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Exploring the Convergence Properties of a New Modified Newton-Raphson Root Method

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Abstract

We examine the convergence properties of a modified Newton-Raphson root method, by using a simple complex polynomial equation, as a test example. In particular, we numerically investigate how a parameter, entering the iterative scheme, affects the efficiency and the speed of the method. Color-coded polynomiographs are deployed for presenting the regions of convergence, as well as the fractality degree of the complex plane. We demonstrate that the behavior of the modified Newton-Raphson method is correlated with the numerical value of the parameter α_1 . Additionally, there are cases for which the method works flawlessly, while in some other cases we encounter the phenomena of ill-convergence or even non-convergence.

Keywords: Newton-Raphson method; Iterative methods; Convergence basins; Basin entropy

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1. Introduction

Using numerical techniques for locating the position of roots of equations is, without any doubt, one of the most interesting and inspiring problems of applied mathematics. For many years, the issue of finding numerical solutions of nonlinear equations has been a very active field (Chun et al. (2014); Chun and Neta (2015a); Chun and Neta (2015b); Chun and Neta (2015c); Chun and Neta (2016); Chun and Neta (2017a); Chun and Neta (2017b); Chun and Neta (2018); Neta et al. (2012)). The roots of an equation are directly associated with the convergence basins of the iterative method. These convergence basins correspond to sets of points for which the numerical method leads to the same numerical attractor (final state), or in other words to the same root of the equation.

In Susanto and Karjanto (2009) (hereafter Paper I) the convergence properties of a modified Newton-Raphson (NR) method have been numerically investigated. The family of the modified NR method contains a free parameter which determines the behavior and the overall shape of the convergence basins. In the same paper, it was demonstrated that the convergence basins, corresponding even to the simplest iterative scheme, can be highly complicated.

However, our numerical experiments indicate that Paper I contains a couple of typos, along with some errors regarding the behavior of the modified NR method. More precisely, the results presented in Paper I suggest that the modified NR iterative scheme works almost equally well for all chosen initial conditions, which is not correct. The aim of our paper is to demonstrate the correct behavior of the numerical method, by conducting a systematic and thorough examination of its convergence properties. The novelty of our work lies in the fact we will present quantitative outcomes regarding the speed as well as the efficiency of the modified NR method. At the same time, we will also determine how one of the main parameters of the iterative scheme influences the degree of fractality of the complex plane, containing all the starting conditions of the numerical iterator.

This new modified Newton-Raphson method converges faster, with respect to the classical version of the Newton-Raphson method. In our analysis, we will conduct a systematic and thorough examination of the convergence properties of this new numerical method (using the same test example as in Paper I) in an attempt to determine its behavior. In particular, we shall investigate how the free parameter, entering the iterative scheme, affects the speed as well as the accuracy of the method, by testing large sets of starting conditions and this is exactly the novelty of our work.

Our article's organizations is the following: Section 2 illustrates the convergence basins of the modified NR method, for several values of the free parameter, while in Section 3 we present the influence of the free parameter on the fractal degree of the complex plane, by using quantitative arguments, such as the (boundary) basin entropy. The last section of our paper is Section 4, where the most interesting conclusions of our exploration are discussed and emphasized.



Figure 1. The locations of the roots of the equation $f(z) = z^7 - 1 = 0$ are pinpointed by blue dots. The red dashed line indicates the circle of unity radius, which includes all the roots. (Color figure online).

2. Basins of convergence

The iterative scheme of the modified NR method is given by

$$z_{n+1} = z_n - \frac{f(z_n) f'(z_n)}{\alpha_0 (f'(z_n))^2 + \alpha_1 f(z_n) f''(z_n)},$$
(1)

where α_0 and α_1 are free real parameters.

Obviously, when $\alpha_0 = 1$ and $\alpha_1 = 0$ the iterative formula (1) is reduced to

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)},$$
(2)

which is the classical NR method, while when $\alpha_0 = 1$ and $\alpha_1 = -1/2$ it is reduced to

$$z_{n+1} = z_n - \frac{2f(z_n) f'(z_n)}{2(f'(z_n))^2 - f(z_n) f''(z_n)},$$
(3)

which is the Halley's method.

Our aim is to elucidate the influence of α_1 on the behavior of the modified NR method. For reasons of simplicity and without the loss of the generality throughout the paper we set $\alpha_0 = 1$. As in Paper I, the polynomial function $f(z) = z^7 - 1$ will be used as a test example. The equation f(z) = 0 has one real root and six complex conjugate roots, which all lie on a circle of unity radius R = 1 (see Figure 1).

The iterative procedure works as follows: First, we provide an initial condition on the complex plane in the form of a complex number $z = \mathcal{R} + i\mathcal{I}$. This initial condition actually activates the code, which continues the iterations until we reach to a root of the polynomial function, of course



Figure 2. Basin color diagrams for the (a-upper left): classical Newton-Raphson method and (c-lower left): Halley method. The locations of the seven roots are pinpointed by black dots. The colors, corresponding to the seven roots, is: R_1 (green); R_2 (orange); R_3 (teal); R_4 (purple); R_5 (blue); R_6 (magenta); R_7 (red). (b) and (d): The distributions of the number of required iterations. (Color figure online).

with the predefined accuracy. Here we would like to emphasize that generally not all starting points lead to a root of the function. For obtaining the regions of convergence on the complex numbers plane, we follow the numerical techniques and procedures that are described in detail in (Zotos (2017); Zotos (2018)).

In parts (a) and (c) of Figure 2 we depict the basin diagrams for the classical NR and Halley methods, respectively. As expected, the basin boundaries in the case of the Halley method are more smooth and distinct, with respect to that of the NR method. The distributions of the respective required number of iterations N are shown in parts (b) and (d) of Figure 2. It is seen, that for starting conditions in the close vicinity of the roots both methods require only a couple of iterations for landing on a numerical attractor (root). On the contrary, for starting points with coordinates near the basin boundaries the number of required iterations is very high (more than 15 iterations in the case of the Halley method and more than 40 iterations in the case of the NR method). Especially



Figure 3. Basin color diagrams for the modified Newton-Raphson method, when $\alpha_1 > 0$. (a): $\alpha_1 = 1.5$; (b): $\alpha_1 = 0.5$; (c): $\alpha_1 = 0.1$; (d): $\alpha_1 = 0.08$; (e): $\alpha_1 = 0.075$; (f): $\alpha_1 = 0.01$. The color choice is the same as in Figure 2. False converging starting conditions are shown in cyan, while for non-converging starting points we use white color. (Color figure online).

initial conditions lying at the basin boundaries will never converge (although the points in the basin boundaries are a set of Lebesgue measure zero (Royden (1988))). It should be noted that both these methods display excellent convergence properties if we take into account that all the tested starting



Figure 4. Basin diagrams where the color of the starting conditions is according to the number of required iterations, for the six cases of Figure 3. For false and non-converging starting conditions we use white color. (Color figure online).

conditions on the complex numbers plane finally reach to one of the numerical attractors (roots). This means, that for the particular test function for both methods there are no non-converging or false-converging starting points.



Figure 5. Probability histograms for the six cases of Figure 3. The most probable number of iterations is indicated by dashed, vertical, red lines, while the blue lines correspond to the best fitting curves. (Color figure online).

2.1. Case I: $\alpha_1 > 0$

The basin diagrams for the modified NR method, when $\alpha_1 > 0$, are depicted in Figure 3(a-f). In panels (a-c) we observe that for relatively large positive numbers of the parameter α_1 a substantial amount of starting points displays a false convergence to zero (cyan regions). Using the term "false convergence to zero", we mean that there exist starting conditions on the complex plane for which



Figure 6. Basin diagrams for the modified Newton-Raphson method, when $\alpha_1 < 0$. (a): $\alpha_1 = -0.1$; (b): $\alpha_1 = -0.8$; (c): $\alpha_1 = -0.95$; (d): $\alpha_1 = -1$; (e): $\alpha_1 = -1.02$; (f): $\alpha_1 = -1.05$; (g): $\alpha_1 = -1.08$; (h): $\alpha_1 = -1.1$; (i): $\alpha_1 = -1.5$. The colors are as in Figure 2. False converging initial conditions are shown in blue, while diverging starting points are shown in yellow. (Color figure online).

the numerical iterator leads to zero. However, the number zero is not a solution of the complex equation $f(z) = z^7 - 1 = 0$. Therefore, for these specific starting conditions the iterative scheme reports, as a final converging point, a false root. With decreasing value of the parameter α_1 the area of the true convergence basins grows, while simultaneously several fractal structures are observed mainly near the boundaries of the converge regions. So far, the use of term "fractal" is used qualitatively, implying that the corresponding regions exhibit a fractal-like geometry. However in the



Figure 7. Basin diagrams where the color of the starting conditions is according to the number of required iterations, for the six cases of Figure 6. For false and non-converging starting conditions we use white color. (Color figure online).

next section quantitative arguments will be presented, regarding the influence of the parameter α_1 on the fractal degree of the complex plane.

When $\alpha_1 = 0.08$ it is seen in part (d) of Figure 3 that all the false convergence regions have disappeared, thus giving place to a very complicated mixture of starting points. Indeed, all the regions between the convergence basins are covered by a complicated mixture of converging and non-converging starting points. With a further decrease of the parameter α_1 the amount of nonconverging starting conditions almost vanishes. In panel (e) of Figure 3, when $\alpha_1 = 0.075$ one can observe that non-converging starting points are present only on the horizontal axis with $\mathcal{I} = 0$. In panel (f), where $\alpha_1 = 0.01$, the modified NR method works equally well for all tested starting conditions, since there is no numerical indication of no or false convergence. Remember, that $\alpha_1 = 0.01$ is very close to the value $\alpha_1 = 0$ which corresponds to the classical version of the NR method.



Figure 8. Probability histograms for the six cases of Figure 6. The most probable number of iterations is indicated by dashed, vertical, red lines, while the blue lines correspond to the best fitting curves. (Color figure online).

In Figure 4(a-f) it is shown how the number of required iterations N is distributed on the complex numbers plane, for several values of α_1 . Figure 5(a-f) shows the distributions of the corresponding probability $P = N_0/N_t$, where N_0 is the number of the starting points which converge, while N_t is the total number of starting points. According to the probability histograms, N^* (the number of iterations with the highest probability) is constantly reduced, as α_1 decreases. For the fitting curves of the histograms we use blue lines (in the following subsection 2.3 we will present more details regarding the best fit of the histograms).



Figure 9. (a-upper left): Basin diagram on the (\mathcal{R}, α_1) plane, for the modified Newton-Raphson methods, when $-3 \le \alpha_1 \le +3$. (c-lower left): Local magnification of the plot shown in panel (a). The colors are as in Figs. 3 and 6. Parts (b) and (d): Basin diagrams where the color of the starting conditions is according to the number of required iterations. (Color figure online).

2.2. Case II: $\alpha_1 < 0$

Our numerical experiments show that the convergence properties of the modified NR method are much more interesting in the case where $\alpha_1 < 0$. In Figure 6 we see that for negative values, close to zero, the basin diagrams are very organized, while the regions with the fractal-like geometry are reduced. When $\alpha_1 = -1$ we observe in panel (d) of Figure 6 a unique configuration of the basin of convergence. However, according to panels (e) and (f) of the same figure, the convergence pattern on the complex plane drastically changes.

For $\alpha_1 = -1.08$ a similar phenomenon to that seen in panel (d) of Figure 3 is observed. More specifically, for a substantial amount of starting conditions on the complex numbers plane the modified NR method does not converge, to any of the seven roots of the complex equation, even



Figure 10. Parametric evolution of (a-c): |z| and (d-lower right): \mathcal{R} , as a function of the number N of iterations. Panel (a): $\alpha_1 = 2$, $\mathcal{R} = -1.5$, $\mathcal{I} = 0$, Panel (b): $\alpha_1 = -2$, $\mathcal{R} = -1.5$, $\mathcal{I} = 0$, Panels (c) and (d): $\alpha_1 = -1.08$, $\mathcal{R} = 1.92$, $\mathcal{I} = 0$.

after 1000 iterations. In panels (h) and (i) of Figure 6 one can observe that most of the plane is dominated by points for which the modified NR method displays a strange behavior. In particular, for all these starting conditions the iterative scheme quickly diverges (Kellison (1975)), which is a numerical indication of divergence to infinity. As the parameter α_1 continues to drop ($\alpha_1 < 1$) the area of the convergence basins is effectively reduced, while simultaneously all the fractal regions disappear.

In panels (a-i) of Figure 7 we show the distributions of the number of required iterations, while in panels (a-i) of Figure 8 the respective probability histograms are given. Now, the value of N^* does not seem to display a constant trend, since it fluctuates, as the value of the parameter α_1 changes.

2.3. An overview analysis

For obtaining a more general view of the convergence properties of the modified NR method we classified initial conditions on the (\mathcal{R}, α_1) plane, that is when the initial conditions of the iterative method are real numbers. In part (a) of Figure 9 we illustrate the corresponding basin diagram. We observe, that the plane is divided into the following four types of areas:



Figure 11. Evolution of the (a): N^* , (b): diversity d, (c): differential entropy h, and (d): percentage of the area A(%) on the complex numbers plane, covered by false and non-converging initial conditions, as a function of α_1 . The dashed, vertical, red lines indicate the positions, where non-converging initial conditions mostly appear. (Color figure online).

- (1) Regions where the iterative formula converges to the real root z = 1 (green regions).
- (2) Regions where the iterative formula displays a false convergence to z = 0 (blue regions).
- (3) Regions where the iterative formula diverges quickly to extremely large numbers (yellow regions).
- (4) Regions where the iterative formula does not show any sign of convergence (white regions).

In part (c) of Figure 9 we illustrate a local magnification of the (\mathcal{R}, α_1) , near the regions where the



Figure 12. Evolution of the (a): S_b (basin entropy) and (b): S_{bb} (boundary basin entropy), as a function of α_1 . The dashed, vertical, red lines indicate the positions, where non-converging initial conditions mostly appear, while the critical value of $\log 2$ is denoted by a horizontal, blue, dashed line. (Color figure online).



Figure 13. Evolution of the fractal dimension D_0 , as a function of α_1 . The dashed, vertical, red lines indicate the positions, where non-converging initial conditions mostly appear. (Color figure online).

convergence properties change. Additional numerical computations indicate that the areas where the numerical method displays non-convergence exist at about $-1.084 < \alpha_1 < -1.064$ and $0.064 < \alpha_1 < 0.085$.

It should be emphasized, that in Paper I similar diagrams are presented in Figure 2. However, the regions where the modified NR method displays ill behavior are not discussed. More precisely,

the authors do not provide any evidence of ill behavior (false convergence of divergence to large numbers) of the modified method. After communicating with the authors of Paper I it was revealed that the observed differences are due to the fact that the stopping conditions of the iterative scheme are different. In particular, in Paper I instead of $|f(z)| < \epsilon$ the authors used $|g(z) = f(z)^{a_0} \times$ $f'(z)^{a_1}| < \epsilon$, which of course came with the expense that the zero of f'(z), which is not necessarily the zero of f(z), would be counted as well. Nevertheless, our outcomes coincide at least partially with those of Paper I, since there are similar areas between our plots (see Figure 9) and those of Figure 2 of Paper I.

In part (a) of Figure 10 the evolution of the complex norm |z| of the modified NR method is presented, as a function of the iterations. It is seen, that for this initial condition the iterative scheme tends smoothly to zero. In part (b) we have the case of an initial condition for which the iterative formula quickly diverges. In this case, the iterative procedure was stoped when $|z| = 10^{10}$. Finally, in panel (c) of Figure 10 we have the case of an initial condition for which the modified method fails to converge. We observe that the complex norm oscillates without showing any signs of convergence. In part (d) of Figure 10 we display the evolution of the real part of z. Here it becomes more obvious why the method cannot converge. This is because the iterative procedure constantly oscillates between the same positive and negative real numbers. The same behavior continues even for larger values of iterations ($N \gg 1000$).

The numerical analysis suggests that when the parameter α_1 is positive, there is a damping effect that prevents some initial conditions from reaching one of the roots of the complex equation. On the other hand, for negative values of the parameter α_1 , there is an anti-damping effect leading to oscillatory or diverging behavior of some starting conditions.

The histograms shown in Figs. 5 and 8, with the probability distributions, may give additional results about the properties of the modified NR method. For example, the right-hand side of the histograms can be fitted by using the well-known Laplace distribution, which is the simplest and most suitable choice (Motter and Lai (2001); Seoane et al. (2006); Seoane et al. (2008)).

The probability density function (PDF) of the Laplace distribution reads

$$P(N|l,d) = \frac{1}{2d} \begin{cases} \exp\left(-\frac{l-N}{d}\right), & \text{if } N < l, \\ \exp\left(-\frac{N-l}{d}\right), & \text{if } N \ge l, \end{cases}$$
(4)

where the quantities l and d > 0 are known as the location parameter and the diversity, respectively. Since we are interested only for the tails of the probability histograms we need only the $N \ge l$ part of the PDF. In the following Table 1, we provide the numerical values of both the location parameter l as well as the diversity d, for all the cases presented in this section.

Note that the smaller the value of d the best is the fit of the Laplace distribution. For example, in panels (d) and (g) of Figure 8 we have seen that the Laplace PDF does not fit very satisfactorily the tails of the histograms. This is because for this cases the values of d are 12.41 and 11.38, respectively.

Figure	N^*	l	d
5 (a)	24	$N^{*} - 2$	4.14
5 (b)	17	N^*-1	2.71
5 (c)	14	N^*	2.55
5 (d)	14	N^*	4.98
5 (e)	14	N^*	6.32
5 (f)	13	$N^* + 1$	4.31
8 (a)	12	N^*	2.28
8 (b)	8	N^*	1.44
8 (c)	11	N^*	2.46
8 (d)	37	$N^* - 5$	12.41
8 (e)	13	$N^* + 1$	3.57
8 (f)	9	$N^{*} + 2$	4.02
8 (g)	12	$N^{*} + 6$	11.38
8 (h)	8	$N^* + 2$	3.12
8 (i)	9	$N^* + 2$	3.59

Table 1. Table containing the values of N^* , l, and d for all the probability histograms presented in this section.

3. Influence of the parameter α_1

For determining how exactly the parameter α_1 influences the convergence properties of the modified NR method we classified 200 sets of starting conditions $(\mathcal{R}, \mathcal{I})$ regularly distributed inside the square region $R = [-3,3] \times [-3,3]$ on the complex numbers plane, for the range $\alpha_1 \in [-2,2]$. Our outcomes are given in Figure 11(a-d).

In part (a) of Figure 11 we see how N^* evolves, as a function of α_1 . It is seen, that when $\alpha_1 < -1$ it fluctuates mostly around 8 and 9, while the most interesting behavior is displayed for $\alpha_1 > -1/2$. Indeed, when $\alpha_1 > -1/2$ we observe that the value of N^* increase gradually by following a ladder-type growth. It should be noted, that the increase of N^* starts exactly at the value of α_1 which corresponds to the Halley method ($\alpha_1 = -1/2$). The evolution of the diversity d as well of the differential entropy $h = 1 + \ln(2b)$ (Lazo and Rathie (1978)) is shown in parts (b) and (c) of Figure 11, respectively. Both quantities exhibit a very similar pattern. In particular, the lowest values of d and h exist in the interval $-1 < \alpha_1 < 0$, while at the boundaries of this interval we observe sudden peaks. These two peaks appear for values of α_1 where there exist non-converging initial conditions. In part (d) of Figure 11 we show how the percentage A(%) of the area on the plane of complex numbers, where the modified NR method displays an ill behavior, evolves as a function of α_1 . One can observe that for $\alpha_1 < -1$ more than 90% of the plane is covered by points for which the iterator quickly diverges. In the same vein, for $\alpha_1 > 0$ we have the appearance of initial conditions which lead to false convergence to zero (z = 0). Apparently, in the region $-1 < \alpha_1 < 0$ the modified method displays the optimum performance, as there is no evidence of ill or non-converging starting conditions.

For obtaining a quantitative measure about the fractality of the convergence regions we will use

the so-called basin entropy S_b tool (Daza et al. (2016); Daza et al. (2018)), which computes the fractality of a diagram. The evolution of S_b , as a function of α_1 , is given in part (a) of Figure 12. We see that for $\alpha_1 < -1$ and $\alpha_1 > 0$, where the complex plane is filled by unified areas of ill converging points the degree of fractality is low, while in the range $-1 < \alpha_1 < 0$ the unpredictability is higher. Once more, we note that in the boundaries of this range there are sudden peaks of S_b , simply because of the presence of the highly complicated mixture of converging and non-converging starting points (see e.g., panel (d) of Figure 3 and panel (g) of Figure 6).

Another useful tool for the quantitative estimation of the fractal degree of a basin diagram is the boundary basin entropy S_{bb} (Daza et al. (2016)). The evolution of S_{bb} is shown in part (b) of Figure 12. Furthermore, using the so-called "log 2 criterion", we can obtain quantitative information on whether or not the basin boundaries are certainly fractal. This is true only when $S_{bb} > \log 2$, while the opposite argument is not always valid. As we see in part (b) of Figure 12 the boundaries of the basins are highly fractal, when $-1 < \alpha_1 < 0$, while the overall pattern of the evolution of S_{bb} is very similar to that of the basin entropy S_b .

Finally, one of the most convenient ways of measuring the degree of fractality of a basin diagram is by computing the uncertainty or fractal dimension D_0 (Ott (1993)), thus following the computational methodology used in (Aguirre et al. (2001); Aguirre et al. (2009)). At this point, we would like to emphasize that the degree of fractality is an intrinsic property of the system and therefore it does not depend on the particular initial conditions we use for its calculation. Figure 13 shows the parametric evolution of the uncertainty dimension, as a function of the parameter α_1 . When the fractal dimension is close to 1 it implies zero fractality, while when its value tends to 2 it suggests total fractality of the respective basin diagram. It is seen, that D_0 displays two main peaks near the boundaries of the interval $-1 < \alpha_1 < 0$. In addition, one should certainly discuss the large similarity on the parametric evolutionary pattern of D_0 with respect to that of the basin entropy S_b . We also note, that for a two-dimensional space, such as the complex plane the value of D_0 lies in the interval [1, 2].

4. Conclusion

The scope of this work was to unveil the behavior of the modified NR method. Using a simple complex polynomial function as a test example, we obtained the convergence regions on the complex numbers plane, by means of polynomiographs. Additionally, we investigated how the parameter α_1 , entering the iterative scheme, influences the main characteristics (e.g., speed and the accuracy) of the method. Moreover, we found how α_1 affects the fractal degree of the complex plane, by deploying modern indices, such as the (boundary) basin entropy and the fractal dimension.

A similar numerical analysis has been conducted in Paper I. However, as we described in detail in our work, that paper contains some errors, about the convergence properties of the modified NR method. In our article, we point out these mistakes, while we supplement the analysis given in Paper I, by presenting additional quantitative novel information on the behavior of the modified NR method.

The most interesting outcomes of our exploration are listed below:

- (1) When $\alpha_1 < -1$ most of the complex numbers plane is dominated by starting conditions for which the numerical method diverges very quickly to large numbers (shoot to infinity).
- (2) When $\alpha_1 > 0$ there are regions composed of points for which the numerical methods displays a false convergence to zero (z = 0).
- (3) In the intervals $-1.084 < \alpha_1 < -1.064$ and $0.064 < \alpha_1 < 0.085$ there exists a considerable amount of starting conditions for which the iterator does not converge.
- (4) Our computations suggest that there are no values of the parameter α_1 for which the method to display simultaneously false and non-convergence. Each ill behavior exists in different ranges, regarding the value of α_1 .
- (5) We concluded that the optimum behavior of the method appears roughly when $-1 < \alpha_1 < 0$, where there is no evidence of false or non-convergence.

The iterative NR algorithm was coded in the standard version of FORTRAN 77 (Press et al. (1992)), while additional tests have been also conducted in other programming languages, for verifying our results. For the classification of the sets of starting points on the complex numbers plane, we required about 3 minutes of CPU time, per grid, using a Quad-Core i7 4.2 GHz processor. All the diagrams of the article have been constructing by using version 12.0 of Mathematica[®] (Wolfram (2003)).

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REFERENCES

- Aguirre, J., Vallejo, J.C. and Sanjuán, M.A.F. (2001). Wada basins and chaotic invariant sets in the Hénon-Heiles system, Phys. Rev. E, Vol. 64, 066208.
- Aguirre, J., Viana, R.L. and Sanjuán, M.A.F. (2009). Fractal Structures in nonlinear dynamics, Rev. Mod. Phys., Vol. 81, pp. 333-386.
- Chun, C., Neta, B. and Kim, S. (2014). On Jarratt's family of optimal fourth-order iterative methods and their dynamics, Fractals, Vol. 22, 1450013.
- Chun, C. and Neta, B. (2015a). An analysis of a family of Maheshwari-based optimal eighth order methods., Appl. Math. Comput., Vol. 253, pp. 294-307.
- Chun, C. and Neta, B. (2015b). Basins of attraction for several third order methods to find multiple roots of nonlinear equations, Appl. Math. Comput., Vol. 268, pp. 129-137.

- Chun, C. and Neta, B. (2015c). Comparing the basins of attraction for Kanwar-Bhatia-Kansal family to the best fourth order method, Appl. Math. Comput., Vol. 266, pp. 277-292.
- Chun, C. and Neta, B. (2016). Comparison of several families of optimal eighth order methods, Appl. Math. Comput., Vol. 274, pp. 762-773.
- Chun, C. and Neta B. (2017a). On the dynamics of a triparametric family of optimal fourth-order multiple-zero finders with a weight function of the principal mth root of a function-to function ratio, Appl. Math. Comput., Vol. 315, pp. 564-590.
- Chun, C. and Neta, B. (2017b). Comparative study of eighth-order methods for finding simple roots of nonlinear equations, Numerical Algorithms, Vol. 74, pp. 1169-1201.
- Chun, C. and Neta, B. (2018). Comparative study of methods of various orders for finding repeated roots of nonlinear equations, J. Comput. Appl.Math., Vol. 340, pp. 11-42.
- Daza, A., Wagemakers, A., Georgeot, B., Guéry-Odelin, D. and Sanjuán, M.A.F. (2016). Basin entropy: a new tool to analyze uncertainty in dynamical systems, Scientific Reports, Vol. 6, 31416.
- Daza, A., Wagemakers, A., Georgeot, B., Guéry-Odelin, D. and Sanjuán, M.A.F. (2018). Basin Entropy, a Measure of Final State Unpredictability and Its Application to the Chaotic Scattering of Cold Atoms. M. Edelman et al. (eds.), Chaotic, Fractional, and Complex Dynamics: New Insights and Perspectives, Understanding Complex Systems, Springer International Publishing AG.
- Kellison, S.G. (1975). Fundamentals of Numerical Analysis, R.D. Irwin, Inc., Homewood, Illinois.
- Lazo, A. and Rathie, P. (1978). On the entropy of continuous probability distributions, IEEE Transactions on Information Theory, Vol. 24, pp. 120-122.
- Motter, A.E. and Lai, Y.C. (2001). Dissipative chaotic scattering, Phys. Rev. E, Vol. 65, 015205.
- Neta, B., Scott, M. and Chun, C. (2012). Basins of attraction for several methods to find simple roots of nonlinear equations, Appl. Math. Comput., Vol. 218, pp. 10548-10556.
- Ott, E. (1993). Chaos in Dynamical Systems, Cambridge University Press, Cambridge.
- Press, H.P., Teukolsky, S.A., Vetterling, W.T. and Flannery, B.P. (1992). Numerical Recipes in FORTRAN 77, 2nd Ed., Cambridge Univ. Press, Cambridge, USA.
- Royden, H.L. (1988). Real Analysis, 3rd edition, New York, Macmillan.
- Seoane, J.M., Aguirre, J., Sanjuán, M.A.F. and Lai, Y.C. (2006). Basin topology in dissipative chaotic scattering, Chaos, Vol. 16, 023101.
- Seoane, J.M. and Sanjuán, M.A.F. (2008). Exponential decay and scaling laws in noisy chaotic scattering, Phys. Let. A, Vol. 372, pp. 110-116.
- Susanto, H. and Karjanto, N. (2009). Newton's methods's basins of attraction revisited, Appl. Math. Comput., Vol. 215, pp. 1084-1090 (Paper I).
- Wolfram, S. (2003). The Mathematica Book, Fifth Edition, Wolfram Media, Champaign.
- Zotos, E.E. (2017). Comparing the fractal basins of attraction in the Hill problem with oblateness and radiation, Astrophys. Space Sci., Vol. 362, 190.
- Zotos, E.E. (2018). Comparing the basins of attraction for several methods in the circular Sitnikov problem with spheroid primaries, Astrophys. Space Sci., Vol. 363, 113.