 12. Service rate vs. μ_{system} - **HEXS**

δ	ERLA	EXPA	HEXA
20	0.1478730489	0.1693056550	0.2033226507
21	0.1421912213	0.1625729973	0.1943485575
22	0.1370312210	0.1564695272	0.1862761794
23	0.1323237761	0.1509108435	0.1789769842
24	0.1280114157	0.1458270284	0.1723456795
25	0.1240460504	0.1411596608	0.1662951959
26	0.1203871267	0.1368595412	0.1607529148
27	0.1170002013	0.1328849382	0.1556577958
28	0.1138558285	0.1292002203	0.1509581619

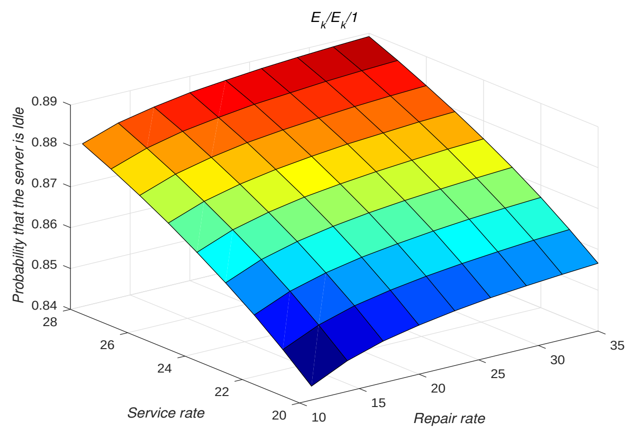

Figure 2. Repair rate (Ψ) and Service rate(δ) vs. P_{IDLE} - $E_k/E_k/1$

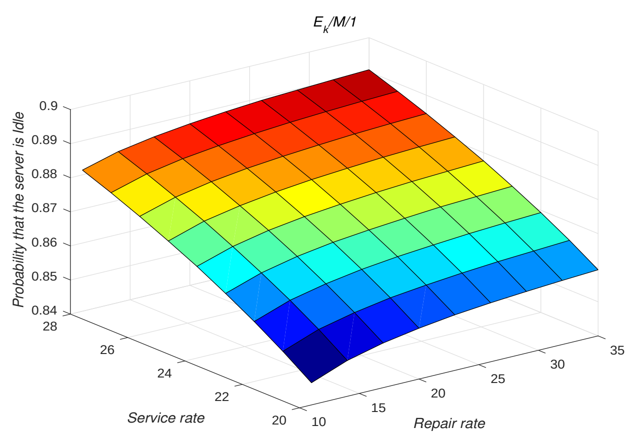

Figure 3. Repair rate (Ψ) and Service rate(δ) vs. P_{IDLE} - $E_k/M/1$

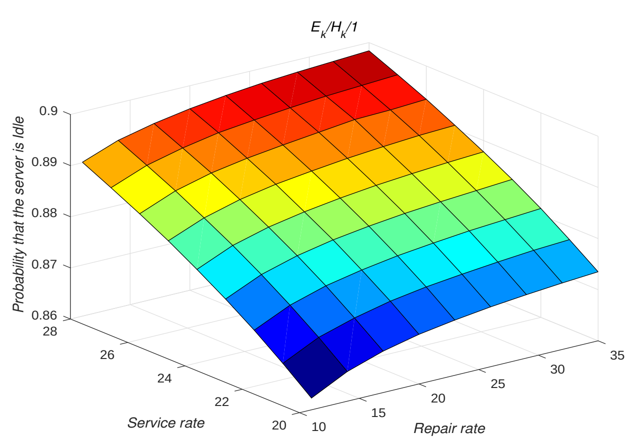

Figure 4. Repair rate (Ψ) and Service rate(δ) vs. P_{IDLE} - $E_k/H_k/1$

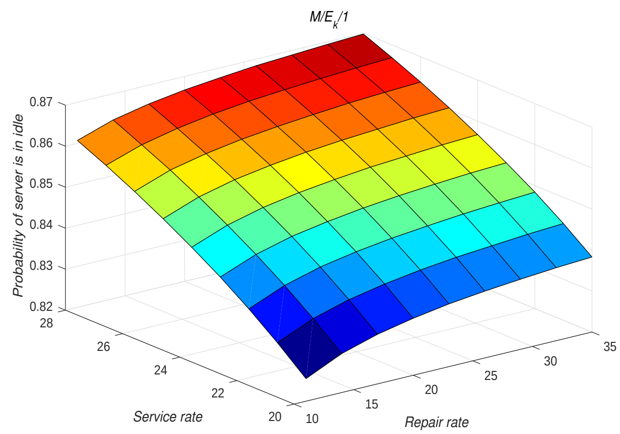

Figure 5. Repair rate (Ψ) and Service rate(δ) vs. P_{IDLE} - $M/E_k/1$

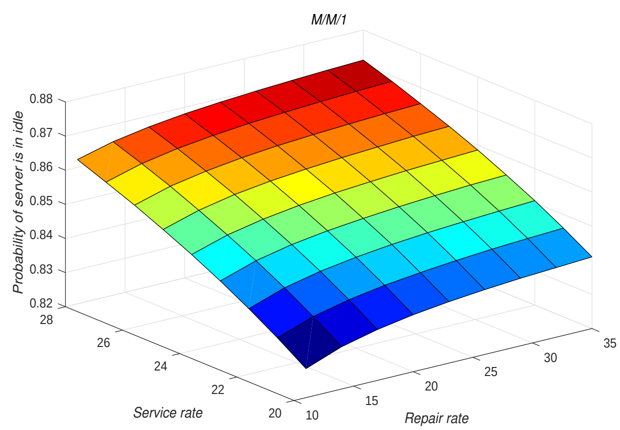

Figure 6. Repair rate (Ψ) and Service rate(δ) vs. P_{IDLE} - $M/M/1$

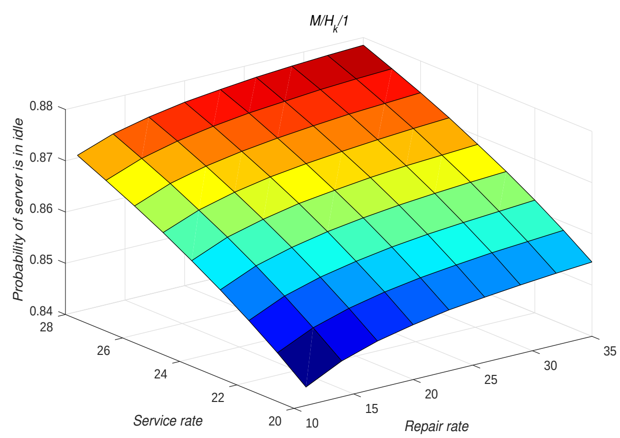

Figure 7. Repair rate (Ψ) and Service rate(δ) vs. P_{IDLE} - $M/H_k/1$

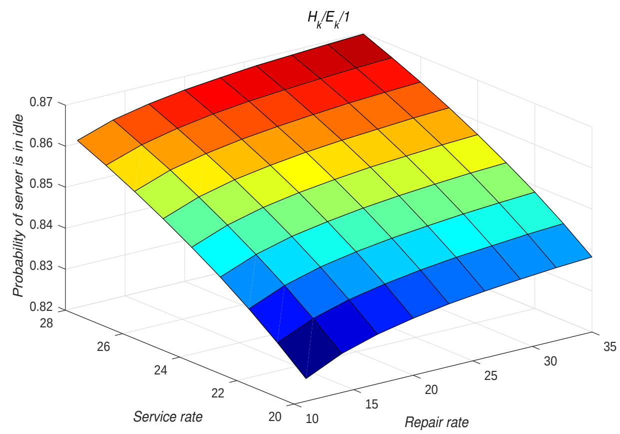

Figure 8. Repair rate (Ψ) and Service rate(δ) vs. P_{IDLE} - $H_k/E_k/1$

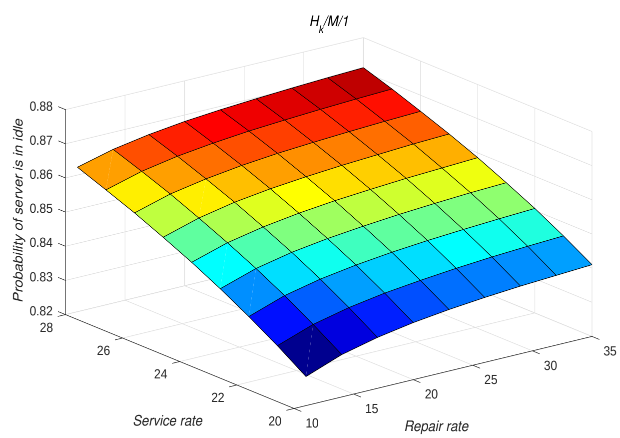

Figure 9. Repair rate (Ψ) and Service rate(δ) vs. P_{IDLE} - $H_k/M/1$

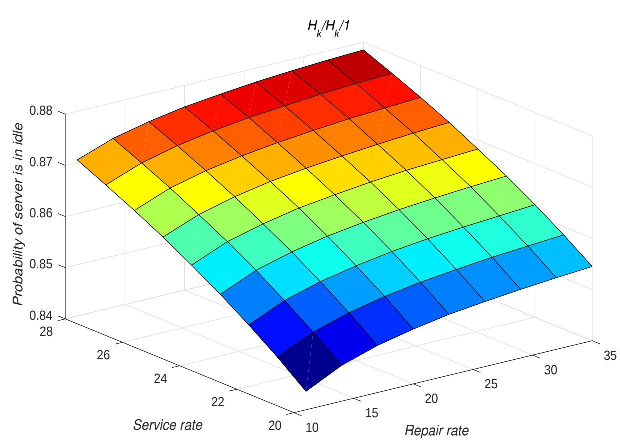

Figure 10. Repair rate (Ψ) and Service rate(δ) vs. P_{IDLE} - $H_k/H_k/1$

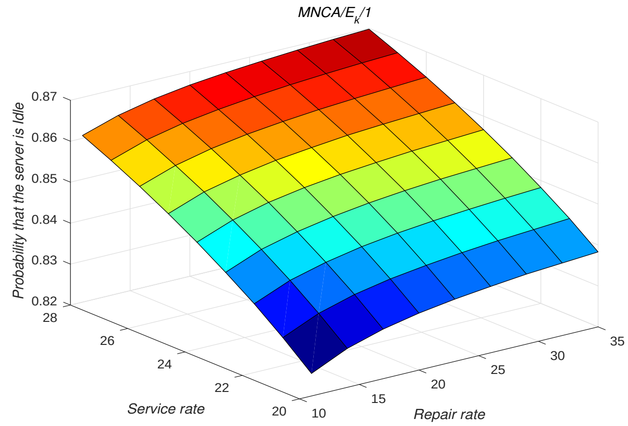

Figure 11. Repair rate (Ψ) and Service rate(δ) vs. P_{IDLE} - $MNCA/E_k/1$

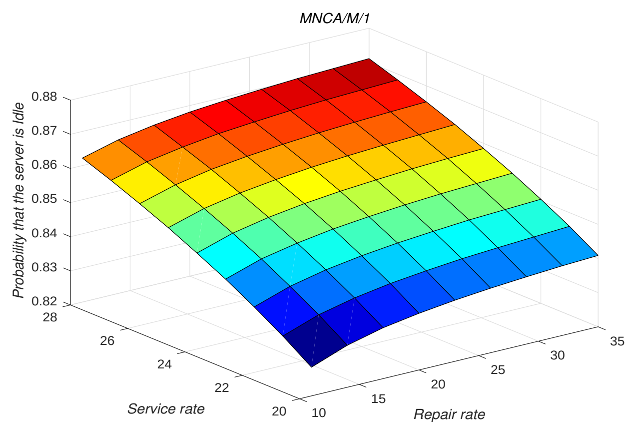

Figure 12. Repair rate (Ψ) and Service rate(δ) vs. P_{IDLE} - $MNCA/M/1$

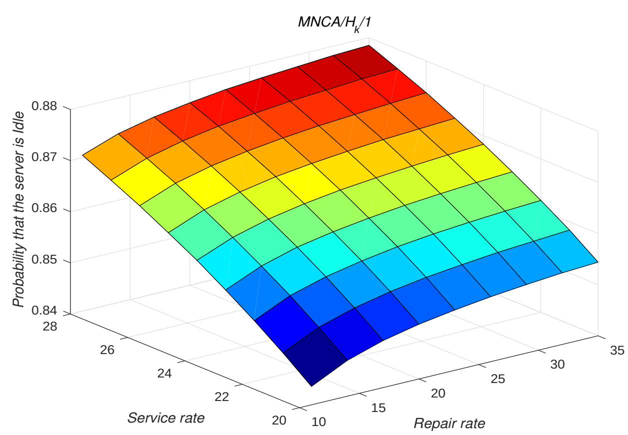

Figure 13. Repair rate (Ψ) and Service rate(δ) vs. P_{IDLE} - $MNCA/H_k/1$

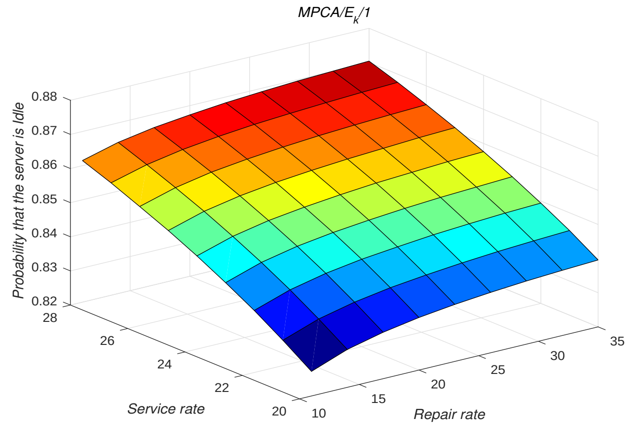

Figure 14. Repair rate (Ψ) and Service rate(δ) vs. P_{IDLE} - $MPCA/E_k/1$

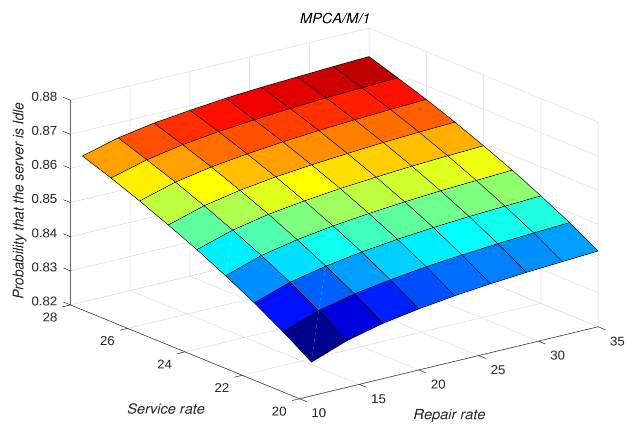

Figure 15. Repair rate (Ψ) and Service rate(δ) vs. P_{IDLE} - $MPCA/M/1$

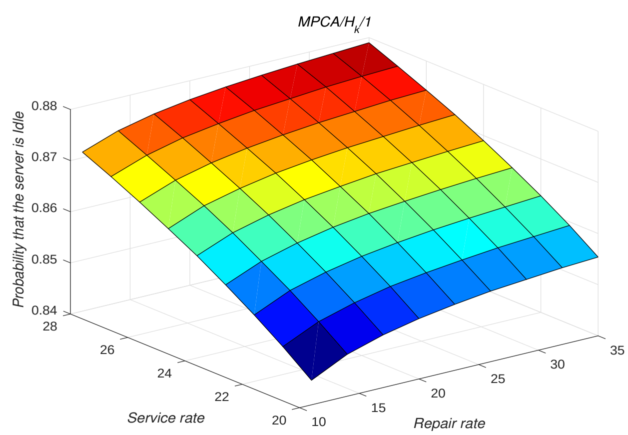

Figure 16. Repair rate (Ψ) and Service rate(δ) vs. P_{IDLE} - $MPCA/H_k/1$

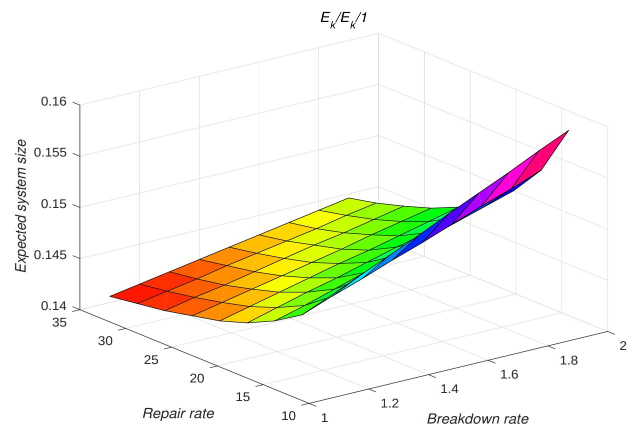

Figure 17. Breakdown rate (σ) and Repair rate (Ψ) vs. μ_{system} - $E_k/E_k/1$

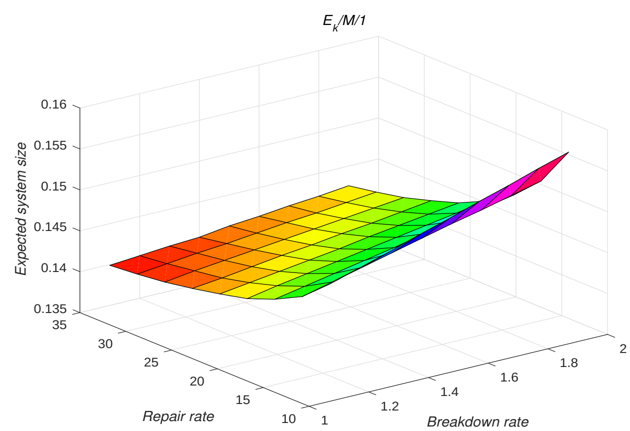

Figure 18. Breakdown rate (σ) and Repair rate (Ψ) vs. μ_{system} - $E_k/M/1$

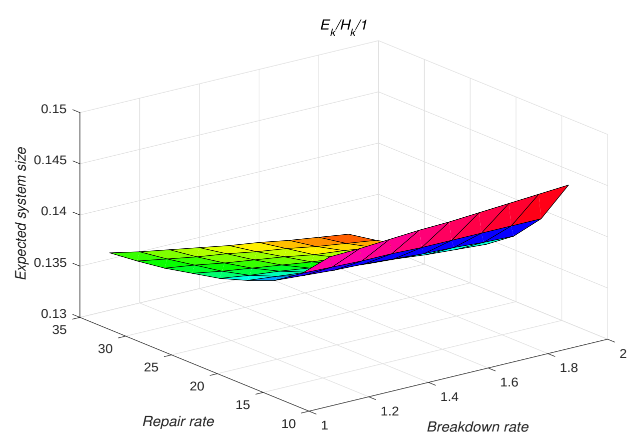

Figure 19. Breakdown rate (σ) and Repair rate (Ψ) vs. μ_{system} - $E_k/H_k/1$

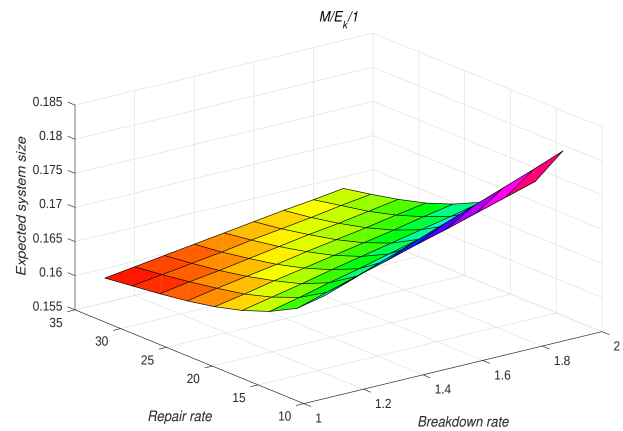

Figure 20. Breakdown rate (σ) and Repair rate (Ψ) vs. μ_{system} - $M/E_k/1$

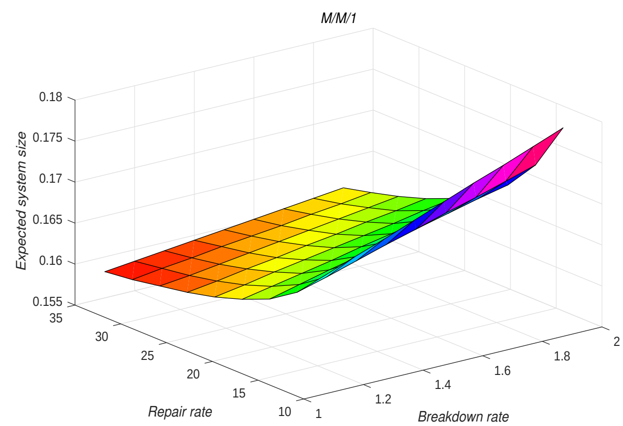

Figure 21. Breakdown rate (σ) and Repair rate (Ψ) vs. μ_{system} - $M/M/1$

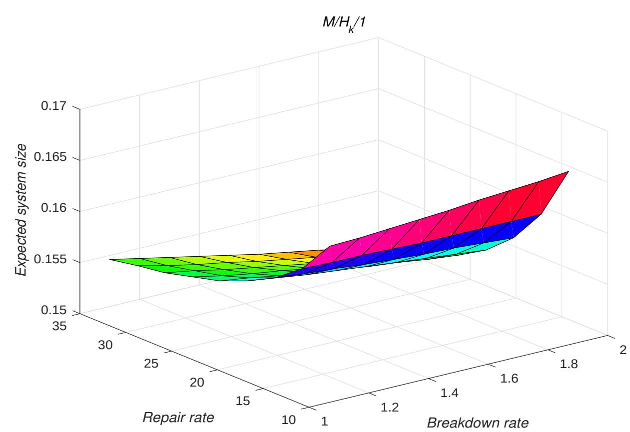

Figure 22. Breakdown rate (σ) and Repair rate (Ψ) vs. μ_{system} - $M/H_k/1$

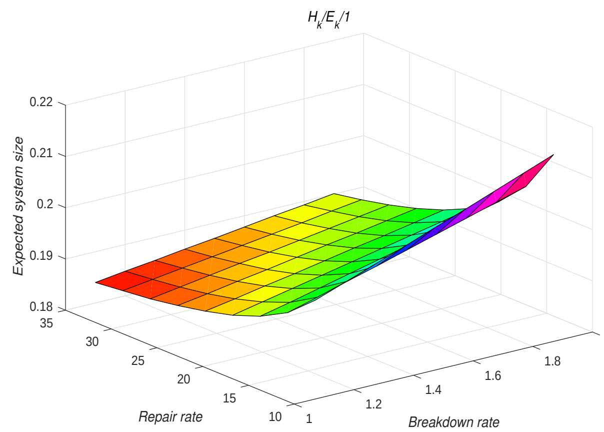

Figure 23. Breakdown rate (σ) and Repair rate (Ψ) vs. μ_{system} - $H_k/E_k/1$

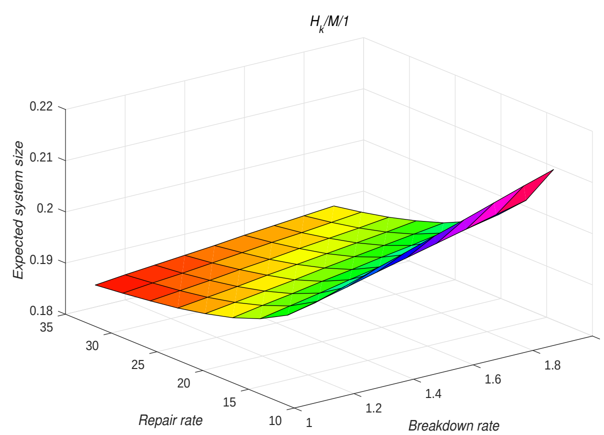

Figure 24. Breakdown rate (σ) and Repair rate (Ψ) vs. μ_{system} - $H_k/M/1$

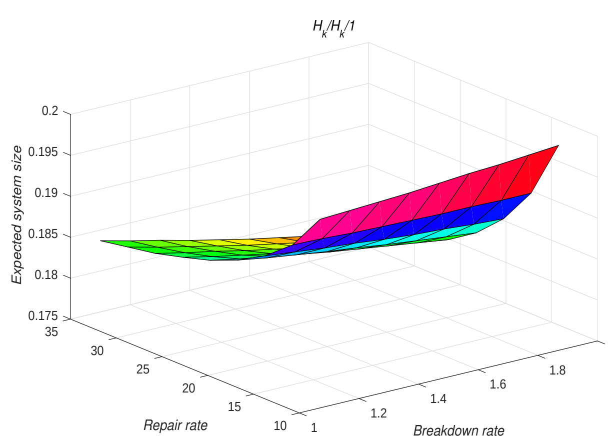

Figure 25. Breakdown rate (σ) and Repair rate (Ψ) vs. μ_{system} - $H_k/H_k/1$

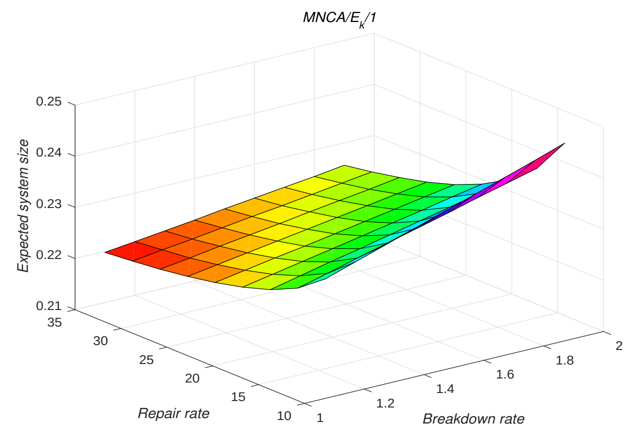

Figure 26. Breakdown rate (σ) and Repair rate (Ψ) vs. μ_{system} - $MNCA/E_k/1$

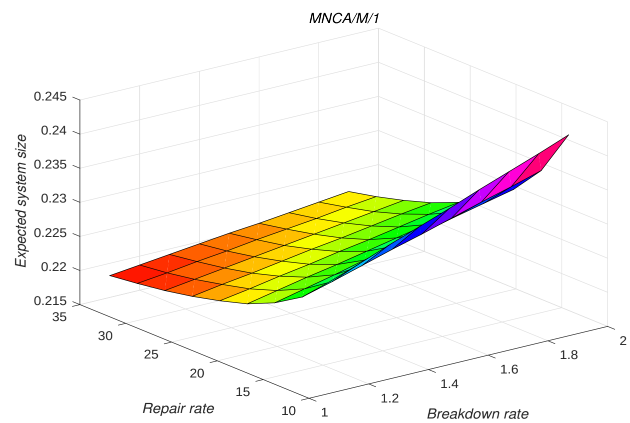

Figure 27. Breakdown rate (σ) and Repair rate (Ψ) vs. μ_{system} - $MNCA/M/1$

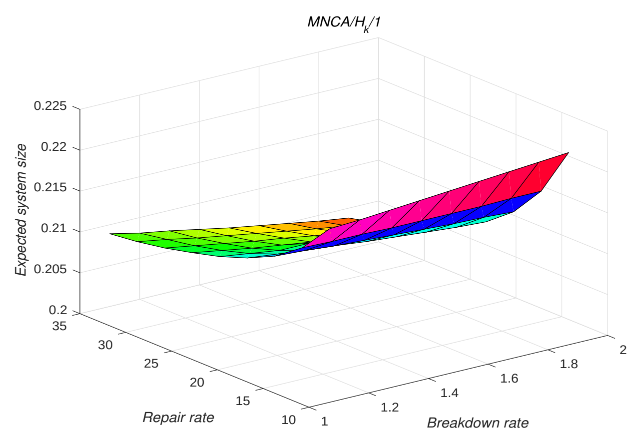

Figure 28. Breakdown rate (σ) and Repair rate (Ψ) vs. μ_{system} - $MNCA/H_k/1$

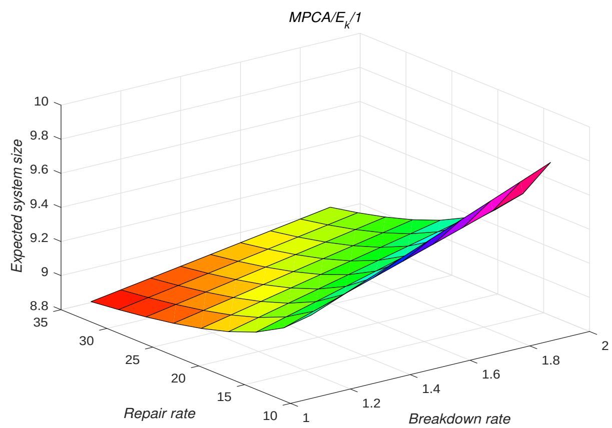

Figure 29. Breakdown rate (σ) and Repair rate (Ψ) vs. μ_{system} - $MPCA/E_k/1$

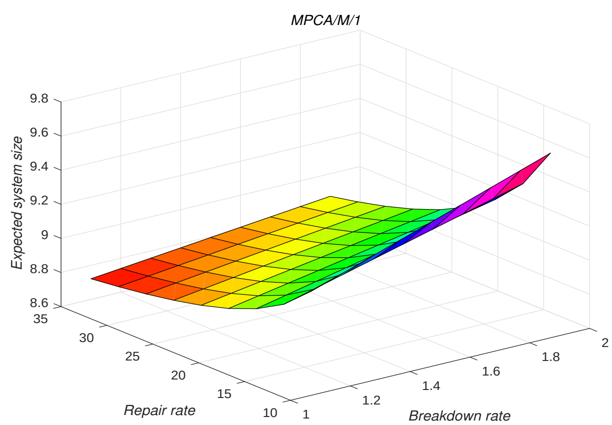

Figure 30. Breakdown rate (σ) and Repair rate (Ψ) vs. μ_{system} - $MPCA/M/1$

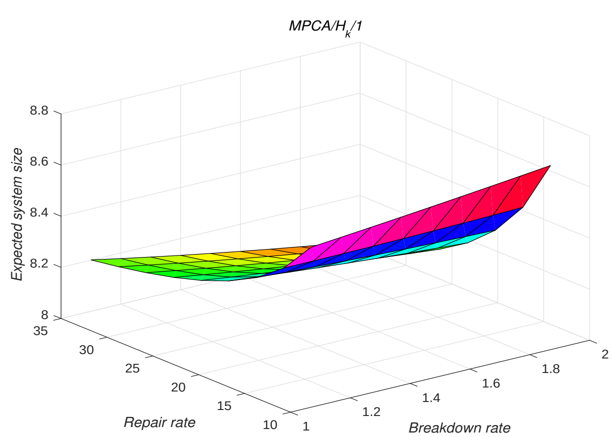

Figure 31. Breakdown rate (σ) and Repair rate (Ψ) vs. μ_{system} - $MPCA/H_k/1$

9. Conclusion

In this manuscript, we consider customer's arrival follows Markovian arrival process and the service times follows phase-type distribution in which the server may affect by breakdown while offering service to the customers. During the server repair times, the customer who is getting service from the server who is in a frozen state up to the server repair completion and the server starts fresh service to that customer. If some customers would like to get feedback, they will get feedback immediately. In our work, we also compute the busy period analysis. Using the numerical values of arrival and service times, we tabulated the breakdown rate, repair rate, service rate and vacation rate versus expected system size numerically. We compared the service rate and repair rate versus the probability of the server is in idle and also compared breakdown and repair rate versus expected system size showed through graphical demonstrations.

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