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Some Contiguous Relation on *k*-Generalised Hypergeometric Function

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Abstract

In this research work our aim is to determine some contiguous relations and some integral transform of the *k*-generalised hypergeometric functions, by using the concept of "*k*-Gamma and *k*-Beta function". "Obviously if $k \to 1$", then the contiguous function relations become Gauss contiguous relations.

Keywords: *k*-Gamma function; *k*-Beta function; *k*-Hypergeometric function; Contiguous function

MSC 2010 No.: 33B15, 33Cxx, 44A10, 44A20

1. Introduction

The hypergeometric function, acting as a strong tool in mathematical analysis and its applications allows us to solve many interesting problems. Various extensions and modifications of the hypergeometric function have been given by many researchers like Rainville (1965). In past one-decade Diaz et al. (2005, 2007) introduced *k*-Gamma function and *k*-Beta functions and proved a number of their properties. They also studied *k*-hypergeometric functions based on Pochhammer *k*-symbols for factorial functions. These studies were extended by Mansour (2009), Kokologiannaki (2010) etc. and explained and established the scope of *k*-gamma and *k*-beta functions. Mubeen et al. (2014) defined some contiguous relations of k-hypergeometric functions. Mittal et al. (2018) defined some results on k-wright functions. Joshi et al. (2018) defined an Integral Representation of Generalized k-Mittag-Leffler functions of the type $GE^{\gamma,q}_{k,\alpha,\beta,\delta,p}(z)$.The number of useful results have been investigated by several authors. The purpose of this paper is to present contiguous relations with integral transform of the k-generalized hypergeometric functions.

This paper has been systematized as follows. The Section 2 is dedicated to some definitions and preliminaries. The section 3 is devoted to Contiguous functions for k-generalised hypergeometric function with more than one parameter. Section 4 contains the Main result in which we define nine theorems with proofs in detail and the last section 5 consists of the conclusion.

2. Definitions and Preliminaries:

The integral representation of k-gamma function and k-beta function studied by Diaz et al.

(2005, 2007) are

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}}\, dt = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right),\ Re(z) > 0, t > 0, k > 0\ , \tag{1}$$

where

$$\Gamma_k(z+k) = z\Gamma_k(z)$$

and

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt,\ \mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0. \tag{2}$$

The k-Hypergeometric function defined by Mubeen and Habibullah (2012) is

$$ {}_2F_{1,k}(a,b;c;z) = {}_2F_{1,k}\left(\begin{matrix}(a,1),(b,k) \\ \\ (c,k)\end{matrix} ;z\right)$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_{n,k}}{(c)_{n,k}}\frac{z^n}{n!}, \{|z| < 1, c \neq 0,-1,-2,...\}. \tag{3}$$

The generalised k-hypergeometric function is given as

for $\mathrm{Re}(a) > 0, \mathrm{Re}(c) > 0, \mathrm{Re}(b) > 0, k > 0, m \geq 1, m \in Z^+$,

$$ {}_{m+1}F_{m,k}\left(\begin{matrix}(a,k),\left(\frac{b}{m},k\right),\left(\frac{b+k}{m},k\right),...,\left(\frac{b+(m-1)k}{m},k\right) \\ \\ \left(\frac{c}{m},k\right),\left(\frac{c+k}{m},k\right),...,\left(\frac{c+(m-1)k}{m},k\right)\end{matrix} ;z\right)$$

$$= \sum_{n=0}^{\infty} \frac{(a)_{n,k} \left(\frac{b}{m}\right)_{n,k} \left(\frac{b+k}{m}\right)_{n,k} \cdots \left(\frac{b+(m-1)k}{m}\right)_{n,k}}{\left(\frac{c}{m}\right)_{n,k} \left(\frac{c+k}{m}\right)_{n,k} \cdots \left(\frac{c+(m-1)k}{m}\right)_{n,k}} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(a)_{n,k} (b)_{mn,k}}{(c)_{mn,k}} \frac{z^n}{n!}. \quad (4)$$

and

for $\mathrm{Re}(a) > 0, \mathrm{Re}(c) > 0, \mathrm{Re}(b) > 0, k > 0, m \geq 1, m \in Z^+$,

$${}_mF_{m,k}\left(\begin{matrix} \left(\frac{b}{m},k\right),\left(\frac{b+k}{m},k\right),\ldots,\left(\frac{b+(m-1)k}{m},k\right) \\ \\ \left(\frac{c}{m},k\right),\left(\frac{c+k}{m},k\right),\ldots,\left(\frac{c+(m-1)k}{m},k\right) \end{matrix} ; z\right)$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{b}{m}\right)_{n,k} \left(\frac{b+k}{m}\right)_{n,k} \cdots \left(\frac{b+(m-1)k}{m}\right)_{n,k}}{\left(\frac{c}{m}\right)_{n,k} \left(\frac{c+k}{m}\right)_{n,k} \cdots \left(\frac{c+(m-1)k}{m}\right)_{n,k}} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(b)_{mn,k}}{(c)_{mn,k}} \frac{z^n}{n!}. \quad (5)$$

3. Contiguous functions for *k*-generalised hypergeometric function with more than one parameter

If we increase or decrease one and only one of the parameters of *k*-generalised hypergeometric function by $\pm k$; then the resultant function is said to be contiguous to ${}_4F_{3,k}$. If we substitute $m = 3$ in equation (4) then it becomes

$$F_k = {}_4F_{3,k}\left(\begin{matrix} (a,k),\left(\frac{b}{3},k\right),\left(\frac{b+k}{3},k\right),\left(\frac{b+2k}{3},k\right) \\ \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3},k\right),\left(\frac{c+2k}{3},k\right) \end{matrix} ; z\right)$$

$$= \sum_{n=0}^{\infty} \frac{(a)_{n,k} \left(\frac{b}{3}\right)_{n,k} \left(\frac{b+k}{3}\right)_{n,k} \left(\frac{b+2k}{3}\right)_{n,k}}{\left(\frac{c}{3}\right)_{n,k} \left(\frac{c+k}{3}\right)_{n,k} \left(\frac{c+2k}{3}\right)_{n,k}} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(a)_{n,k} (b)_{3n,k}}{(c)_{3n,k}} \frac{z^n}{n!}. \quad (6)$$

Further,

$$F_k(a+) = \sum_{n=0}^{\infty} \frac{(a+k)_{n,k} (b)_{3n,k}}{(c)_{3n,k}} \frac{z^n}{n!}. \quad (7)$$

$$F_k(a-) = \sum_{n=0}^{\infty} \frac{(a-k)_{n,k} (b)_{3n,k}}{(c)_{3n,k}} \frac{z^n}{n!}. \quad (8)$$

In the similar manner we can also define $F_k(c-), F_k(b-), F_k(b+), F_k(c+)$. For simplicity we rewrite the equation (6) which is as follows:

$$F_k = \sum_{n=0}^{\infty} \delta_{n,k}, \tag{9}$$

where

$$\delta_{n,k} = \frac{(a)_{n,k}\left(\frac{b}{3}\right)_{n,k}\left(\frac{b+k}{3}\right)_{n,k}\left(\frac{b+2k}{3}\right)_{n,k}}{\left(\frac{c}{3}\right)_{n,k}\left(\frac{c+k}{3}\right)_{n,k}\left(\frac{c+2k}{3}\right)_{n,k}}\frac{z^n}{n!} = \frac{(a)_{n,k}(b)_{3n,k}}{(c)_{3n,k}}\frac{z^n}{n!}, \tag{10}$$

Since $a(a+k)_{n,k} = (a)_{n,k}(a+nk)$, by using this result in equation (7) we obtain

$$F_k(a+) = \sum_{n=0}^{\infty} \frac{(a+nk)}{a}\delta_{n,k}. \tag{11}$$

Similarly, we rearrange the term in equation (8) by $(a-k)(a)_{n,k} = (a+(n-1)k)(a-k)_{n,k}$, then it becomes

$$F_k(a-) = \sum_{n=0}^{\infty} \frac{(a-k)}{(a+(n-1)k)}\delta_{n,k}. \tag{12}$$

Again, by the help of differential operator $k\theta = kz(d/dz)$, we get the following result

$$(k\theta+a)F_k = (k\theta+a)\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{3n,k}}{(c)_{3n,k}}\frac{z^n}{n!} = aF_k(a+). \tag{13}$$

Similarly, for other parameters, we have

$$(3k\theta+b) = bF_k(b+). \tag{14}$$

$$(3k\theta+c-k)F_k = (c-k)F_k(c-). \tag{15}$$

4. Main Results: Contiguous Relations for *k*-Hypergeometric Function

Since there are five contiguous functions to a given function ${}_4F_{3,k}$, therefore, we obtain the different type of contiguous function relations for *k*-hypergeometric function ${}_4F_{3,k}$ for all $a,b,c \in C, \mathrm{Re}(a) > 0, \mathrm{Re}(b) > 0, \mathrm{Re}(c) > 0, |z| < 1$ and $k > 0$. By subtracting equations (14) and (15) from equation (13), we obtain the following two relations given in equations (16) and (17) respectively.

Relation 4.1.

$$(3a-b)F_k = aF_k(a+) - bF_k(b+). \tag{16}$$

Relation 4.2.

$$(3a-c+k)F_k = aF_k(a+)-(c-k)F_k(c-). \tag{17}$$

***Proof*:**

Relation 4.1 and 4.2 can be by using equation (11) and equation (12).

Relation 4.3.

$$(a-k)(\Omega_k-\xi_k)=kz\frac{(a-k)(b-2k)(b-k)}{(c-2k)(c-k)}\left(F_k+c^{-1}(b-c)F_k(c+)\right), \tag{18}$$

where

$$\xi_k = {}_4F_{3,k}\left(\begin{matrix}(a-k,k),\left(\frac{b}{3},k\right),\left(\frac{b+k}{3}-k,k\right),\left(\frac{b+2k}{3}-k,k\right) \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3}-k,k\right),\left(\frac{c+2k}{3}-k,k\right)\end{matrix};z\right),$$

and

$$\Omega_k = {}_4F_{3,k}\left(\begin{matrix}(a,k),\left(\frac{b}{3},k\right),\left(\frac{b+k}{3}-k,k\right),\left(\frac{b+2k}{3}-k,k\right) \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3}-k,k\right),\left(\frac{c+2k}{3}-k,k\right)\end{matrix};z\right).$$

***Proof*:**

Let us consider

$$k\,\theta\xi_k = kz\frac{d}{dz}\sum_{n=0}^{\infty}\frac{(a-k)_{n,k}(b-2k)_{3n,k}}{(c-2k)_{3n,k}}\frac{z^n}{n!}$$

$$= k\ z\sum_{n=1}^{\infty}\frac{(a-k)_{n,k}(b-2k)_{3n,k}}{(c-2k)_{3n,k}}\frac{z^{n-1}}{(n-1)!}, \tag{19}$$

By changing the index $n+1$ to n in above expression, we have

$$= k\ z\sum_{n=0}^{\infty}\frac{(a-k)_{n+1,k}(b-2k)_{3(n+1),k}}{(c-2k)_{3(n+1),k}}\frac{z^n}{(n)!}$$

$$= k\ z\frac{(a-k)(b-2k)(b-k)}{(c-2k)(c-k)}\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{3n,k}}{(c)_{3n,k}}\frac{(b+3nk)}{(c+3nk)}\frac{z^n}{(n)!}$$

$$= k\ z\frac{(a-k)(b-2k)(b-k)}{(c-2k)(c-k)}\left(F_k + c^{-1}(b-c)F_k(c+)\right). \qquad (20)$$

If we replace

$$a \to a-k, \frac{(b+k)}{3} \to \frac{(b+k)}{3}-k,$$
$$\frac{(b+2k)}{3} \to \frac{(b+2k)}{3}-k, \frac{(c+k)}{3} \to \frac{(c+k)}{3}-k,$$
$$\frac{(c+2k)}{3} \to \frac{(c+2k)}{3}-k,$$

in equation (13), we obtain

$$(k\theta)\xi_k = (a-k)(\Omega_k - \xi_k). \qquad (21)$$

Finally, equating the equations (19) and (21) we get the desired result.

Relation 4.4.

$$(a-k)(\varsigma_k - \Delta_k) = kz\frac{(a-k)(b-2k)(b-k)}{(c-2k)(c-k)}\left((c-k)F_k(c-) + 2(b-c+k)F_k\right)$$
$$+ kz\frac{(a-k)(b-2k)(b-k)}{(c-2k)(c-k)}c^{-1}(b-c)(k+b-c)F_k(c+), \qquad (22)$$

where

$$\Delta_k = {}_4F_{3,k}\left(\begin{matrix}(a-k,k),\left(\frac{b}{3},k\right),\left(\frac{b+k}{3},k\right),\left(\frac{b+2k}{3}-k,k\right) \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3}-k,k\right),\left(\frac{c+2k}{3}-k,k\right)\end{matrix};z\right),$$

and

$$\varsigma_k = {}_4F_{3,k}\left(\begin{matrix}(a,k),\left(\frac{b}{3},k\right),\left(\frac{b+k}{3},k\right),\left(\frac{b+2k}{3}-k,k\right) \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3}-k,k\right),\left(\frac{c+2k}{3}-k,k\right)\end{matrix};z\right).$$

***Proof*:**

Let us consider

$$k\theta\Delta_k = kz\frac{d}{dz}\sum_{n=0}^{\infty}\frac{(a-k)_{n,k}(b-k)_{3n,k}}{(c-2k)_{3n,k}}\frac{z^n}{n!}$$
$$= kz\sum_{n=1}^{\infty}\frac{(a-k)_{n,k}(b-k)_{3n,k}}{(c-2k)_{3n,k}}\frac{z^{n-1}}{(n-1)!},$$

changing index $n+1$ to n, we get

$$= kz\sum_{n=0}^{\infty}\frac{(a-k)_{n+1,k}(b-k)_{3(n+1),k}}{(c-2k)_{3(n+1),k}}\frac{z^n}{(n)!}$$

$$= kz\frac{(a-k)(b-k)}{(c-2k)(c-k)}\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{3n,k}}{(c)_{3n,k}}\frac{(b+3nk)(b+(3n+1)k)}{(c+3nk)}\frac{z^n}{(n)!}$$

$$= kz\frac{(a-k)(b-2k)(b-k)}{(c-2k)(c-k)}\sum_{n=0}^{\infty}\left(3nk+(2b-c+k)\frac{(b-c)(k+b-c)}{(c+3nk)}\right)\delta_{n,k}$$

$$= kz\frac{(a-k)(b-2k)(b-k)}{(c-2k)(c-k)}$$

$$\times\left(3k\theta F_k+(2b-c+k)F_k+c^{-1}(b-c)(k+b-c)F_k(c+)\right)$$

$$= kz\frac{(a-k)(b-2k)(b-k)}{(c-2k)(c-k)}$$

$$\times\left((3k\theta+c-k)F_k+2(b-c+k)F_k+c^{-1}(b-c)(k+b-c)F_k(c+)\right),$$

using the result $(3k\theta+c-k)F_k=(c-k)F_k(c-)$, in above expression, we have

$$= kz\frac{(a-k)(b-2k)(b-k)}{(c-2k)(c-k)}$$

$$\times\left((c-k)F_k(c-)+2(b-c+k)F_k+c^{-1}(b-c)(k+b-c)F_k(c+)\right). \quad (23)$$

Finally replacing

$$a\to a-k,\frac{(b+2k)}{3}\to\frac{(b+2k)}{3}-k,\frac{(c+k)}{3}\to\frac{(c+k)}{3}-k,\frac{(c+2k)}{3}\to\frac{(c+2k)}{3}-k,$$

we obtain

$$(k\theta)\Delta_k=(a-k)\varsigma_k-(a-k)\Delta_k, \quad (24)$$

after comparing equations (23) and (24) we get the desired result.

Relation 4.5.

$$(a)\mathfrak{I}_k-(a)\aleph_k$$
$$=kz\frac{(b-2k)(b-k)}{(c-2k)(c-k)}\left(aF_k(a+)-aF_k+\frac{1}{3}(b-c+3a)F_k+\frac{1}{3}c^{-1}(c-b)(c-3a)F_k(c+)\right),$$

(25)

where

$$\aleph_k={}_4F_{3,k}\left(\begin{matrix}(a,k),\left(\frac{b}{3},k\right),\left(\frac{b+k}{3}-k,k\right),\left(\frac{b+2k}{3}-k,k\right)\\ \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3}-k,k\right),\left(\frac{c+2k}{3}-k,k\right)\end{matrix};z\right),$$

and

$$\mathfrak{I}_k={}_4F_{3,k}\left(\begin{matrix}(a+k,k),\left(\frac{b}{3},k\right),\left(\frac{b+k}{3}-k,k\right),\left(\frac{b+2k}{3}-k,k\right)\\ \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3}-k,k\right),\left(\frac{c+2k}{3}-k,k\right)\end{matrix};z\right).$$

***Proof*:**

We follow the same process which we applied in relations 4.3 and 4.4.

Further, we also presenting few interesting and useful theorems which are as follows:

Theorem 4.6.

If $\mathrm{Re}(b)>0,\mathrm{Re}(c)>0,|z|<1$ and $k>0$ then the following result holds true.

$$e^{-z}\left[{}_3F_{3,k}\left(\begin{matrix}\left(\frac{b}{3},k\right),\left(\frac{b+k}{3},k\right),\left(\frac{b+2k}{3},k\right)\\ \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3},k\right),\left(\frac{c+2k}{3},k\right)\end{matrix};z\right)\right]$$
$$=\sum_{s=0}^{\infty}{}_4F_3\left(\begin{matrix}(-s,1),\left(\frac{b}{3},k\right),\left(\frac{b+k}{3},k\right),\left(\frac{b+2k}{3},k\right)\\ \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3},k\right),\left(\frac{c+2k}{3},k\right)\end{matrix};1\right)\frac{(-z)^s}{s!}. \qquad (26)$$

***Proof*:**

Consider left hand side of the equation (26) and using equation (5) by putting $m=3$, we have

$$=\sum_{s=0}^{\infty}\frac{(-1)^s z^s}{s!}\sum_{n=0}^{\infty}\frac{(b)_{3n,k}}{(c)_{3n,k}}\frac{z^n}{n!}$$

$$=\sum_{s=0}^{\infty}\sum_{n=0}^{s}\frac{(-1)^{s-n}}{(s-n)!}\frac{(b)_{3n,k}}{(c)_{3n,k}}\frac{(z)^s}{n!}$$

$$=\sum_{s=0}^{\infty}\sum_{n=0}^{s}\frac{(-1)^s(-s)_n}{s!}\frac{(b)_{3n,k}}{(c)_{3n,k}}\frac{z^s}{n!}$$

$$=\sum_{s=0}^{\infty}{}_4F_3\left(\begin{matrix}(-s,1),\left(\frac{b}{3},k\right),\left(\frac{b+k}{3},k\right),\left(\frac{b+2k}{3},k\right)\\ \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3},k\right),\left(\frac{c+2k}{3},k\right)\end{matrix};1\right)\frac{(-z)^s}{s!}.$$

Theorem 4.7. (*k*-Beta Transform)

If $k>0,\operatorname{Re}(a)>0,\operatorname{Re}(b)>0,\operatorname{Re}(c)>0,|z|<1,$ then the subsequent result holds true.

$$B_k\left({}_4F_{3,k}\left(\begin{matrix}(a+b,k),\left(\frac{b}{3},k\right),\left(\frac{b+k}{3},k\right),\left(\frac{b+2k}{3},k\right)\\ \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3},k\right),\left(\frac{c+2k}{3},k\right)\end{matrix};z\right);a,b\right)$$

$$=B_k(a,b)\left({}_4F_{3,k}\left(\begin{matrix}(a,k),\left(\frac{b}{3},k\right),\left(\frac{b+k}{3},k\right),\left(\frac{b+2k}{3},k\right)\\ \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3},k\right),\left(\frac{c+2k}{3},k\right)\end{matrix};z\right)\right). \tag{27}$$

***Proof*:**

Applying Beta function in the left side of the equation (27), we have

$$\frac{1}{k}\int_0^1 t^{\frac{a}{k}-1}(1-t)^{\frac{b}{k}-1}\sum_{n=0}^{\infty}\frac{(a+b)_{n,k}(b)_{3n,k}}{(c)_{3n,k}}\frac{(z\ t)^n}{n!}dt.$$

Interchanging the order of summation and integration and using the definition of beta function

$$\sum_{n=0}^{\infty} \frac{B_k\left(a+nk,b\right)\left(a+b\right)_{n,k}\left(b\right)_{3n,k}}{\left(c\right)_{3n,k}} \frac{\left(z\right)^n}{n!},$$

and after simplification, we get the desired result.

Theorem 4.8. (Kummer's confluent *k*-generalised hypergeometric function)

If $a,b,c \in C, k>0, \operatorname{Re}(a)>0, \operatorname{Re}(b)>0, \operatorname{Re}(c)>0, |z|<1,$ then the subsequent result holds true.

$$ {}_4F_3\left(\begin{matrix} (a,k),\left(\frac{b}{3},k\right),\left(\frac{b+k}{3},k\right),\left(\frac{b+2k}{3},k\right) \\ \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3},k\right),\left(\frac{c+2k}{3},k\right) \end{matrix};z\right)$$

$$= {}_3F_3\left(\begin{matrix} \left(\frac{b}{3},k\right),\left(\frac{b+k}{3},k\right),\left(\frac{b+2k}{3},k\right) \\ \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3},k\right),\left(\frac{c+2k}{3},k\right) \end{matrix};z\right). \tag{28}$$

***Proof*:**

Consider left hand side of the equation (28), we have

$$ {}_4F_3\left(\begin{matrix} (a,k),\left(\frac{b}{3},k\right),\left(\frac{b+k}{3},k\right),\left(\frac{b+2k}{3},k\right) \\ \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3},k\right),\left(\frac{c+2k}{3},k\right) \end{matrix};z\right) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}(b)_{3n,k}}{(c)_{3n,k}} \frac{z^n}{n!},$$

Replacing $z = \frac{z}{a}$ and taking limit $a \to \infty$ in above equation, we have

$$= Lim_{a\to\infty} \sum_{n=0}^{\infty} \frac{(a)_{n,k}(b)_{3n,k}}{(c)_{3n,k}} \frac{\left(\frac{z}{a}\right)^n}{n!},$$

further simplification reaches to desired result.

Theorem 4.9. Laplace Transform of *k*-generalized Hypergeometric Function:

If $a,b,c \in C, k>0, \operatorname{Re}(a)>0, \operatorname{Re}(b)>0, \operatorname{Re}(c)>0, \left|\frac{z}{s}\right|<1,$ then the following result holds true.

$$L\left\{ {}_4F_3\left(\begin{matrix} (a,k),\left(\frac{b}{3},k\right),\left(\frac{b+k}{3},k\right),\left(\frac{b+2k}{3},k\right) \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3},k\right),\left(\frac{c+2k}{3},k\right) \end{matrix} ;z\,t \right) \right\}$$

$$= \frac{1}{s}\,{}_5F_3\left(\begin{matrix} (a,k),(k,k),\left(\frac{b}{3},k\right),\left(\frac{b+k}{3},k\right),\left(\frac{b+2k}{3},k\right) \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3},k\right),\left(\frac{c+2k}{3},k\right) \end{matrix} ;\frac{z}{s} \right). \tag{29}$$

***Proof*:**

Using the Laplace transforming left side of the equation (29), we get

$$\int_0^\infty e^{-st}\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{3n,k}}{(c)_{3n,k}}\frac{(zt)^n}{n!}dt = \sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{3n,k}}{(c)_{3n,k}}\frac{z^n}{n!}\int_0^\infty e^{-st}t^n dt,$$

Put $st=\frac{u^k}{k}$, in above expression and simplify, we get

$$=\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{3n,k}}{(c)_{3n,k}}\frac{z^n}{n!}\frac{1}{k^n s^{n+1}}\Gamma_k(nk+k)$$

$$= \frac{1}{s}\,{}_5F_3\left(\begin{matrix} (a,k),(k,k),\left(\frac{b}{3},k\right),\left(\frac{b+k}{3},k\right),\left(\frac{b+2k}{3},k\right) \\ \left(\frac{c}{3},k\right),\left(\frac{c+k}{3},k\right),\left(\frac{c+2k}{3},k\right) \end{matrix} ;\frac{z}{s} \right).$$

5. Conclusion

In this current investigation, we established and evaluated some contiguous function relation by using *k*-parameter in hypergeometric function. We also established some useful and interested integral transforms in terms of *k*-hypergeometric function. The approach presented in this investigation is general but can be extended to establish other properties of special functions.

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