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A Comparative Study of Shehu Variational Iteration Method and Shehu Decomposition Method for Solving Nonlinear Caputo Time-Fractional Wave-like Equations with Variable Coefficients

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Abstract

In this paper, a comparative study between two different methods for solving nonlinear Caputo time-fractional wave-like equations with variable coefficients is conducted. These two methods are called the Shehu variational iteration method (SVIM) and the Shehu decomposition method (SDM). To illustrate the efficiency and accuracy of the proposed methods, three different numerical examples are presented. The results obtained show that the two methods are powerful and efficient methods which both give approximations of higher accuracy and closed form solutions if existing. However, the SVIM has an advantage over SDM that it solves the nonlinear problems without using the Adomian polynomials. Furthermore, the SVIM enables us to overcome the difficulties arising in identifying the general Lagrange multiplier and it may be considered as an added advantage of this technique over the SDM.

Keywords: Nonlinear Caputo time-fractional wave-like equations; Shehu transform; Variational iteration method; Adomian decomposition method; Series solutions

MSC 2010 No.: 35R11, 26A33, 35C05, 35C10

1. Introduction

Fractional differential equations are widely used in interpretation and modeling of many of realism matters appeared in applied mathematics and physics including fluid mechanics, viscoelasticity, chemistry, electrical circuits, diffusion, damping laws, relaxation processes, mathematical biology, and so on (Atanackovic et al. (2016), Fitt et al. (2009), Khalouta et al. (2019b), Kilbas et al. (2006), Lakshmikantham et al. (2008), Podlubny (1999), Vinagr et al. (2000), Zhou et al. (2017)). Recently, many researchers have been interested in studying solutions of fractional differential equations by using various methods, where the Adomian decomposition method (ADM) (Dhaigude et al. (2014), El-Borai et al. (2015), Guo (2019)), variational iteration method (VIM) (Abolhasani et al .(2017), Sontakke et al. (2019), Wu et al. (2017)), homotopy analysis method (HAM) (Atchi et al. (2017), Das (2015), Odibat (2019)), and homotopy perturbation method (HPM) (Al-Khaled et al. (2014), Hamdi Cherif et al. (2016), Javeed et al. (2019)), are the most popular ones that are used to solve both fractional ordinary differential equations as well as fractional partial differential equations.

The aim of this paper is to extend the obtained results in (see Khalouta et al. (2019a)) and to solve nonlinear Caputo time-fractional wave-like equation with variable coefficients by using two powerful method called the Shehu variational iteration method (SVIM) which is the combination of the Shehu transform method and the variational iteration method and the Shehu decomposition method (SDM) which is the combination of the Shehu transform method and the Adomian decomposition method and the comparison between these two methods with numerical results. The nonlinear Caputo time-fractional wave-like equations with variable coefficients is presented as follows,

$$D_t^{\alpha} v = \sum_{i,j=1}^{n} F_{1ij}(X,t,v)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(v_{x_i}, v_{x_j}) + \sum_{i=1}^{n} G_{1i}(X,t,v)\frac{\partial^{p}}{\partial x_i^p} G_{2i}(v_{x_i}) + H(X,t,v) + S(X,t), \tag{1}$$

with the initial conditions

$$v(X,0) = a_0(X), v_t(X,0) = a_1(X), \tag{2}$$

where D_t^{α} is the fractional derivative operator in the sense of Caputo of order α and $1 < \alpha \leq 2$, $v = v(X,t), X = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, t \geq 0$, F_{1ij}, G_{1i} $i, j \in \{1, 2, ..., n\}$ are nonlinear functions of X, t and v, F_{2ij}, G_{2i} $i, j \in \{1, 2, ..., n\}$, are nonlinear functions of derivatives of v with respect to x_i and x_j $i, j \in \{1, 2, ..., n\}$, respectively. Also, H, S are nonlinear functions and k, m, p are integers.

Note that, when $\alpha = 2$, the equation (1) reduces to the classical wave-like equations with variable coefficients. These types of equations are of considerable significance in various fields of applied sciences, mathematical physics, nonlinear hydrodynamics, engineering physics, biophysics, human movement sciences, astrophysics and plasma physics. These equations describe the evolution of erratic motions of small particles that are immersed in fluids, fluctuations of the intensity of laser light, velocity distributions of fluid particles in turbulent flows.

2. Basic Definitions and Results

In this section, we present necessary definitions and preliminary results about fractional calculus and Shehu transform, which are used further in this paper. For more details, see Kilbas et al. (2006).

Definition 2.1.

Let $f:[0,T]\longrightarrow\mathbb{R}$ be a continuous function. The left sided Riemann-Liouville fractional integral of order $\alpha\geq 0$ is defined by

$$I^{\alpha}f(t)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\int\limits_0^t(t-\xi)^{\alpha-1}f(\xi)d\xi, & \alpha>0,\\ f(t), & \alpha=0,\end{cases}\tag{3}$$

where

$$\Gamma(\alpha)=\int\limits_0^{\infty}t^{\alpha-1}e^{-t}dt,\alpha>0,$$

is the Euler gamma function.

Definition 2.2.

Let $f:[0,T]\longrightarrow\mathbb{R}$ be a continous function. The left sided Caputo fractional derivative of order $\alpha\geq 0$ is defined by

$$D^{\alpha}f(t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\int\limits_0^t(t-\xi)^{n-\alpha-1}f^{(n)}(\xi)d\xi, & n-1<\alpha<n,\\ f^{(n)}(t), & \alpha=n,\end{cases}\tag{4}$$

where $n=[\alpha]+1$ with $[\alpha]$ being the integer part of α.

Definition 2.3.

The Mittag-Leffler function is defined as follows

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)},\alpha\in\mathbb{C},Re(\alpha)>0.\tag{5}$$

A further generalization of (6) is given in the form

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+\beta)},\alpha,\beta\in\mathbb{C},Re(\alpha)>0,Re(\beta)>0.\tag{6}$$

For the Riemann-Liouville fractional integral and Caputo fractional derivative, we have the following relation

$$I^{\alpha}D^{\alpha}f(t)=f(t)-\sum_{k=0}^{n-1}f^{(k)}(0^+)\frac{t^k}{k!},\ t>0.$$

Remark 2.4.

In this paper, we consider the time-fractional derivative in the Caputo sense. When $\alpha \in \mathbb{R}^+$, the Caputo time-fractional derivative is defined by

$$D_t^\alpha v(X,t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \int\limits_0^t (t-\xi)^{n-\alpha-1} v^{(n)}(X,\xi)d\xi, & n-1<\alpha<n, \\ v^{(n)}(X,t), & \alpha = n, n \in \mathbb{N}^*. \end{cases}$$

Definition 2.5.

The Shehu transform of the function $f(t)$ of exponential order is defined over the set of functions (Shehu et al. (2019))

$$A = \left\{ f(t)/\exists N, \eta_1, \eta_2 > 0, |f(t)| < N \exp\left(\frac{|t|}{\eta_j}\right), \text{ if } t \in (-1)^i \times [0,\infty) \right\},$$

by the following integral

$$\mathbb{S}\left[f(t)\right] = F(s,u) = \int_0^\infty \exp\left(-\frac{st}{u}\right) f(t)dt, t > 0.$$

Theorem 2.6.

Let $n \in \mathbb{N}^*$ and $\alpha > 0$ be such that $n-1 < \alpha \leq n$ and $F(s,u)$ be the Shehu transform of the function $f(t)$, then the Shehu transform denoted by $F_\alpha(s,u)$ of the Caputo fractional derivative of $f(t)$ of order α, is given by

$$\mathbb{S}\left[D^\alpha f(t)\right] = F_\alpha(s,u) = \frac{s^\alpha}{u^\alpha} F(s,u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{\alpha-(k+1)} \left[D^k f(t)\right]_{t=0}. \tag{7}$$

Proof:

See Khalouta et al. (2019a). ∎

3. Solution of nonlinear Caputo time-fractional wave-like equations by the Shehu variational iteration method (SVIM)

Theorem 3.1.

Consider the nonlinear Caputo time-fractional wave-like equations (1) with the initial conditions (2). Then, by the SVIM the exact solution of Equations (1) and (2) is given as a limit of the successive approximations $v_n(X,t), n = 0,1,2,...$, in other words

$$v(X,t) = \lim_{n \longrightarrow \infty} v_n(X,t).$$

Proof:

In order to achieve our goal, we consider the following nonlinear Caputo time-fractional wave-like equations (1) with the initial conditions (2). First we define

$$
\begin{aligned}
Nv &= \sum_{i,j=1}^{n} F_{1ij}(X,t,v)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(v_{x_i}, v_{x_j}),\\
Mv &= \sum_{i=1}^{n} G_{1i}(X,t,v)\frac{\partial^{p}}{\partial x_i^p} G_{2i}(v_{x_i}),\\
Kv &= H(X,t,v).
\end{aligned} \tag{8}
$$

Equation (1) is written in the form

$$
D_t^\alpha v(X,t) = Nv(X,t) + Mv(X,t) + Kv(X,t) + S(X,t). \tag{9}
$$

Applying the Shehu transform on both sides of (9) and using the Theorem 2.6, we get

$$
\begin{aligned}
\mathbb{S}\left[v(X,t)\right] = \frac{u}{s}a_0(X) + \left(\frac{u}{s}\right)^2 a_1(X) + \frac{u^\alpha}{s^\alpha}\mathbb{S}\left[S(X,t)\right]\\
+\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[Nv(X,t) + Mv(X,t) + Kv(X,t)\right].
\end{aligned} \tag{10}
$$

After that, let us take the inverse Shehu transform on both sides of (10). We have

$$
u(X,t) = L(X,t) + \mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[Nv(X,t) + Mv(X,t) + Kv(X,t)\right]\right), \tag{11}
$$

where $L(X,t)$ is a term arising from the source term and the prescribed initial conditions. Take the first partial derivative with respect to t of Equation (11) to obtain

$$
\frac{\partial}{\partial t}v(X,t) - \frac{\partial}{\partial t}\mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[Nv(X,t) + Mv(X,t) + Kv(X,t)\right]\right) - \frac{\partial}{\partial t}L(X,t) = 0. \tag{12}
$$

According to the variational iteration method (Biazar et al. (2010)), we can construct a correct functional as follows

$$
v_{n+1}(X,t) = v_n(X,t) - \int_0^t \left[\frac{\partial v_n}{\partial \xi} - \frac{\partial}{\partial \xi}\mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[Nv_n + Mv_n + Kv_n\right]\right) - \frac{\partial L}{\partial \xi}\right] d\xi, \tag{13}
$$

or

$$
v_{n+1}(X,t) = L(X,t) + \mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[Nv_n(X,t) + Mv_n(X,t) + Kv_n(X,t)\right]\right). \tag{14}
$$

Finally, the exact solution of Equations (1) and (2) is given as a limit of the successive approximations $v_n(X,t), n = 0, 1, 2, ...$, in other words

$$
v(X,t) = \lim_{n\to\infty} v_n(X,t).
$$

This completes the proof. ■

4. Solution of nonlinear Caputo time-fractional wave-like equations by the Shehu decomposition method (SDM)

Theorem 4.1.

Consider the following nonlinear Caputo time-fractional wave-like equations (1) with the initial conditions (2). Then, by SDM the solution of Equations (1) and (2) is given in the form of infinite series which converges rapidly to the exact solution as follows

$$v(X,t)=\sum_{n=0}^{\infty} v_n(X,t).$$

Proof:

Similar to the proof of the Theorem 3.1, we have

$$v(X,t)=L(X,t)+\mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[Nv(X,t)+Mv(X,t)+Kv(X,t)\right]\right). \tag{15}$$

Now, we represent the solution in an infinite series form

$$v(X,t)=\sum_{n=0}^{\infty} v_n(X,t), \tag{16}$$

and the nonlinear terms can be decomposed as

$$Nv(X,t)=\sum_{n=0}^{\infty} A_n, Mv(X,t)=\sum_{n=0}^{\infty} B_n, Kv(X,t)=\sum_{n=0}^{\infty} C_n, \tag{17}$$

where A_n, B_n and C_n are Adomian polynomials of $v_0, v_1, v_2, .., v_n$, and it can be calculated by formula given below (Hosseini et al. (2012), Moradweysi et al. (2018)),

$$A_n=B_n=C_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i v_i\right)\right]_{\lambda=0}, n=0,1,2,.... \tag{18}$$

Using Equations (16) and (17), we can rewrite Equation (15) as

$$\sum_{n=0}^{\infty} v_n(X,t)=L(X,t)+\mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[\sum_{n=0}^{\infty}A_n+\sum_{n=0}^{\infty}B_n+\sum_{n=0}^{\infty}C_n\right]\right). \tag{19}$$

By comparing both sides of Equation (19) we have the following relation

$$\begin{aligned} v_0(X,t)&=L(X,t),\\ v_1(X,t)&=\mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[A_0+B_0+C_0\right]\right),\\ v_2(X,t)&=\mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[A_1+B_1+C_1\right]\right),\\ v_3(X,t)&=\mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[A_2+B_2+C_2\right]\right),\\ &\vdots \end{aligned} \tag{20}$$

In general the recursive relation is given by

$$\begin{aligned} v_0(X,t) &= L(X,t), \\ v_{n+1}(X,t) &= \mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[A_n + B_n + C_n\right]\right), n \geq 0. \end{aligned} \tag{21}$$

Then, the solution of Equations (1) and (2) is given in the form of infinite series as follows

$$v(X,t) = \sum_{n=0}^{\infty} v_n(X,t).$$

This completes the proof. ■

5. Illustrative examples and numerical results

In this section, we apply the SVIM and SDM to solve three examples of nonlinear Caputo time-fractional wave-like equations with variable coefficients and then compare our approximate solutions with the exact solutions.

Example 5.1.

Consider the 2-dimensional nonlinear Caputo time-fractional wave-like equation with variable coefficients

$$D_t^\alpha v = \frac{\partial^2}{\partial x \partial y}(v_{xx}v_{yy}) - \frac{\partial^2}{\partial x \partial y}(xyv_xv_y) - v, t > 0, 1 < \alpha \leq 2, \tag{22}$$

with the initial conditions

$$v(x,y,0) = e^{xy}, v_t(x,y,0) = e^{xy}, (x,y) \in \mathbb{R}^2. \tag{23}$$

5.1. Application of the SVIM

By applying the steps involved in the SVIM as presented in Section 3 to Equations (22) and (23), we obtain the iteration formula as follows

$$v_{n+1}(x,y,t) = e^{xy} + te^{xy} + \mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[\frac{\partial^2}{\partial x \partial y}(v_{nxx}v_{nyy}) \; - \frac{\partial^2}{\partial x \partial y}(xyv_{nx}v_{ny}) - v_n\right]\right),$$

and

$$\begin{aligned} v_0(x,y,t) &= (1+t)\,e^{xy}, \\ v_1(x,y,t) &= \left(1 + t - \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^{xy}, \\ v_2(x,y,t) &= \left(1 + t - \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^{xy}, \\ &\vdots \end{aligned}$$

Then, the general term in successive approximation is given by

$$v_n(x,y,t)=\sum_{k=0}^{n}\left(\frac{(-1)^k t^{k\alpha}}{\Gamma(k\alpha+1)}+\frac{(-1)^k t^{k\alpha+1}}{\Gamma(k\alpha+2)}\right)e^{xy}.$$

Finally, the exact solution of Equations (22) and (23) is given by

$$\begin{aligned} v(x,y,t)=\lim_{n\to\infty}v_n(x,y,t)&=\lim_{n\to\infty}\sum_{k=0}^{n}\left(\frac{(-1)^k t^{k\alpha}}{\Gamma(k\alpha+1)}+\frac{(-1)^k t^{k\alpha+1}}{\Gamma(k\alpha+2)}\right)e^{xy}\\ &=\left(E_\alpha(-t^\alpha)+tE_{\alpha,2}(-t^\alpha)\right)e^{xy}. \end{aligned} \tag{24}$$

5.2. Application of the SDM

By applying the steps involved in the SDM as presented in Section 4 to Equations (22) and (23), we have

$$\sum_{n=0}^{\infty}v_n(x,y,t)=e^{xy}+te^{xy}+\mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[\frac{\partial^2}{\partial x\partial y}\left(\sum_{n=0}^{\infty}A_n\right)-\frac{\partial^2}{\partial x\partial y}\left(xy\sum_{n=0}^{\infty}B_n\right)-\sum_{n=0}^{\infty}u_n\right]\right),$$

where $v_{xx}v_{yy}=\sum_{n=0}^{\infty}A_n$, $v_xv_y=\sum_{n=0}^{\infty}B_n$, are the Adomian polynomials that represents the nonlinear terms, and

$$\begin{aligned} v_0(x,y,t)&=(1+t)e^{xy},\\ v_1(x,y,t)&=-\left(\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^{xy},\\ v_2(x,y,t)&=\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^{xy},\\ &\vdots \end{aligned}$$

So, the solution of Equations (22) and (23) can be expressed by

$$\begin{aligned} v(x,y,t)&=\left(1+t-\frac{t^\alpha}{\Gamma(\alpha+1)}-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+...\right)e^{xy}\\ &=\left(E_\alpha(-t^\alpha)+tE_{\alpha,2}(-t^\alpha)\right)e^{xy}, \end{aligned} \tag{25}$$

where $E_\alpha(-t^\alpha)e^{xy}$ and $E_{\alpha,2}(-t^\alpha)$ are the Mittag-Leffler functions defined by Equations (5) and (6).

In the special case $\alpha=2$, (24) and (25) becomes

$$v(x,y,t)=\left(E_2(-t^2)+tE_{2,2}(-t^2)\right)e^{xy}=(\cos t+\sin t)e^{xy}.$$

The solution is the same as that obtained by the FRDTM (Khalouta et al. (2019c)) and FRPSM (Khalouta et al. (2020)).

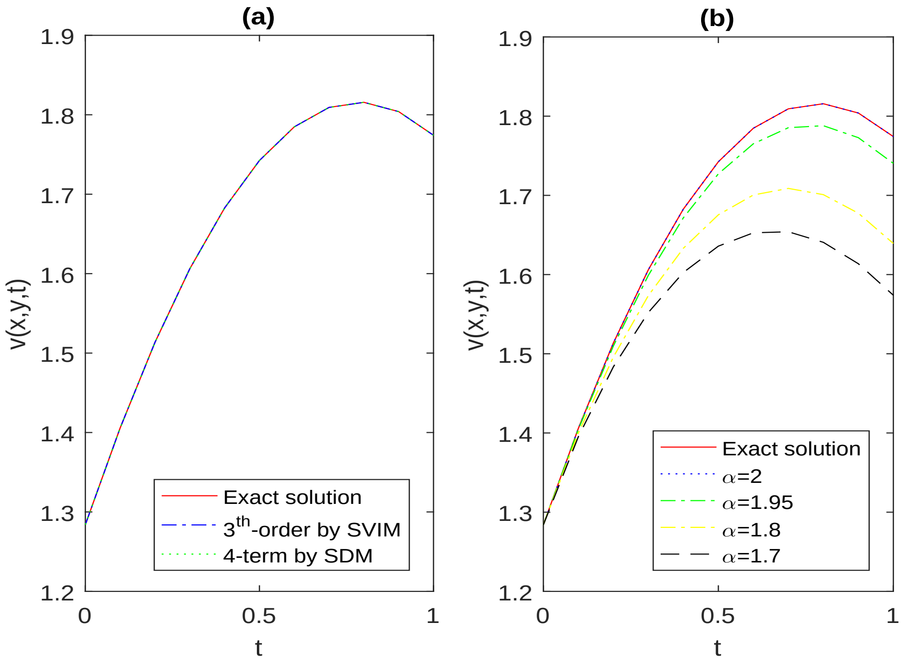

Figure 1. (a) The comparison of the exact solution and the approximate solutions by SVIM and SDM, when $\alpha = 2$ and $x = y = 0.5$, (b) The behavior of the exact solution and the approximate solutions by SVIM and SDM for different values of α when $x = y = 0.5$.

Table 1. The absolute errors for differences between the exact solution and the approximate solutions by SVIM and SDM for Example 5.1 when $n = 3, m = 4$ and $\alpha = 2$.

	$\lvert v_{exact} - v_{SVIM}\rvert$	$\lvert v_{exact} - v_{SDM}\rvert$	$\lvert v_{exact} - v_{SVIM}\rvert$	$\lvert v_{exact} - v_{SDM}\rvert$
$t/x,y$	0.5	0.5	0.7	0.7
0.1	3.2196×10^{-13}	3.2196×10^{-13}	4.0929×10^{-13}	4.0929×10^{-13}
0.3	2.1569×10^{-9}	2.1569×10^{-9}	2.7420×10^{-9}	2.7420×10^{-9}
0.5	1.3095×10^{-7}	1.3095×10^{-7}	1.6647×10^{-7}	1.6647×10^{-7}
0.7	1.9680×10^{-6}	1.9680×10^{-6}	2.5019×10^{-6}	2.5019×10^{-6}
0.9	1.4947×10^{-5}	1.4947×10^{-5}	1.9001×10^{-5}	1.9001×10^{-5}

Example 5.2.

Consider the following nonlinear Caputo time-fractional wave-like equation with variable coefficients

$$D_t^{\alpha} v = v^2 \frac{\partial^2}{\partial x^2}(v_x v_{xx} v_{xxx}) + v_x^2 \frac{\partial^2}{\partial x^2}(v_{xx}^3) - 18v^5 + v, t > 0, 1 < \alpha \leq 2, \tag{26}$$

with the initial conditions

$$v(x,0) = e^x, v_t(x,0) = e^x, x \in \,]0,1[\,. \tag{27}$$

5.3. Application of the SVIM

By applying the steps involved in the SVIM as presented in Section 3 to Equations (26) and (27), we obtain the iteration formula as follows

$$v_{n+1}(x,t) = e^x + te^x + \mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[v_n^2\frac{\partial^2}{\partial x^2}(v_{nx}v_{nxx}v_{nxxx}) \ +v_{nx}^2\frac{\partial^2}{\partial x^2}(v_{nxx}^3) - 18v_n^5 + v_n\right]\right),$$

and

$$\begin{aligned}
v_0(x,t) &= (1+t)\,e^x,\\
v_1(x,t) &= \left(1+t+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^x,\\
v_2(x,t) &= \left(1+t+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^x,\\
&\vdots
\end{aligned}$$

Then, the general term in successive approximation is given by

$$v_n(x,t) = \sum_{k=0}^{n}\left(\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}+\frac{t^{k\alpha+1}}{\Gamma(k\alpha+2)}\right)e^x.$$

Finally, the exact solution of Equations (26) and (27) is given by

$$\begin{aligned}
v(x,t) = \lim_{n\to\infty} v_n(x,t) &= \sum_{k=0}^{\infty}\left(\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}+\frac{t^{k\alpha+1}}{\Gamma(k\alpha+2)}\right)e^x\\
&= \left(E_\alpha(t^\alpha)+tE_{\alpha,2}(t^\alpha)\right)e^x. \qquad (28)
\end{aligned}$$

5.4. Application of the SDM

By applying the steps involved in the SDM as presented in Section 4 to Equations (26) and (27), we have

$$\sum_{n=0}^{\infty} v_n(x,t) = e^x + te^x + \mathbb{S}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathbb{S}\left[\sum_{n=0}^{\infty}A_n + \sum_{n=0}^{\infty}B_n - 18\sum_{n=0}^{\infty}C_n + \sum_{n=0}^{\infty}u_n\right]\right),$$

where $v^2\frac{\partial^2}{\partial x^2}(v_xv_{xx}v_{xxx}) = \sum_{n=0}^{\infty}A_n, v_x^2\frac{\partial^2}{\partial x^2}(v_{xx}^3) = \sum_{n=0}^{\infty}B_n$ and $v^5 = \sum_{n=0}^{\infty}C_n$, are the Adomian polynomials that represents the nonlinear terms, and

$$\begin{aligned}
v_0(x,t) &= (1+t)\,e^x,\\
v_1(x,t) &= \left(\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^x,\\
v_2(x,t) &= \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^x,\\
&\vdots
\end{aligned}$$

So, the solution of Equations (26) and (27) can be expressed by

$$\begin{aligned} v(x,t) &= \left(1 + t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \ldots\right) e^{x} \\ &= \left(E_{\alpha}(t^{\alpha}) + tE_{\alpha,2}(t^{\alpha})\right) e^{x}, \end{aligned} \tag{29}$$

where $E_{\alpha}(t^{\alpha})$ and $E_{\alpha,2}(t^{\alpha})$ are the Mittag-Leffler functions, defined by Equations (5) and (6).

In the special case $\alpha = 2$, (28) and (29) becomes

$$\begin{aligned} v(x,t) &= \left(E_2(t^2) + tE_{2,2}(t^2)\right) e^{x} \\ &= e^{x+t}. \end{aligned}$$

The solution is the same as that obtained by the FRDTM (Khalouta et al. (2019c)) and FRPSM (Khalouta et al. (2020)).

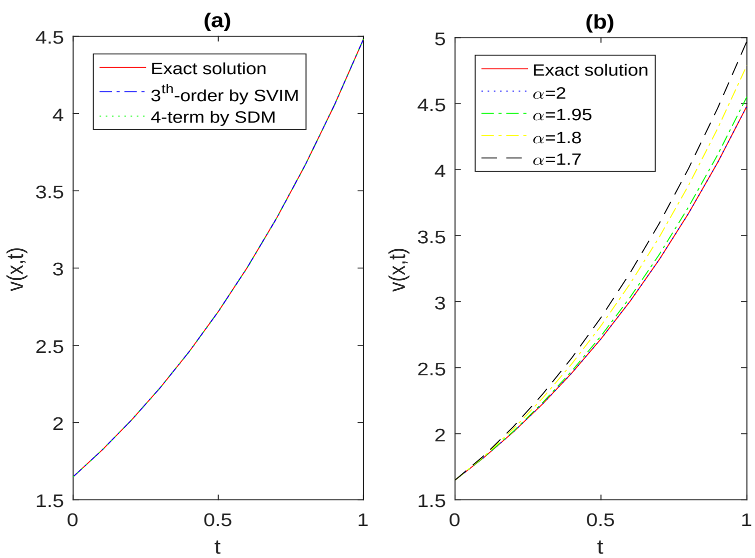

Figure 2. (a) The comparison of the exact solution and the approximate solutions by SVIM and SDM, when $\alpha = 2$ and $x = 0.5$, (b) The behavior of the exact solution and the approximate solutions by SVIM and SDM for different values of α when $x = 0.5$.

Table 2. The absolute errors for differences between the exact solution and the approximate solutions by SVIM and SDM for Example 5.2 when $n = 3, m = 4$ and $\alpha = 2$.

	$\lvert v_{exact} - v_{SVIM}\rvert$	$\lvert v_{exact} - v_{SDM}\rvert$	$\lvert v_{exact} - v_{SVIM}\rvert$	$\lvert v_{exact} - v_{SDM}\rvert$
t/x	0.5	0.5	0.7	0.7
0.1	4.1350×10^{-13}	4.1350×10^{-13}	5.0505×10^{-13}	5.0505×10^{-13}
0.3	2.7750×10^{-9}	2.7750×10^{-9}	3.3894×10^{-9}	3.3894×10^{-9}
0.5	1.6907×10^{-7}	1.6907×10^{-7}	2.0650×10^{-7}	2.0650×10^{-7}
0.7	2.5543×10^{-6}	2.5543×10^{-6}	3.1199×10^{-6}	3.1199×10^{-6}
0.9	1.9535×10^{-5}	1.9535×10^{-5}	2.3860×10^{-5}	2.3860×10^{-5}

Example 5.3.

Consider the following one dimensional nonlinear Caputo time-fractional wave-like equation with variable coefficients

$$D_t^{\alpha} v = x^2 \frac{\partial}{\partial x}(v_x v_{xx}) - x^2 (v_{xx})^2 - v, t > 0, 1 < \alpha \leq 2, \tag{30}$$

with the initial conditions

$$v(x,0) = 0, v_t(x,0) = x^2, x \in \left]0,1\right[. \tag{31}$$

5.5. Application of the SVIM

By applying the steps involved in the SVIM as presented in Section 3 to Equations (30) and (31), we obtain the iteration formula as follows

$$v_{n+1}(x,t) = tx^2 + \mathbb{S}^{-1}\left(\frac{u^{\alpha}}{s^{\alpha}}\mathbb{S}\left[x^2\frac{\partial}{\partial x}(v_{nx}v_{nxx}) - x^2(v_{nxx})^2 - v_n\right]\right),$$

and

$$\begin{aligned} v_0(x,t) &= tx^2, \\ v_1(x,t) &= \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)x^2, \\ v_2(x,t) &= \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)x^2, \\ &\vdots \end{aligned}$$

Then, the general term in successive approximation is given by

$$v_n(x,t) = x^2\left(\sum_{k=0}^{n}\frac{(-1)^k t^{k\alpha+1}}{\Gamma(k\alpha+2)}\right).$$

Finally, the exact solution of Equations (30) and (31) is given by

$$v(x,t) = \lim_{n\to\infty} v_n(x,t) = x^2\left(\sum_{i=0}^{\infty}\frac{(-1)^k t^{k\alpha+1}}{\Gamma(k\alpha+2)}\right) = x^2\left(tE_{\alpha,2}(-t^{\alpha})\right). \tag{32}$$

5.6. Application of the SDM

By applying the steps involved in the SDM as presented in Section 4 to Equations (30) and (31), we have

$$\sum_{n=0}^{\infty} v_n(x,t) = tx^2 + \mathbb{S}^{-1}\left(\frac{u^{\alpha}}{s^{\alpha}}\mathbb{S}\left[x^2\frac{\partial}{\partial x}\sum_{n=0}^{\infty}A_n - x^2\sum_{n=0}^{\infty}B_n - \sum_{n=0}^{\infty}u_n\right]\right),$$

where $u_x u_{xx} = \sum_{n=0}^{\infty} A_n$ and $(u_{xx})^2 = \sum_{n=0}^{\infty} B_n$, are the Adomian polynomials that represents the nonlinear terms, and

$$\begin{aligned} v_0(x,t) &= tx^2, \\ v_1(x,t) &= -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}x^2, \\ v_2(x,t) &= \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}x^2, \\ &\vdots \end{aligned}$$

So, the solution of Equations (30) and (31) can be expressed by

$$\begin{aligned} v(x,t) &= x^2 \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + ... \right) \\ &= x^2 \left(tE_{\alpha,2}(-t^{\alpha}) \right), \end{aligned} \tag{33}$$

where $E_{\alpha,2}(-t^{\alpha})$ is the Mittag-Leffler function, defined by Equation (5).

In the special case, $\alpha = 2$, Equations (32) and (33) become

$$\begin{aligned} v(x,t) &= x^2 \left(tE_{2,2}(-t^2) \right) \\ &= x^2 \sin t. \end{aligned}$$

The solution is the same as that obtained by the the FRDTM (Khalouta et al. (2019c)) and FRPSM (Khalouta et al. (2020)).

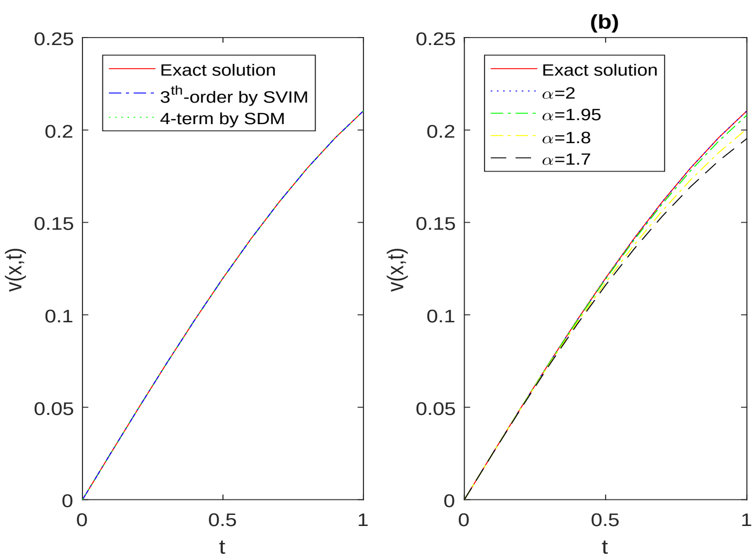

Figure 3. (a) The comparison of the exact solution and the approximate solutions by SVIM and SDM, when $\alpha = 2$ and $x = 0.5$, (b) The behavior of the exact solution and the approximate solutions by SVIM and SDM for different values of α when $x = 0.5$.

Table 3. The absolute errors for differences between the exact solution and the approximate solutions by SVIM and SDM for Example 5.3 when $n = 3, m = 4$ and $\alpha = 2$.

	$\lvert v_{exact} - v_{SVIM}\rvert$	$\lvert v_{exact} - v_{SDM}\rvert$	$\lvert v_{exact} - v_{SVIM}\rvert$	$\lvert v_{exact} - v_{SDM}\rvert$
t/x	0.5	0.5	0.7	0.7
0.1	6.8887×10^{-16}	6.8887×10^{-16}	1.3502×10^{-15}	1.3502×10^{-15}
0.3	1.3549×10^{-11}	1.3549×10^{-11}	2.6556×10^{-11}	2.6556×10^{-11}
0.5	1.3425×10^{-9}	1.3425×10^{-9}	2.6313×10^{-9}	2.6313×10^{-9}
0.7	2.7677×10^{-8}	2.7677×10^{-8}	5.4248×10^{-8}	5.4248×10^{-8}
0.9	2.6495×10^{-7}	2.6495×10^{-7}	5.1930×10^{-7}	5.1930×10^{-7}

6. Numerical results and discussion

Figures 1, 2 and 3(a) represent the comparison of the 3^{th} order approximate solution obtained by SVIM and the 4-term approximate solution obtained by SDM and the exact solution at $\alpha = 2$ when $x = y = 0.5$ for Example 5.1 and $x = 0.5$ for Examples 5.2 and 5.3. The numerical results show the SVIM and SDM are highly accurate. Figure 3(b) represents the behavior of the exact solutions and the 3^{th} order approximate solution by SVIM and the $4-$term approximate solution by SDM at $\alpha = 1.7, 1.8, 1.95, 2$. These results affirm that when α approaches 2, our results approach the exact solutions. In Tables 1, 2 and 3, we compute the absolute errors for differences between the exact solutions and the 3^{th} order approximate solution by SVIM and the $4-$term approximate solution by SDM at $\alpha = 2$. The absolute errors obtained by SVIM are the same results obtained by SDM.

7. Conclusion

In this paper, we have compared between the Shehu variational iteration method (SVIM) and the Shehu decomposition method (SDM) for solving nonlinear Caputo time-fractional wave-like equations with variable coefficients. These two methods are powerful and efficient methods that both give approximations of higher accuracy and closed form solutions. The comparison between the third iteration solution of the SVIM and fourth terms of the SDM constitutes an excellent agreement. However, the SVIM has an advantage over SDM that it solves the nonlinear problems without using the Adomian polynomials. The SVIM enables us to overcome the difficulties arising in identifying the general Lagrange multiplier and it may be considered as an added advantage of this technique over the decomposition method. It is concluded that these methods are very powerful mathematical tool for solving different kinds nonlinear fractional differential equations.

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