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On the Asymptotic Stability of a Nonlinear Fractional-order System with Multiple Variable Delays

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Abstract

In this paper, we consider a nonlinear differential system of fractional-order with multiple variable delays. We investigate asymptotic stability of zero solution of the considered system. We prove a new result, which includes sufficient conditions, on the subject by means of a suitable Lyapunov functional. An example with numerical simulation of its solutions is given to illustrate that the proposed method is flexible and efficient in terms of computation and to demonstrate the feasibility of established conditions by MATLAB-Simulink.

Keywords: Asymptotic stability, Lyapunov functional, fractional-order, variable delay

MSC 2020 No.: 34A08; 34A09; 34K20, 34K40

1. Introduction

The theory and applications of fractional calculus emerging with various applications in science and engineering fields have attracted great attention of researchers during the last 20 years. At the same time, we note that fractional differential equations/systems with time-varying delays often occur in many scientific areas such as engineering techniques fields, physics, polymer rheology, neural networks, biophysics, blood flow phenomena, capacitor theory, electrical circuits, signal processing, biology, control theory and etc. (see Agarwal et al. (2010), Altun (2019), Chen et al. (2012), Chen et al. (2014), Deng and Deng (2014), Deng (2010), Diethelm (2010), Duarte-Mermoud et al. (2015), Hristova and Tunç (2019), Hu et al. (2015), Khan et al. (2018), Kilbas et al. (2006), Li et al. (2015), Li et al. (2010), Liu et al. (2017), Liu et al. (2016, 2017), Lu and Chen (2009), Matignon (1996), Podlubny (1999), Qian et al. (2010) and the references therein).

This paper, motivated by the results of Liu et al. (2016) and that can be found in Agarwal et al. (2010), Alidousti (2017), Altun (2019), Altun and Tunç (2019), Chen et al. (2012), Chen et al. (2014), Deng and Deng (2014), Deng (2010), Diethelm (2010), Duarte-Mermoud et al. (2015), Gözen and Tunç (2020), Grace et al. (2019), Graef et al. (2017), Hristova and Tunç (2019), Hu et al. (2015), Khan et al. (2018), Kilbas et al. (2006), Li et al. (2015), Li et al. (2010), Liu et al. (2017), Lu and Chen (2009), Matignon (1996), Podlubny (1999), Qian et al. (2010), Slyn'ko and Tunç (2019), Tan (2008), Tunç and Mohammed (2019), Tunç and Tunç (2016a, 2016b), Wang et al. (2012), Zhou et al.(2014), Zhang et al. (2018) and the sources therein.

2. Preliminaries

In this section, several basic definitions and lemmas related to fractional calculus are presented.

The fractional integral (Riemann–Liouville integral) ${}_{t_0}D_t^{-q}$ with fractional-order $q \in R^+$ of a function $x(t)$ is defined by

$${}_{t_0}D_t^{-q}x(t) = \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-\tau)^{q-1}x(\tau)d\tau,$$

(see Podlubny (1999)).

The fractional derivative (Riemann–Liouville derivative) of fractional-order q of a function $x(t)$ is defined by

$${}_{t_0}D_t^{q}x(t) = \frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\int_{t_0}^{t}(t-\tau)^{n-q-1}x(\tau)\,d\tau, \quad (n-1 \le q < n),$$

where $\Gamma(.)$ is the Gamma function, (see Liu et al. (2016)).

If $p > q > 0,$ then the equality

$${}_{t_0}D_t^{p}({}_{t_0}D_t^{-q}f(t))={}_{t_0}D_t^{p-q}f(t)$$

holds for sufficiently good functions $f(t)$. In particular, this relation holds if $f(t)$ is integrable (see Hu et al. (2015)).

Suppose $x(t) \in R^n \to R^n$ is a vector of differentiable functions. Then, for any $t \geq t_0$, the following relationship holds:

$$ {}_{t_0}D_t^q (x^T(t)Mx(t)) \leq 2x^T(t)M\, {}_{t_0}D_t^q x(t), \ 0 < q < 1, $$

where $M \in R^{n\times n}$ is a constant, square, symmetric and positive semi-definite matrix (see Duarte-Mermoud et al. 2015)).

For any $x, y \in R^n$, $\alpha > 0$, $\alpha \in R$, the following inequality holds:

$$ 2x^T y \leq \alpha x^T x + \frac{1}{\alpha} y^T y, $$

(see Liu et al. (2016)).

Suppose $U > 0$ and $V \geq 0$ are real symmetric matrices and ε is a positive number. Then,

$$ \varepsilon U > V \quad \Leftrightarrow \quad \lambda_{\max}(VU^{-1}) < \varepsilon \quad \Leftrightarrow \quad \lambda_{\max}(U^{-\frac{1}{2}} VU^{-\frac{1}{2}}) < \varepsilon, $$

(see Liu et al. (2016)).

3. Stability

Liu et al. (2016) considered the following fractional system with unbounded delay:

$$ {}_{t_0}D_t^q x(t) = Ax(t) + Bx(t-\tau(t)) + F_1(x(t)) + F_2(x(t-\tau(t))). $$

Liu et al. (2016), applying the Lyapunov second method, sufficient conditions on asymptotic stability of zero solution of this fractional nonlinear system with variable delay is obtained. The advantage of the employed method is that one may directly calculate integer-order derivative of the Lyapunov function used therein.
In this paper, we consider the following fractional-order nonlinear differential system with multiple variable delays:

$$ {}_{t_0}D_t^q x(t) = A_0 x(t) + G_0(x(t)) + \sum_{i=1}^{m} A_i x(t-h_i(t)) + \sum_{i=1}^{m} G_i(x(t-h_i(t))), \qquad (1) $$

with the initial conditions

$$ {}_{t_0}D_t^{-(1-q)} x(t) = \vartheta(t), \qquad t \in [-\infty, 0], $$

where ${}_{t_0}D_t^q$ is the Riemann–Liouville fractional derivative of order $0 < q < 1$, $t \in R^+$, $R^+ = [0, \infty)$, $x(t) \in R^n$, $x(t)$ is the state vector and A_0, $A_i \in R^{n\times n}$ are known constant matrices. The nonlinear functions $G_i \in R^n$ are continuous and satisfy $G_i(0) = 0.$ The variable delays $h_i(t) \geq 0$ are differentiable and they satisfy

$$\dot{h}_i(t) \leq \delta_i < 1, \quad i = 1, 2, \ldots, m, \tag{2}$$

where $\delta_i \in R,\ \delta_i > 0.$

Suppose the functions $G_j(x)$ are the higher order term in x such that

$$\lim_{\|x\| \to 0} \frac{\|G_j(x)\|}{\|x\|} = 0, \qquad j = 0, 1, \ldots, m. \tag{3}$$

It is clear that fractional system (1) includes and improves the fractional system discussed by Liu et al. (2016). The aim of this paper is to generalize and improve the work of Liu et al. (2016). These are the contributions of this paper to the subject and the relevant literature.

Throughout this paper, we use the following notations. R^n denotes the $n-$dimensional Euclidean space; $R^{n\times n}$ is the set of all $n \times n$ real matrices; $\|\,.\,\|$ denotes the Euclidean norm; L^T means the transpose of matrix L; U is symmetric if $U = U^T$; H is positive definite (or negative definite) if $\langle Hx, x\rangle > 0$ (or $\langle Hx, x\rangle < 0$) for all $x \neq 0$; $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$ denotes the spectral norm of matrix A; $\lambda_{\min}(B)$ and $\lambda_{\max}(B)$ denote the minimal and maximal eigenvalues of matrix B, respectively.

Theorem 3.1.

The zero solution of fractional differential equation system (1) is asymptotically stable if there exist a positive-definite matrix $P \in R^{n\times n}$ and a symmetric positive-definite matrix $Q \in R^{n\times n}$ such that the following relations hold:

$$\begin{cases} P^T A_0 + A_0^T P + (m+1)Q = 0, \\ \displaystyle\sum_{i=1}^{m} \frac{1}{\sqrt{1-\delta_i}} \|PA_i\| - \lambda_{\min}(Q) < 0. \end{cases} \tag{4}$$

Proof:

We define following Lyapunov functional:

$$V(t) = {}_{t_0}D_t^{q-1}(x^T(t)Px(t)) + \sum_{i=1}^{m} \int_{t-h_i(t)}^{t} x^T(s)Qx(s)ds. \tag{5}$$

In view of Lemma 2.1, Lemma 2.2 and the inequality (2), from the time derivative $V(t)$ along solutions of fractional system (1), we obtain the following:

$$\begin{aligned}
\dot{V}(t) &= {}_{t_0}D_t^q(x^T(t)Px(t)) + mx^T(t)Qx(t) - \sum_{i=1}^{m}(1-\dot{h}_i(t))x^T(t-h_i(t))Qx(t-h_i(t)) \\
&\le 2x^T(t)P\,{}_{t_0}D_t^q(x(t)) + mx^T(t)Qx(t) - \sum_{i=1}^{m}(1-\delta_i)x^T(t-h_i(t))Qx(t-h_i(t)) \\
&= 2x^T(t)P\{A_0x(t) + G_0(x(t)) + \sum_{i=1}^{m}A_ix(t-h_i(t)) + \sum_{i=1}^{m}G_i(x(t-h_i(t)))\} \\
&\quad + mx^T(t)Qx(t) - \sum_{i=1}^{m}(1-\delta_i)x^T(t-h_i(t))Qx(t-h_i(t)) \\
&= x^T(t)(P^TA_0 + A_0^TP + mQ)x(t) + 2x^T(t)P\sum_{i=1}^{m}A_ix(t-h_i(t)) \\
&\quad + 2x^T(t)PG_0(x(t)) + 2x^T(t)P\sum_{i=1}^{m}G_i(x(t-h_i(t)))\,, \\
&\quad - \sum_{i=1}^{m}(1-\delta_i)x^T(t-h_i(t))Qx(t-h_i(t)). \qquad (6)
\end{aligned}$$

For the terms included in (6), except the first one, using Lemma 2.3, we get the following inequalities, respectively:

$$\begin{aligned}
2\sum_{i=1}^{m}x^T(t)PA_ix(t-h_i(t)) &= 2\sum_{i=1}^{m}x^T(t)PA_iQ^{-\frac{1}{2}}Q^{\frac{1}{2}}x(t-h_i(t)) \\
&\le \sum_{i=1}^{m}\frac{1}{\alpha(1-\delta_i)}x^T(t)PA_iQ^{-1}A_i^TP^Tx(t) \\
&\quad + \sum_{i=1}^{m}\alpha(1-\delta_i)x^T(t-h_i(t))Qx(t-h_i(t)), \qquad (7)
\end{aligned}$$

$$2x^T(t)PG_0(x(t)) \le \frac{1}{\beta}x^T(t)P^2x(t) + \beta G_0^T(x(t))G_0(x(t)), \qquad (8)$$

$$\begin{aligned}
2x^T(t)P\sum_{i=1}^{m}G_i(x(t-h_i(t))) &\le \frac{1}{\mu}x^T(t)mP^2x(t) \\
&\quad + \mu\sum_{i=1}^{m}G_i^T(x(t-h_i(t)))G_i(x(t-h_i(t))), \qquad (9)
\end{aligned}$$

where α, β and μ are some positive constants.

If we gather the inequalities (7)-(9) into (6) and use the equality in (4), then it follows that:

$$\begin{aligned}
\dot{V}(t) &\le x^T(t)\left[P(\sum_{i=1}^{m}\frac{1}{\alpha(1-\delta_i)}A_iQ^{-1}A_i^T)P^T - Q + (\frac{1}{\beta}+\frac{m}{\mu})P^2\right]x(t) \\
&\quad + \sum_{i=1}^{m}(\alpha-1)(1-\delta_i)x^T(t-h_i(t))Qx(t-h_i(t)) + \beta G_0^T(x(t))G_0(x(t))
\end{aligned}$$

$$+\mu\sum_{i=1}^{m}G_i^T(x(t-h_i(t)))G_i(x(t-h_i(t))). \tag{10}$$

By the definition of the spectral norm, we derive that

$$\begin{aligned}
&\left[\lambda_{\max}\left(Q^{-\frac{1}{2}}P\left(\sum_{i=1}^{m}\frac{1}{1-\delta_i}A_iQ^{-1}A_i^T\right)P^TQ^{-\frac{1}{2}}\right)\right]^{\frac{1}{2}}\\
&=\left[\sum_{i=1}^{m}\frac{1}{1-\delta_i}\lambda_{\max}\left(Q^{-\frac{1}{2}}PA_iQ^{-1}A_i^TP^TQ^{-\frac{1}{2}}\right)\right]^{\frac{1}{2}}\\
&=\left(\sum_{i=1}^{m}\frac{1}{1-\delta_i}\left\|Q^{-\frac{1}{2}}PA_iQ^{-\frac{1}{2}}\right\|^2\right)^{\frac{1}{2}}\\
&\le\sum_{i=1}^{m}\frac{1}{\sqrt{1-\delta_i}}\left\|Q^{-\frac{1}{2}}\right\|^2\|PA_i\|\\
&=\frac{1}{\lambda_{\min}(Q)}\sum_{i=1}^{m}\frac{1}{\sqrt{1-\delta_i}}\|PA_i\|.
\end{aligned} \tag{11}$$

In view of the inequality given by (4), it follows that

$$\lambda_{\max}\left(Q^{-\frac{1}{2}}P\left(\sum_{i=1}^{m}\frac{1}{1-\delta_i}A_iQ^{-1}A_i^T\right)P^TQ^{-\frac{1}{2}}\right)<1. \tag{12}$$

Hence, from (12), it is seen that there exists a positive constant ε such that

$$\lambda_{\max}\left(Q^{-\frac{1}{2}}P\left(\sum_{i=1}^{m}\frac{1}{1-\delta_i}A_iQ^{-1}A_i^T\right)P^TQ^{-\frac{1}{2}}\right)<\varepsilon<1. \tag{13}$$

Since $A_iQ^{-1}A_i^T\ge 0,$ owing to Lemma 2.4, it follows from (9) that

$$P\left(\sum_{i=1}^{m}\frac{1}{1-\delta_i}A_iQ^{-1}A_i^T\right)P^T<\varepsilon Q. \tag{14}$$

Let $\alpha<1$ such that $0<\dfrac{\varepsilon}{\alpha}<1$. Hence, in view of (14), we have

$$P\left(\sum_{i=1}^{m}\frac{1}{\alpha(1-\delta_i)}A_iQ^{-1}A_i^T\right)P^T-Q<(\frac{\varepsilon}{\alpha}-1)Q<0. \tag{15}$$

Let us choose suitable positive constants β and μ such that the following inequality holds:

$$K_0=P\left(\sum_{i=1}^{m}\frac{1}{\alpha(1-\delta_i)}A_iQ^{-1}A_i^T\right)P^T-Q+(\frac{1}{\beta}+\frac{m}{\mu})P^2<0. \tag{16}$$

Since $\alpha<1,\ 0<\delta_i<1$ and Q is positive, then it follows that

$$(\alpha-1)(1-\delta_i)Q<0, \quad i=1,\ 2,\ldots,m.$$

Let

$$K_i=(\alpha-1)(1-\delta_i)Q. \tag{17}$$

Hence, from (10)-(17), we obtain the following inequality:

$$\dot{V}(t)\le x^T(t)K_0x(t)+\sum_{i=1}^{m}x^T(t-h_i(t))K_ix(t-h_i(t))$$
$$+\beta\|G_0(x(t))\|^2+\mu\sum_{i=1}^{m}\|G_i(x(t-h_i(t)))\|^2 . \tag{18}$$

By noting the inequalities (16) and (17), we can choose a constant $\sigma>0$ such that

$$K_i+\sigma I<0, \quad i=0,1,2,\ldots,m. \tag{19}$$

From (7), it follows that there exists a constant $\vartheta(\sigma,\beta,\mu)>0$ such if $\|x(t)\|<\vartheta, \quad t\ge t_0$, then the following two inequalities hold:

$$\|G_0(x(t))\|^2\le\frac{\sigma}{\beta}\|x(t)\|^2, \tag{20}$$

$$\sum_{i=1}^{m}\|G_i(x(t-h_i(t)))\|^2\le\frac{\sigma}{\mu}\sum_{i=1}^{m}\|x(t-h_i(t))\|^2 . \tag{21}$$

Substituting (20) and (21) into (18), we obtain

$$\dot{V}(t)\le x^T(t)(K_0+\sigma I)x(t)+\sum_{i=1}^{m}x^T(t-h_i(t))(K_i+\sigma I)x(t-h_i(t)).$$

Taking into account the inequality (19), we can conclude that $\dot{V}(t)$ is negative definite. Therefore, the zero solution of nonlinear fractional system (1) is asymptotically stable. This result completes the proof of the Theorem3.1.

Remark.

Let us consider nonlinear fractional differential system (1) for the scalar linear constant coefficients, when $A_0=a_0,\ a_0\in\Re,\ G_0=0\ A_i=0,\ G_i=0$ and $q=1$. In this case, we have the following scalar linear differential equation of first order:

$$\frac{dx}{dt}=a_0x.$$

It is well known that all solutions of this equation are asymptotically stable if and only if $a_0<0$. This condition coincides with the condition (4) of Theorem 3.1. Here, we would not like to give the details of the discussion for the sake of the brevity.

We now give a numerical example to analyze the stability a fractional system with variable delay. Hence, we see the efficiency and applicability of the proposed method.

Example 3.2.

In a special case of system (1), we consider the following fractional-order nonlinear system with variable delay:

$$ {}_{t_0}D_t^q x(t) = A_0 x(t) + A_1 x(t-h_1(t)) + G_0(x(t)) + G(x(t-h_1(t))), \tag{22}$$

where

$$q \in (0,1),\ x(t) = [x_1(t),\ x_2(t)]^T,\ A_0 = \begin{bmatrix} -3 & 0 \\ 0 & -6 \end{bmatrix},\ A_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 1.1 \end{bmatrix},$$

$$h_1(t) = 0.25t,\ G_0(x) = [x_1 \cos x_2,\ x_2 \sin x_1]^T,$$

$$G(x(t-h_1(t))) = [0.2x_1(t-h_1(t))e^{-x_1^2(t-h_1(t))}, 0.1x_2(t-h_1(t))e^{-x_2^2(t-h_1(t))}]^T.$$

Since $\dot{h}_1(t) = 0.25$, it is clear that $\delta_1 = 0.25$. Let $Q = 3I_2$. Then, it follows from condition (3.8) that $P = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$. Next, a simple calculation yields that $\|PA_1\| = 0.55$ and $\sqrt{1-\delta_1}\lambda_{\min}(Q) = 2.5980$. Thus, the condition (4) holds. That is, the trivial solution of fractional-order nonlinear system (22) is asymptotic stable.

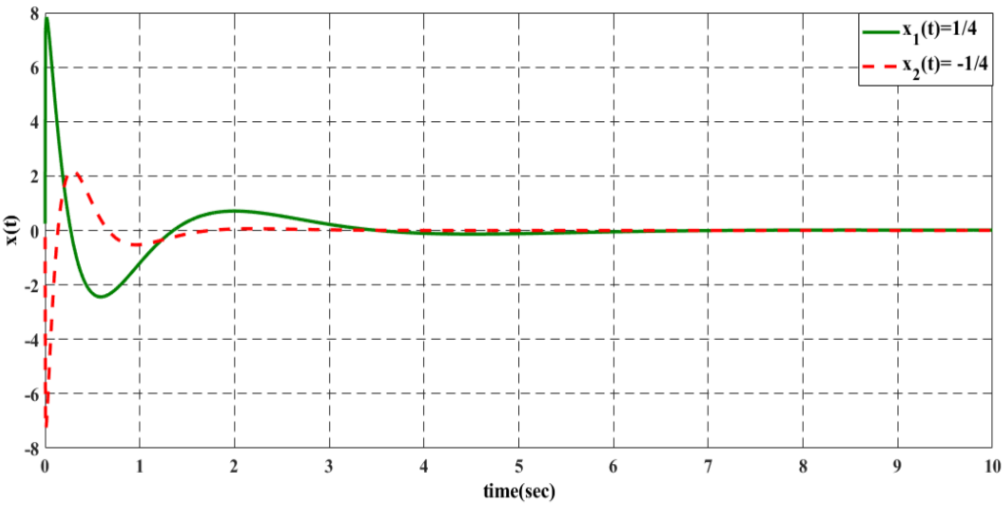

Figure 1. Numeric simulation for the asymptotic stability of solutions of equation (22) for $h_1(t) = 0.25t$.

4. Conclusions

In this paper, we derive certain new sufficient conditions on the asymptotic stability of zero solution of a nonlinear fractional system with multiple variable delays. We prove a new theorem on the subject by constructing a meaningful Lyapunov functional and using matrix inequalities. The proposed method avoids computing Riemann-Liouville fractional-order derivative of the given Lyapunov functional. When we compare our criteria with the stability criteria can be found in the relevant literature, it is seen that our criteria are simple and suitable for applications. Further, a numerical example is given with numerical simulation (see Figure 1) to demonstrate the effectiveness of the stability criteria of the considered system.

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