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Generalized Growth and Faber Polynomial Approximation of Entire Functions of Several Complex Variables in Some Banach Spaces

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Abstract

In this paper the relationship between the generalized order of growth of entire functions of many complex variables $m(m \geq 2)$ over Jordan domains and the sequence of Faber polynomial approximations in some Banach spaces has been investigated. Our results improve the various results shown in the literature.

Keywords: Generalized order of growth; Faber polynomial approximation; Unit polydisk; Banach spaces and several complex variables

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1. Introduction

We denote the space of complex numbers $z = x + iy$ by $\mathbb{C}$ and

$$\mathbb{C}^m = \{\mathbf{z} = (z_1, \dots, z_m) : z_j \in \mathbb{C}, j = \overline{1, m}\},$$

an m-dimensional complex space. Let $\Gamma_j, j = \overline{1, m}$ be given Jordan curves in the complex plane $\mathbb{C}$ and D_j, E_j be the interior and exterior respectively of Γ_j. Let φ_j map E_j conformally onto $\{w_j : |w_j| > 1\}$ such that $\varphi_j(\infty) = \infty$ and $\varphi'_j(\infty) > 0$. Then, in view of Faber (1979), for sufficiently large $|z_j|, \varphi_j(z_j)$ can be expressed as

$$\begin{aligned} w_1 = \varphi_1(z_1) &= \frac{z_1}{d_1} + c_0 + \frac{c_1}{z_1} + \frac{c_2}{z_1^2} + \dots, \\ w_2 = \varphi_2(z_2) &= \frac{z_2}{d_2} + c'_0 + \frac{c'_1}{z_2} + \frac{c'_2}{z_2^2} + \dots, \\ &\dots, \\ w_m = \varphi_m(z_m) &= \frac{z_m}{d_m} + c_0^{(m-1)\prime} + \frac{c_1^{(m-1)\prime}}{z_m} + \frac{c_2^{(m-1)\prime}}{z_m^2} + \dots, \end{aligned}$$

where $d_j > 0$.

Let us put $D = D_1 \times D_2 \times \dots \times D_m$ and $E = E_1 \times E_2 \times \dots \times E_m$ in $\mathbb{C}^m$ and let the function φ map E conformally onto the unit polydisk $U^m = \{\mathbf{z} \in \mathbb{C}^m : |w_j| > 1, j = \overline{1, m}\}$ such that $\varphi(\mathbf{z}) = \varphi_1(z_1)\varphi_1(z_1)\dots\varphi_m(z_m)$ satisfies the conditions $\varphi(\infty) = \infty : \infty = (\infty, \dots, \infty)$ and $\varphi'(\infty) > 0$. Then, for sufficiently large value of $|z_j|, \varphi(\mathbf{z})$ can be expressed as

$$w_1 w_2 \dots w_m = \varphi(\mathbf{z}) = \frac{\mathbf{z}}{\mathbf{d}} + \sum_{|\mathbf{k}|=0}^{\infty} \frac{\mathbf{c_k}}{\mathbf{z^k}},$$

where $\frac{\mathbf{z}}{\mathbf{d}} = \frac{z_1}{d_1} \dots \frac{z_m}{d_m}, \mathbf{k} = (k_1, \dots, k_m) \in z_+^m, |\mathbf{k}| = k_1 + k_2 + \dots + k_m, \mathbf{z^k} = z_1^{k_1} \dots z_m^{k_m}$ and $\mathbf{c_k} = c_{k_1,\dots,k_m}$. It is known that an arbitrary Jordan curve can be approximated from the inside as well as from the outside by analytic Jordan curves. Since $\Gamma = \Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_m$ is analytic and φ is holomorphic on Γ as well the k^{th} Faber polynomial $F_{\mathbf{k}}(\mathbf{z})$ of Γ_j is the principal part of $(\varphi(\mathbf{z}))^{|\mathbf{k}|}$ at (∞), so that

$$F_{\mathbf{k}}(\mathbf{z}) = (\frac{\mathbf{z}}{\mathbf{d}})^{\mathbf{k}} + \dots \quad .$$

Following the one variable case of Faber (1979), it can be seen that

$$F_{\mathbf{k}}(\mathbf{z}) \sim (\varphi_1(z_1))^{k_1} (\varphi_2(z_2))^{k_2} \dots (\varphi_m(z_m))^{k_m} \sim (\frac{\mathbf{z}}{\mathbf{d}})^{\mathbf{k}}, \tag{1}$$

uniformally for $z_j \in E_j$ and

$$\lim_{|\mathbf{k}| \to \infty} (\max_{\mathbf{z} \in \Gamma} |F_{\mathbf{k}}(\mathbf{z})|)^{\frac{1}{|\mathbf{k}|}} = 1. \tag{2}$$

A function f holomorphic in D can be represented by its Faber series

$$f(\mathbf{z}) = \sum_{|\mathbf{k}|=0}^{\infty} \mathbf{a_k} F_{\mathbf{k}}(\mathbf{z}),$$

where

$$\mathbf{a_k} = \frac{1}{(2\pi i)^m} \int_{|w_1|=r_1} \cdots \int_{|w_m|=r_m} f(\varphi_1^{-1}(w_1), \dots, \varphi_m^{-1}(w_m)) w_1^{-(k_1+1)} \dots w_m^{-(k_m+1)} \mathbf{dw},$$

$\mathbf{a_k} = a_{k_1,\dots,k_m}$, $\mathbf{dw} = dw_1 dw_2 \dots dw_m$ and $0 < r_1, \dots, r_m < 1$ are sufficiently close to 1 so that for $j = 1, \dots, m, \varphi_j^{-1}$ are holomorphic and univalent in $|w_j| \geq r_j$ respectively, the series converging uniformly on compact subsets of D.

Consider a unit polydisk $U^m = \{\mathbf{z} \in \mathbb{C}^m : |z_j| < 1, j = \overline{1,m}\}$ and let $\mathbf{\Gamma}^m = \{\mathbf{z} \in \mathbb{C}^m : |z_j| = 1, j = \overline{1,m}\}$ be its skeleton. By $\mathbb{R}$, we denote the set of all finite real numbers. Also, let $\mathbb{R}^m$ be an m-dimensional real space. Further, by

$$\mathbb{T}^m = \{\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m : 0 \leq x_j \leq 2\pi, j = \overline{1,m}\},$$

and

$$\Pi^m = \{\mathbf{r} = (r_1, \dots, r_m) \in \mathbb{R}^m : 0 \leq r_j < 1, j = \overline{1,m}\},$$

we denote m-dimensional cubes in $\mathbb{R}^m$.

Now we consider the following Banach spaces X formed by the functions analytic in a unit polydisk U^m with finite norm.

(1) The space B of functions analytic in the set U^m and continuous in its skeleton $\mathbf{\Gamma}^m$ with the norm

$$\|f\|_B = \max_{\mathbf{z} \in \mathbf{\Gamma}^m} |f(\mathbf{z})| < \infty.$$

(2) Let $A(U^m)$ be the set of all functions f analytic in the set U^m. By $H_p(U^m), 0 < p \leq \infty$, we denote a Hardy space formed by functions $f \in A(U^m)$ with the norm

$$\|f\|_{H_p} = \sup\{M_p(\mathbf{r}, f) : \mathbf{r} \in \Pi^m\} < \infty,$$
$$M_p(\mathbf{r}, f) = \left\{\frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} |f(\mathbf{r}e^{i\mathbf{t}})|^p d\mathbf{t}\right\}^{1/p}, p \in [1, \infty),$$
$$M_\infty(\mathbf{r}, f) = \max\left\{|f(\mathbf{r}e^{i\mathbf{t}})| : t \in \mathbb{T}^m\right\}, r \in \Pi^m,$$

where

$$f(\mathbf{r}e^{i\mathbf{t}}) = f(r_1 e^{it_1}, \dots, r_m e^{it_m}), d\mathbf{t} = dt_1 \dots dt_m.$$

$H_p(U^m)$ is a Banach space for $p \geq 1$.

Remark 1.1.

For $m = 1$, the spaces $H_p(U) \equiv H_p(U^1), 0 < p \leq \infty$, were studied for the first time by Hardy (1914). Further, these spaces are completely investigated by Littlewood, Privalov, F. Riesz, M. Riesz, Zygmund, etc.

(3) The Bergman spaces $H'_p(U^m)$ of functions $f \in A(U^m)$ for $p \in [1, \infty)$ with the norm

$$\|f\|_{H'_p} = \left(\frac{1}{(\pi)^m} \int \int_{\mathbf{z} \in \mathbf{\Gamma}^m} |f(x+iy)|^p dxdy \right)^{1/p},$$

where $f(x+iy) = f(x_1+iy_1, \ldots, x_m+iy_m), dxdy = dx_1dy_1 \ldots dx_mdy_m$.

(4) The spaces $A_p(U^m), p \in (0,1)$, of functions $f \in A(U^m)$ with the norm

$$\|f\|_{A_p(U^m)} = \int_{\Pi^m} (1-\mathbf{r})^{1/p-1} M_1(\mathbf{r}, f) d\mathbf{r}.$$

These spaces were first studied by Hardy and Littlewood (1931) and later by Duren et al.(1969) for $m = 1$.

(5) The spaces $B_m(p, q, \lambda), 0 < p < q \leq \infty, \lambda > 0$, of functions $f \in A(U^m)$ with the norm

$$\|f\|_{p,q,\lambda} = \left\{ \int_{\Pi^m} (1-\mathbf{r})^{\lambda(\frac{1}{p}-\frac{1}{q})-1} M_q^\lambda(\mathbf{r}, f) d\mathbf{r} \right\}^{\frac{1}{\lambda}}, 0 < \lambda < \infty,$$

and

$$\|f\|_{p,q,\infty} = \sup \left\{ (1-\mathbf{r})^{\frac{1}{p}-\frac{1}{q}} M_q(\mathbf{r}, f) : \mathbf{r} \in \Pi^m \right\} < \infty, \lambda = \infty.$$

Remark 1.2.

The space $B_m(p, q, \lambda)$ is a Banach space if $min(q, \lambda) \geq 1$. Gvaradze (1977) considered the spaces of analytic functions $B(p, q, \lambda), 0 < p < q \leq \infty, 0 < \lambda \leq \infty$, to generalize the Hardy spaces and established the properties of functions from these spaces for functions $f \in A(U) \equiv A(U^1)$, with weaker restrictions imposed on their behavior near boundary of a unit disk and latter Gvaradze (1975, 1980) introduced the spaces $B_m(p, q, \lambda)$ to the case of polydisk $U^m, m \in \mathbb{N}\backslash\{1\}$.

Let X be one of the Banach spaces of analytic functions of m-complex variables. By P_n we denote a subspace of algebraic polynomials of m-complex variables of the form

$$P_n = \left\{ \sum_{|k|=0}^{n} \mathbf{a_k} \mathbf{z}^k : \mathbf{a_k} \in \mathbb{C} \right\}, n \in \mathbb{Z}_+.$$

By $E_n(f, X)$, we denote the value of the best polynomial approximation of the function $f \in X$ by elements of the subspace P_n, i.e.,

$$E_n(f, X) = \inf \left\{ \|f - p_n\|_X : p_n \in P_n \right\}.$$

To study the generalized growth in single complex variable of entire function $f(z) = \sum_{n=0}^{\infty} a_n F_n(z)$, Ganti and Srivastava (2009) defined the general functions as follows.

Let L^0 denote the class of functions h satisfying the following conditions.

(i) $h(x)$ is defined on $[a, \infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$,

(ii) $\lim_{x\to\infty} \frac{h\{(1+1/\xi(x))x\}}{h(x)} = 1$, for every function $\xi(x)$ such that $\xi(x) \to \infty$ as $x \to \infty$.

Let Δ denote the class of functions h satisfying condition (i)and $\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1$, provided $c > 0$, that is, $h(x)$ is slowly increasing.

For functions $\alpha(x) \in \Delta$, $\beta(x) \in L^0$, Ganti and Srivastava (2009) defined generalized order as

$$\rho(\alpha, \beta, f) = \limsup_{r\to\infty} \frac{\alpha[\log M(r, f)]}{\beta(\log r)},$$

where $M(r, f) = \max_{|z|=r} |f(z)|$, and obtained coefficients characterizations over Jordan domains.

The necessary and sufficient conditions of generalized order of entire functions in terms of approximation errors also have been studied. All these results have been obtained by using the condition.

$$\frac{d(\beta^{-1}(c\alpha(x)))}{d(\log x)} \le b, x \ge a, 0 < c < \infty, \tag{3}$$

where a and b are positive constants. It has been observed that the condition (3) does not hold for $\alpha = \beta$. To overcome, this problem Kumar (2013) used the approach introduced by Kapoor and Nautiyal (1981) and defined generalized order $\rho(\alpha, \alpha, f)$ of slow growth with the help of general functions as follows.

Let Ω be the class of functions $h(x)$ satisfying (i) and

(iv) there exists a $\eta(x) \in \Omega$ and x_0, K_1 and K_2 such that

$$0 < K_1 \le \frac{d(h(x))}{d(\eta(\log x))} \le K_2 < \infty \texttt{ for all } x > x_0.$$

Let $\overline{\Omega}$ be the class of functions $h(x)$ satisfying (i) and
(v)

$$\lim_{x\to\infty} \frac{d(h(x))}{d(\log x)} = K \quad , \quad 0 < K < \infty.$$

Kapoor and Nautiyal (1981) showed that classes Ω and $\overline{\Omega}$ are contained in Δ. Further, $\Omega \cap \overline{\Omega} = \phi$ and they defined the generalized order $\rho(\alpha, \alpha, f)$ for entire function $f(z)$ of slow growth as

$$\rho(\alpha, \alpha, f) = \limsup_{r\to\infty} \frac{\alpha(\log M(r, f))}{\alpha(\log r)},$$

where $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$.

Kumar (2013) improved the results studied by Ganti and Srivastava (2009). Generalized α-logarithmic order has been studied in the spaces 1,2,4,5 mentioned above in finite domain by Vakarchuk and Zhir (2011). The coefficient characterizations of classical order and type of entire functions of two complex variables over Jordan domains by using Faber polynomials have been obtained in terms of the approximation error in L^p-norm $(2 \leq p \leq \infty)$ by Ganti and Srivastava (2009). The aim of this paper is to extend the above mentioned results in several complex variables using the concept of generalized order of slow growth. Also, Harfaoui (2010, 2014) investigated some results concerning generalized growth and approximation of entire functions in L_p-norm but our results are different from those of Harfaoui.

The paper is organized as follows. Section 1 incorporate a brief introduction of the topic. Section 2, deals with some preliminary results which have been used to prove the main results. In Section 3, the coefficient characterizations of generalized α-order of entire functions represented by Faber series expansion in several complex variables have been obtained in terms of approximation errors in the Banach spaces (1-5).

2. Auxiliary Results

This section contains some auxiliary results which will be used in the sequel.

Lemma 2.1.

Let $f \in X$ and $f(\mathbf{z}) = \sum_{|\mathbf{k}|=0}^{\infty} \mathbf{a_k}(f) F_{\mathbf{k}}(\mathbf{z})$ be an entire function of m-complex variables.

Then,

$$\lim_{|\mathbf{k}|\to\infty} \{\frac{K_X^*(\mathbf{k})}{\mathbf{d^k}}\}^{\frac{1}{|\mathbf{k}|}} = 1, \tag{4}$$

where

$$K_X^*(\mathbf{k}) = \{\|F_{\mathbf{k}}(\mathbf{z})\|_X\}^{-1}.$$

Proof:

In view of (1), we have

$$\|F_{\mathbf{k}}(\mathbf{z})\|_X \sim \frac{\|\mathbf{z^k}\|_X}{\|\mathbf{d^k}\|_X} = \frac{\|\mathbf{z^k}\|_X}{\mathbf{d^k}}.$$

Using the definition of norms

$$\|f(.e^{i\mathbf{t}})\|_X = \|f(.)\|_X,$$

for all $t \in \mathbb{R}^m$ and $f \in X$,

$$\|f(.)\|_X < \infty,$$

for any entire function in the spaces B and $H_p(U^m), 0 < p \leq \infty$, respectively, we get

$$\|\mathbf{z^k}\|_X = 1,$$

and $K_X^*(\mathbf{k}) = \mathbf{d^k}, \mathbf{k} \in \mathbb{Z}_+^m$, i,e., the result (4). In the space $X = H_p'(U^m), p \geq 1$, we have

$$\frac{K^*_{H_p'(U^m)}(\mathbf{k})}{\mathbf{d^k}} = \left\{\Pi_{j=1}^m (k_j p + 2)^{1/p}\right\}^{1/p}, k_j \geq 0, \tag{5}$$

$$\leq \left\{\Pi_{j=1}^{m}\left(k_j p\left(1+\frac{2}{k_j p}\right)\right)^{1/p}\right\}^{1/p}$$

$$\leq \chi_{H_p'(U^m)}|\mathbf{k}|^{m/p^2}, \tag{6}$$

where

$$\chi_{H_p'(U^m)} = \frac{p^{m/p^2}}{m^{m/p^2}}\left(1+\frac{2}{p}\right)^{m/p^2}.$$

From (6) we get the following upper bound

$$\limsup_{|\mathbf{k}|\to\infty}\left(\frac{K^*_{H_p'(U^m)}(\mathbf{k})}{\mathbf{d}^{\mathbf{k}}}\right)^{\frac{1}{|\mathbf{k}|}} \leq 1. \tag{7}$$

For lower bound we use (5) and get

$$\frac{K^*_{H_p'(U^m)}(\mathbf{k})}{\mathbf{d}^{\mathbf{k}}} \geq \frac{p^{m/p^2}}{m^{m/p^2}}\left(\Pi_{j=1}^{m}k_j\right)^{1/p^2}$$
$$\geq \left\{\frac{k_m}{m^{m/p^2}}\right\}^{1/p^2},$$

or

$$\liminf_{|\mathbf{k}|\to\infty}\left\{\frac{K^*_{H_p'(U^m)}(\mathbf{k})}{\mathbf{d}^{\mathbf{k}}}\right\}^{1/|\mathbf{k}|} \geq 1. \tag{8}$$

(7) and (8) together gives the required result.

In the space $X = A_p(U^m), 0 < p < 1$, we have

$$\frac{K^*_{A_p(U^m)}(\mathbf{k})}{\mathbf{d}^{\mathbf{k}}} = (2\pi)^{-m/p}\left(\Pi_{j=1}^{m}B(k_jp+1;\frac{1}{p}-1)\right)^{-1/p^2}.$$

We have the asymptotic relation Lebedev (1963)

$$\frac{\Gamma(x+s_1)}{\Gamma(x+s_2)} = x^{s_1-s_2}\left(1+\frac{(s_1-s_2)(s_1+s_2-1)}{2x}+0(|x^{-2}|)\right),$$

where $|x| \gg 1, x \in \mathbb{R}$, and s_1 and s_2 are arbitrary fixed real numbers.

Also, the relation between the Euler integral of the first kind $B(a,b)$ and Γ-function for $a, b > 0$ is given by

$$B(a,b) = \frac{\Gamma a \Gamma b}{\Gamma(a+b)}.$$

Setting $x = k_j p, s_1 = \frac{1}{p}$ and $s_2 = 1$, for sufficiently large $|\mathbf{k}|$, for $k_j \gg 1, j = \overline{1,m}$ in above relations, we get

$$\frac{K^*_{A_p(U^m)}(\mathbf{k})}{\mathbf{d}^{\mathbf{k}}} = \left[\frac{(2\pi)^{-mp}}{\Gamma^m(\frac{1}{p}-1)}\Pi_{j=1}^m \frac{\Gamma(k_j p + \frac{1}{p})}{\Gamma(k_j p+1)}\right]^{1/p^2}$$

$$= \frac{(2\pi)^{-mp} p^{m/p^2(1/p-1)}}{\Gamma^{m/p^2}(1/p-1)} \left\{\Pi_{j=1}^m k_j \left(1 + \frac{(\frac{1}{p}-1)\left(\frac{1}{p}\right)}{2k_j p} + O\left(k_j^{-2}p^{-2}\right)\right)\right\}^{1/p^2}$$

$$\leq \chi_{A_p(U^m)} |\mathbf{k}|^{m/p^2},$$

where

$$\chi_{A_p(U^m)} = \frac{(2\pi)^{-mp} p^{m/p^2(1/p-1)}}{\Gamma^{m/p^2}(\frac{1}{p}-1) m^{m/p^2}} \left(1 + \left(\frac{1}{p}-1\right)\left(\frac{1}{p}\right) + A\right)^{m/p^2}.$$

Here A is an absolute constant independent of $\mathbf{k}$. Therefore,

$$\limsup_{|\mathbf{k}|\to\infty} \left(\frac{K^*_{A_p(U^m)}(\mathbf{k})}{\mathbf{d}^{\mathbf{k}}}\right)^{1/|\mathbf{k}|} \leq 1. \tag{9}$$

For lower bound

$$\frac{K^*_{A_p(U^m)}(\mathbf{k})}{\mathbf{d}^{\mathbf{k}}} \geq \frac{(2\pi)^{-mp} p^{\frac{m}{p^2}(1/p-1)}}{\Gamma^{m/p^2}(1/p-1)} \left\{\Pi_{j=1}^m k_j\right\}^{1/p^2}$$
$$\geq \left\{\frac{k_m}{\Gamma^m(1/p-1)}\right\}^{1/p^2},$$

or

$$\liminf_{|\mathbf{k}|\to\infty} \left\{\frac{K^* A_p(U^m)(\mathbf{k})}{\mathbf{d}^{\mathbf{k}}}\right\}^{1/|\mathbf{k}|} \geq 1. \tag{10}$$

Combining (9) and (10) we get the required result.

Finally, for the spaces $X = B_m(p,q,\lambda), 0 < p < q \leq \infty, 0 < \lambda \leq \infty$, it follows from Vakarchuk and Zhir (2015) that

$$\lim_{|\mathbf{k}|\to\infty} \left\{\frac{K^*_{B_m(p,q,\lambda)}(\mathbf{k})}{\mathbf{d}^{\mathbf{k}}}\right\}^{1/|\mathbf{k}|} = 1. \qquad \blacksquare$$

Lemma 2.2.

Let $f \in X$ and let

$$f(\mathbf{z}) = \sum_{|\mathbf{k}|=0}^{\infty} \mathbf{a_k}(f) F_{\mathbf{k}}(\mathbf{z}) \texttt{ in } U^m.$$

Then, for $|\mathbf{k}| = n+1$,

$$\mathbf{d^k} |\mathbf{a_k}(f)| \|F_{\mathbf{k}}(\mathbf{z})\|_X \leq E_n(f, X) \leq \|f\|_X. \tag{11}$$

Proof:

Let $\{\mathbf{z} \in \mathbb{C}^m : |z_j \zeta_j| = r_j, j = \overline{1, m}, r_j \in \Pi^m\}$. Using the relation for the Taylor coefficients $\mathbf{a_k}(f), k \in z_+$, of the function $f \in X$, we obtain

$$\mathbf{d^k} |\mathbf{a_k}(f)| \|F_{\mathbf{k}}(\mathbf{z})\|_X = |\mathbf{a_k}| \|\mathbf{z^k}\|_X = \frac{1}{(2\pi i)^m} \int_{|\zeta|=1} \frac{f(\mathbf{z}\zeta) - p_n(\mathbf{z}\zeta)}{\zeta^{n+1}} d\zeta,$$

here $p_n \in P_n$ is a polynomial of the best approximation for $f(\mathbf{z})$ of degree not greater than $n-1$. Now using the conditions of the norm defined in Lemma 2.1, we get (11) for $|\mathbf{k}| = n+1$. ■

Following on the lines of Srivastava and Ganti (2012, Thm. 1,2) we get the following relationship between order, type and moduli of the Faber coefficients $|\mathbf{a_k}(f)|$ of entire function $f(\mathbf{z})$ of m-complex variables:

$$\rho = \limsup_{|\mathbf{k}| \to \infty} \frac{|\mathbf{k}| \log |\mathbf{k}|}{-\log |\mathbf{a_k}(f)|}, \tag{12}$$

$$T = \frac{1}{e\rho} \limsup_{|\mathbf{k}| \to \infty} \frac{|\mathbf{k}|}{\left(\frac{|\mathbf{a_k}(f)|}{\mathbf{d^k}}\right)^{-\rho/|\mathbf{k}|}}. \tag{13}$$

Now we define the quantities $\rho_m(\alpha, f)$ and $\rho'_m(\alpha, f)$ as the generalized α-order of growth as:

$$\rho_m(\alpha, f) = \limsup_{\mathbf{r} \to \infty} \frac{\alpha(\log M(\mathbf{r}, f))}{\alpha(\log \mathbf{r})}, \tag{14}$$

$$\rho'_m(\alpha, f) = \limsup_{|\mathbf{k}| \to \infty} \frac{\alpha(|\mathbf{k}|)}{\alpha(\frac{1}{|\mathbf{k}|} \log |\mathbf{a_k}(f)|^{-1})}. \tag{15}$$

3. Main Results

In this section we shall prove our main results.

Theorem 3.1.

Let $f \in X$ and $f(\mathbf{z}) = \sum_{|\mathbf{k}|=0}^{\infty} \mathbf{a_k}(f) F_{\mathbf{k}}(\mathbf{z})$ be an entire function of m-complex variables with a generalized α-order of growth $\rho_m(\alpha, f)$. Then, the following equality is valid for $\alpha(x) \in \overline{\Omega}$:

$$\rho_m(\alpha, f) = \rho'_m(\alpha, f) + 1. \tag{16}$$

Proof:

From (14), for an arbitrary $\varepsilon > 0$, there exist real numbers $r_{j_0}(\varepsilon) > 1, j = \overline{1, m}$ such that, for all $r > r_{j_0}(\varepsilon)$, the inequality

$$\frac{\alpha(\log M(\mathbf{r}, f))}{\alpha(\log \mathbf{r})} \leq \rho_m(\alpha, f) + \varepsilon \equiv \overline{\rho},$$

holds. It gives

$$M(\mathbf{r}, f) \leq \exp(\alpha^{-1}(\overline{\rho}\alpha(\log \mathbf{r}))). \tag{17}$$

Following Faber (1979), the Faber coefficients $\mathbf{a_k}(f)$ satisfy the relation

$$|\mathbf{a_k}(f)| \leq M(\mathbf{r}, f)\mathbf{r}^{-\mathbf{k}}. \tag{18}$$

Using (17) and (18) we get

$$|\mathbf{a_k}(f)| \leq \mathbf{r}^{-\mathbf{k}} \exp(\alpha^{-1}(\overline{\rho}\alpha(\log \mathbf{r}))). \tag{19}$$

The minimum value of right hand side of (19) is attained at

$$\mathbf{r}(\mathbf{k}) = \exp(G(\frac{|\mathbf{k}| \log \mathbf{r}}{\overline{\rho}}, \frac{1}{\overline{\rho}})), \tag{20}$$

where $G(x, c) = \alpha^{-1}(c\alpha(x))$. Since $\alpha \in \Lambda$, (19) implies that $\mathbf{r}$ increases monotonically with $\mathbf{k}$. Therefore, there exist $n_0(\varepsilon) \in \mathbb{N}$ for which $\mathbf{r}(n_0(\varepsilon)) > \mathbf{r_0}(\varepsilon)$.

Moreover, for all $|\mathbf{k}| > n_0(\varepsilon)$, we have $\mathbf{r}(|\mathbf{k}|) > \mathbf{r_0}(\varepsilon)$. Substituting (20) in (19) we obtain

$$\alpha(\log \mathbf{r}) < \alpha\{\frac{\overline{\rho}}{(\overline{\rho} - 1)|\mathbf{k}|} \log |\mathbf{a_k}(f)|^{-1}\}.$$

Since $\alpha \in \overline{\Omega}$ as $\mathbf{k} \to \infty$, we have

$$\alpha(\log \mathbf{r}) \sim \frac{1}{\overline{\rho} - 1}\alpha(|\mathbf{k}|),$$

for $\mathbf{r} = \mathbf{r}(\mathbf{k})$ satisfying (20). Thus, we obtain

$$|\mathbf{a_k}(f)| \leq \exp(-|\mathbf{k}|(\frac{\overline{\rho} - 1}{\overline{\rho}})G(|\mathbf{k}|, \frac{1}{\overline{\rho} - 1})),$$

or

$$G(|\mathbf{k}|, \frac{1}{\overline{\rho} - 1}) \leq \frac{\overline{\rho}}{\overline{\rho} - 1} \log |\mathbf{a_k}(f)|^{\frac{1}{|\mathbf{k}|}},$$

or

$$\limsup_{|\mathbf{k}\to\infty|} \frac{\alpha(|\mathbf{k}|)}{\alpha(\log(\frac{1}{|\mathbf{a_k}(f)|^{\frac{1}{|\mathbf{k}|}}}))} \leq \overline{\rho} - 1.$$

Conversely, let

$$\rho'_m(\alpha, f) = \limsup_{|\mathbf{k}|\to\infty} \frac{\alpha(|\mathbf{k}|)}{\alpha(\frac{1}{|\mathbf{k}|}\log|\mathbf{a_k}(f)|^{-1})}.$$

Suppose $\rho'_m(\alpha, f) < \infty$. Then, for every $\varepsilon > 0$, there exists a natural number $n_2(\varepsilon)$ such that, for all $\mathbf{k} > n_2(\varepsilon)$, we have

$$|\mathbf{a_k}(f)| \leq (\exp(\alpha^{-1}(\frac{1}{\tilde{\rho}}\alpha(|\mathbf{k}|))))^{-\mathbf{k}}, \;\; \tilde{\rho} = \rho'_m(\alpha, f) + \varepsilon.$$

Let $\{D_R\} \in \mathbb{C}^m, R > 1$ be a family of complete m-circular domains depending on the parameter R such that $\mathbf{z} \in D_R$ if, and only if

$$\frac{\mathbf{z}}{R} = \left(\frac{z_1}{R}, \ldots, \frac{z_m}{R}\right) \in D, D = D_1.$$

Since $f \in X, f_R \in X$ i.e., $f_\zeta(\mathbf{z}) = f(\mathbf{z}\zeta)$ and f is entire function. We have

$$f(\mathbf{z}) = \sum_{|\mathbf{k}|=0}^{\infty} \mathbf{a_k}(f)F_\mathbf{k}(\mathbf{z}),$$

$$f_R(\mathbf{z}) = f(R\mathbf{z}) = \sum_{|\mathbf{k}|=0}^{\infty} \mathbf{a_k}(f)F_\mathbf{k}(R\mathbf{z}),$$

$$\|f_R(\mathbf{z})\|_X = \sum_{|\mathbf{k}|=0}^{\infty} |\mathbf{a_k}(f)|\|F_\mathbf{k}(R\mathbf{z})\|_X.$$

We have

$$\|f_R(R\mathbf{z})\|_X \sim \frac{R^\mathbf{k}\|\mathbf{z^k}\|_X}{\mathbf{d^k}} \sim \frac{R^\mathbf{k}}{\mathbf{d^k}} \quad as \;\; \|\mathbf{z^k}\|_X = 1.$$

By the definition of $\varphi(\mathbf{z})$, for large $z_j, j = \overline{1, m}$, we have

$$\|f_R(\mathbf{z})\|_X \leq \sum_{|\mathbf{k}|=0}^{\infty} (\exp(\alpha^{-1}(\frac{1}{\tilde{\rho}}\alpha(|\mathbf{k}|))))^{-\mathbf{k}} \frac{R^\mathbf{k}}{(\mathbf{d}-\varepsilon)^\mathbf{k}}.$$

Now proceeding on the lines of proof of Bose and Sharma (1963, Thm. IV), we get

$$M(R, f) \leq O\{\exp(\alpha^{-1}(\tilde{\rho} + m\varepsilon)\alpha(\log(\frac{2^m R}{\mathbf{d}-\varepsilon}))\log R)\}.$$

Since ε is arbitrary and independent of R, we have

$$\limsup_{R\to\infty} \frac{\alpha(\log M(R,f))}{\alpha(\log R)} \leq \rho'_m(\alpha,f)+1.$$

Now we have to show that left hand side is equal to $\rho_m(\alpha,f)$ defined by (14). Let $\mathbf{z}=\phi(\mathbf{w})$ be the function inverse to the function $\mathbf{w}=\varphi(\mathbf{z})$ defined in Section 1. The former maps the domains $|w_j|>1$ onto the domains E_j conformally. Its Laurent series expansion in a vicinity of the point $w_j=0$ has the form

$$z_1=\phi_1(w_1)=\nu w_1+\nu_0+\frac{\nu_1}{w_1^2}+\ldots,$$
$$z_2=\phi_2(w_2)=\nu w_2+\nu'_0+\frac{\nu'_1}{w_1^2}+\ldots,$$
$$\ldots$$
$$z_m=\phi_m(w_m)=\nu w_m+\nu_0^{(m-1)\prime}+\frac{\nu^{(m-1)\prime}}{w_m^2}+\ldots,$$

where $\nu=\mathbf{d}$. Using above formulae and the definition of the quantity $\rho_m(\alpha,f)$ and setting $z_j=w_j.\nu$; $R=|w_j|.\nu$, we get

$$\begin{aligned}\limsup_{r_j\to\infty} \frac{\alpha(\log M(\mathbf{r},f))}{\alpha(\log \mathbf{r})} &= \limsup_{r_j\to\infty} \frac{\alpha(\log \max_{|w_j|=r_j}|f(\phi_j(w_j))|)}{\alpha(\log \mathbf{r})}\\ &= \limsup_{r_j\to\infty} \frac{\alpha(\log \max_{|w_j|=r_j}|f(w_j(\nu+\frac{\nu_o^{(j-1)\prime}}{w_j}+\ldots))|)}{\alpha(\log \mathbf{r})}\\ &= \limsup_{r_j\to\infty} \frac{\alpha(\log \max_{|z_j|=R}|f(\mathbf{z})|)}{\alpha(\log \mathbf{r})}\\ &= \limsup_{R\to\infty} \frac{\alpha(\log M(R,f))}{\alpha(\log R)}=\rho_m(\alpha,f).\end{aligned}$$

■

Theorem 3.2.

Let X be one of the Banach spaces $(1-5)$ of functions analytic in U^m. For the function $f\in X$ the condition

$$\lim_{n\to\infty}(E_n(f,X))^{1/n}=0, \tag{21}$$

is necessary and sufficient for the function f to be entire.

Proof:

Let $f(\mathbf{z})=\sum_{|\mathbf{k}|=0}^{\infty}\mathbf{a}_{\mathbf{k}}(f)F_{\mathbf{k}}(\mathbf{z})$ for $\mathbf{z}\in U^m$ and condition (21) holds. First we prove the sufficiency, by using Lemma 2.2, we get

$$|\mathbf{a}_{\mathbf{k}}(f)|\leq \frac{E_n(f,X)}{\mathbf{d}^{\mathbf{k}}\|F_{\mathbf{k}}(\mathbf{z})\|_X},$$

or

$$\lim_{|\mathbf{k}|\to\infty}|\mathbf{a}_{\mathbf{k}}(f)|^{\frac{1}{|\mathbf{k}|}} \leq \lim_{n\to\infty}\Big(\frac{K_X^*(n+1)E_n(f,X)}{\mathbf{d}^{n+1}}\Big)^{1/n+1}=0 \;\; \texttt{for} \;\; |\mathbf{k}|=n+1,$$

in view of Lemma 2.1, f is entire.

For necessary part, we know that

$$E_n(f,X) \le R^{-n} E_n(f_R,X) \le R^{-n} \|f_R\|_X,$$

or

$$\begin{aligned} 0 &\le \lim_{n\to\infty} \left(\frac{E_n(f,X)K_X^*(\mathbf{k})}{\mathbf{d}^{\mathbf{k}}}\right)^{1/n+1} \\ &\le \frac{1}{R} \lim_{|\mathbf{k}|\to\infty} \left(\frac{K_X^*(\mathbf{k})}{\mathbf{d}^{\mathbf{k}}}\right)^{1/|\mathbf{k}|} = \frac{1}{R}. \end{aligned}$$

Since $R > 1$ is arbitrary, it gives

$$\lim_{n\to\infty} (E_n(f,X))^{\frac{1}{n+1}} = 0.$$

This completes the proof. ∎

Theorem 3.3.

Let $f \in X$ and $f(\mathbf{z}) = \sum_{|\mathbf{k}|=0}^{\infty} \mathbf{a}_{\mathbf{k}}(f) F_{\mathbf{k}}(\mathbf{z})$ be an entire function of m-complex variables with a generalized α-order of growth $\rho_m(\alpha,f), 0 < \rho_m(\alpha,f) < \infty$, it is necessary and sufficient that

$$\rho_m(\alpha,f) = 1 + \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha(-\frac{1}{n}\log\left\{\frac{K_X^*(n+1)E_n(f,X)}{\mathbf{d}^{n+1}}\right\}^{\frac{n}{(n+1)}})}, \tag{22}$$

where $\alpha(x) \in \overline{\Omega}$ and

$$\left\{\frac{K_X^*(n+1)}{\mathbf{d}^{n+1}}\right\}^{-1} = \max_{|\mathbf{k}|=n+1} \left\{\frac{K_X^*(\mathbf{k})}{\mathbf{d}^{\mathbf{k}}}\right\}^{-1}.$$

Proof:

Theorem 3.2 prove that f is entire function. Using Lemma 2.2, we obtain

$$\begin{aligned} \rho_m'(\alpha,f) &= \limsup_{|\mathbf{k}|\to\infty} \frac{\alpha(|\mathbf{k}|)}{\alpha(\frac{1}{|\mathbf{k}|}\log|\mathbf{a}_{\mathbf{k}}(f)|^{-1})} \\ &\le \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha(-\frac{1}{n}\log\left\{\frac{K_X^*(n+1)E_n(f,X)}{\mathbf{d}^{n+1}}\right\})^{\frac{n}{(n+1)}}} \\ &= \rho_m(\alpha,f) - 1. \end{aligned} \tag{23}$$

In order to prove reverse inequality in (23), we consider

$$\rho_m'(\alpha,f) = \limsup_{|\mathbf{k}|\to\infty} \frac{\alpha(|\mathbf{k}|)}{\alpha(\frac{1}{|\mathbf{k}|}\log|\mathbf{a}_{\mathbf{k}}(f)|^{-1})}.$$

Then, for any $\varepsilon > 0$, there exists $n_o(\varepsilon) \in \mathbb{N}$ such that

$$|\mathbf{a_k}(f)| \leq \left\{\exp\{|\mathbf{k}|\alpha^{-1}\{\frac{\alpha(|\mathbf{k}|)}{\rho'_m(\alpha, f)+\varepsilon}\}\}\right\}^{-1}, \quad for \quad |\mathbf{k}| > n_o(\varepsilon).$$

Using (2), for sufficiently large $n_o(\varepsilon)$ so that $\|F_\mathbf{k}(\mathbf{z})\|_X \leq (1+\varepsilon)^{|\mathbf{k}|}$ for $|\mathbf{k}| > n_o(\varepsilon)$. Then,

$$\begin{aligned} E_n(f, X) &\leq \| \sum_{|\mathbf{k}|=n+1}^{\infty} \mathbf{a_k}(f) F_\mathbf{k}(\mathbf{z})\|_X \\ &\leq \sum_{|\mathbf{k}|=n+1}^{\infty} |\mathbf{a_k}(f)| \|F_\mathbf{k}(\mathbf{z})\|_X \\ &\leq \sum_{|\mathbf{k}|=n+1}^{\infty} \{\exp\{|\mathbf{k}|\alpha^{-1}\{\frac{\alpha(|\mathbf{k}|)}{\rho'_m(\alpha, f)+\varepsilon}\}\}\}^{-1}(1+\varepsilon)^{|\mathbf{k}|} \\ &= \frac{(1+\varepsilon)^{(n+1)}}{\{\exp\{(n+1)\alpha^{-1}\{\frac{\alpha(n+1)}{\rho'_m(\alpha,f)+\varepsilon}\}\}\}}(1 - \frac{1}{\exp\{\alpha^{-1}(\frac{\alpha(n+1)}{\rho'_m(\alpha,f)+\varepsilon})\}})^{-1}. \end{aligned} \tag{24}$$

Now we have

$$\begin{aligned} &\limsup_{n\to\infty} \frac{\alpha(n)}{\alpha(-\frac{1}{n}\log\{\frac{K_X^*(n+1)E_n(f,X)}{d^{(n+1)}}\}^{\frac{n}{n+1}})} \\ &\leq \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha(\log\{(S_1)^{\frac{1}{n+1}}\exp\{S_2\}(1+\varepsilon)^{-\frac{1}{n+1}}(S_3)^{\frac{1}{n+1}}\})} \\ &= \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha(\log\{(S_1)^{\frac{1}{n+1}} + S_2 - \frac{1}{n+1}\log(1+\varepsilon) + \frac{1}{n+1}\log(S_3)\})} \\ &= \limsup_{n\to\infty} \frac{\alpha(n)(\rho'_m(\alpha, f)+\varepsilon)}{\alpha(n+1)} = \rho'_m(\alpha, f), \end{aligned}$$

where $S_1 = \frac{K_X^*(n+1)}{d^{n+1}}$, $S_2 = \alpha^{-1}\{\frac{\alpha(n+1)}{\rho'_m(\alpha,f)+\varepsilon}\}$ and $S_3 = 1 - \frac{1}{\exp\{\alpha^{-1}(\frac{\alpha(n+1)}{\rho'_m(\alpha,f)+\varepsilon})\}}$. This completes the sufficiency part. To prove the necessity of the condition (22), we assume that $f \in X$ be an entire function of finite generalized order $\rho_m(\alpha, f)$ i.e.,

$$\limsup_{|\mathbf{k}|\to\infty} \frac{\alpha(|\mathbf{k}|)}{\alpha(\frac{1}{|\mathbf{k}|}\log|\mathbf{a_k}(f)|^{-1})} = \rho_m(\alpha, f) - 1.$$

Set

$$\rho'_m(\alpha, f) = \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha(-\frac{1}{n}\log\left\{\frac{K_X^*(n+1)E_n(f,X)}{\mathbf{d}^{n+1}}\right\}^{\frac{n}{(n+1)}})}.$$

In this case the notations $\rho_m(\alpha, f)$ and $\rho'_m(\alpha, f)$ are changed as compared with the proof of sufficiency and we prove equality $\rho_m(\alpha, f) - 1 = \rho'_m(\alpha, f)$. Hence, the proof is complete. ∎

4. Conclusion

Vakarchuk and Zhir investigated the generalized α-logarithmic order in the spaces 1, 2, 4, and 5 mentioned above in finite domain. Ganti and Srivastava obtained coefficient characterizations of growth parameters order and type of entire functions in terms of L^p-approximation errors $(2 \leq p \leq \infty)$ by using Faber polynomials over Jordan domains in two complex variables. Kumar improved the results discussed by Srivastava and Ganti. In the present paper our results extends all results mentioned above in several complex variables using the concept of generalized order of slow growth. When we discuss the dependent problems in $\mathbb{C}^2$ it leads to the study of growth parameters in $\mathbb{C}^n, n \geq 3$. Therefore, our study is reasonable.

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REFERENCES

Bose, S.K. and Sharma, D. (1963). Integral functions of two complex variables, Compositio Math., Vol. 15, pp. 210-216.

Duren, P.L., Romberg, B.W. and Shields, A.L. (1969). Linear functionals in H_p spaces with $0 < p < 1$, J. Reine Angew. Math., Vol. 238, pp. 4-60.

Faber, G. (1979). Uber tschebyscheffsche polynome, J. Reine Angew. Math., Vol. 150, pp. 79-106.

Ganti, R. and Srivastava, G.S. (2009). Approximation of entire functions over Jordan domains, Commun. Math. Anal., Vol. 7, No. 1, 75-93.

Giroux, A. (1980). Approximation of entire functions over bounded domains, J. Approx. Theory, Vol. 28, pp. 45-53.

Gvaradze, M.I. (1975). *On One Class of Species of Analytic Functions [in Russian]*, Candidate - Degree Thesis (Physics and Mathematics) Tbilisi.

Gvaradze, M.I. (1977). On one class of spaces of analytic functions, Mat. Zametki, Vol. 21, No. 2, pp. 141-150.

Gvaradze, M.I. (1980). Factors of one class of analytic function defined on a polydisk, Tr. Tbilis. Mat. Inst., Vol. 66, pp. 15-21.

Hardy, G.H. (1974). The mean values of the modulas of an analytic function, Proc. London Math. Soc., Vol. 14, pp. 269-277.

Hardy, G.H. and Littlewood, J.E. (1931). Some properties of fractional integrals, II, Math. Z., Vol. 34, No. 3, pp. 403-439.

Harfaoui, M. (2010). Generalized order and best approximation of entire function in L_p-norm, Internat. J. Math. Sci., Vol. 2010, pp. 1-15.

Harfaoui, M. (2014). Proximate growth and best approximation in L_p-norm of entire functions, Theory and Appl. Math. Comput. Sci., Vol. 4, No. 2, pp. 154-168.

Kapoor, G.P. and Nautiyal, A. (1981). Polynomial approximation of an entire function of slow growth, J. Approx. Theory, Vol. 32, pp. 64-75.

Kumar, Devendra (2013). Faber polynomial approximation of entire functions of slow growth over Jordan domains, Annali Dell Univ. Ferrara, Vol. 59, No. 2, pp. 331-351.

Lebedev, N.N. (1963). *Special Functgions and Their Applications*, Fizmatgiz, Moscow.

Srivastava, G.S. and Ganti, R. (2012). Approximation of entire functions of two complex variables over Jordan domains, Tamsui Oxford J. Inform. Math. Sci., Vol. 28, No. 4, pp. 349-368.

Vakarchuk, S.B. and Zhir, S.I. (2011). The best polynomial approximations of entire transcendental functions with generalized growth order in Banach spaces $\varepsilon'_p(G)$ and $\varepsilon_p(G), p \geq 1$, Journal Mathematical Sciences, Vol. 179, No. 2, pp. 300-327.

Vakarchuk, S.B. and Zhir, S.I. (2015). On the best polynomial approximation of entire transcendental functions of many complex variables in some Banach spaces, Ukr. Math. J., Vol. 66, No. 12, pp. 1793-1811.