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Study on Solving Two-dimensional Linear and Nonlinear Volterra Partial Integro-differential Equations by Reduced Differential Transform Method

[1]Seyyedeh Roodabeh Moosavi Noori and [2]Nasir Taghizadeh

Department of Pure Mathematics
Faculty of Mathematical Sciences
University of Guilan
P.O.Box 1914
Rasht, Iran
[1]moosavinoori@gmail.com; [2]taghizdeh@guilan.ac.ir

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Abstract

In this article, we study on the analytical and numerical solution of two-dimensional linear and nonlinear Volterra partial integro-differential equations with the appropriate initial condition by means of reduced differential transform method. The advantage of this method is its simplicity in using, it solves the problem directly without the need for linearization, perturbation, or any other transformation and gives the solution in the form of convergent power series with elegantly computed components. The validity and efficiency of this method are illustrated by considering five computational examples.

Keywords: Volterra integral equation; Volterra partial integro-differential equations; Reduced differential transform method; Approximate solutions; Exact solutions

MSC 2010 No.: 35C05, 35E15, 45D05, 45A05, 45G10, 83C15

1. Introduction

The mathematical model of real-life problems usually leads to functional equations, e.g., ordinary and partial differential equations, integral and integro-differential equations and others. In particular, integro-differential equation is an important branch of mathematics which arise in engineering, mechanics, physics, astronomy, fluid dynamics, heat and mass transfer, oscillation theory, economics, potential theory, electrostatics, biological models and chemical kinetics (Jerri (1999); Rashed (2004); Polyanin and Manzhirov (2008); Saadati et al. (2008); Alawneh et al. (2010); Rashidinia and Tahmasebi (2013)). Also, partial integro-differential equation is a good model for viscoelasticity (Dehghan (2006)). Thus the investigation of the analytical and numerical solutions of such integro-differential equations helps us to understand the means of these mathematical models.

The exact solutions of some integro-differential equations cannot be found, thus in recent years, several numerical approaches have been used to estimate the solution of these models such as Adomian Decomposition Method (ADM) (Wazwaz (2010)), Differential Transform Method (DTM) (Arikoglu and Ozkol (2008); Tari and Shahmorad (2011)), ELzaki transform (Elzaki and Elzaki (2011)), Homotopy analysis method (HAM) (Fariborzi Araghi and Behzadi (2011)), Homotopy perturbation method (HPM) (Yusufoğlu (2009); Raftari (2010); Vanani et al. (2011)), Hybrid function method (Hsiao (2009)), Legendre polynomials (Aziz and Khan (2017)), Tau method (Abbasbandy and Taati (2009)), Variational Iteration Method (VIM) (Wang and He (2007); Biazar et al. (2010); Hussain et al. (2016)), etc.

The integro-differential equations have been developed very rapidly in recent years. Our motivation is to apply the reduced differential transform method (RDTM) to solve two-dimensional linear and nonlinear Volterra partial integro-differential equations

$$\begin{aligned} &F_i(D_{11}^{(1)}u_1(x,t)+\ldots+D_{1m}^{(1)}u_m(x,t)) \\ &= f_i(x,t)+\lambda_i\int_0^t\int_0^x K_i(x,t,\omega,\tau)G_i(D_{11}^{(2)}u_1(\omega,\tau),\ldots,D_{1m}^{(2)}u_m(\omega,\tau))\mathrm{d}\omega\mathrm{d}\tau, \end{aligned} \tag{1}$$

with where K_i and f_i are continuous functions and K_i has the following form

$$K_i(x,t,\omega,\tau)=\sum_{j=0}^{p}v_{ij}(x,t)w_{ij}(\omega,\tau), \qquad i=1,2,\ldots,m. \tag{2}$$

The concept of the reduced differential transform method (RDTM) was proposed and applied to solve linear and nonlinear initial value problems by Keskin in his Ph.D (Keskin (2010)). Keskin and Oturanc used RDTM in the study about the analytical solution of linear and nonlinear wave equations (Keskin and Oturanc (2009)) and they showed the effectiveness, and the accuracy of the proposed method. During recent years the reduced differential transform method has been described in a series of papers, for the analytical and numerical solution of ordinary differential equations (Biazar and Eslami (2010)), partial differential equations (Al-Amr (2014); Rawashdeh and Obeidat (2014); Taghizadeh and Moosavi Noori (2017)), fractional differential equations (Gupta (2011); Shahmorad and Khajehnasiri (2014)), Volterra integral equation (Tari et al. (2009); Ziqan et al. (2016)) and integro-differential equations (Abazari and Kilicman (2014)). In this method, the solution is considered as an infinite series which usually converges rapidly to exact solutions.

The rest of this study is presented in the following sections. In Section 2, we describe briefly the reduced differential transform method (RDTM). In Section 3, we outline several important

Theorems. In Section 4, some numerical examples are given to clarify the method. Finally, we give a conclusion in Section 5.

2. Reduced differential transform method (RDTM)

In this section, to illustrate the ideas of this method, let us introduce the following basic definitions and operations of the reduced differential transform method, for more details, see (Keskin and Oturanc (2009)) and the references therein. Now, suppose that function of two variables $u(x,t)$ which is analytic and k-times continuously differentiable with respect to space x and time t in the domain of our interest. Suppose that we can consider this function as a product of two single-variable functions $u(x,t) = f(x).g(t)$. Then $u(x,t)$ can be represented as:

$$u(x,t) = \left(\sum_{i=0}^{\infty} F(i)x^i\right)\left(\sum_{j=0}^{\infty} G(j)t^j\right) = \sum_{k=0}^{\infty} U_k(x)t^k, \tag{3}$$

where the function $U_k(x)$ is called the spectrum of $u(x,t)$.

Definition 2.1.

Let $u(x,t)$ be an analytic function in the domain of interest. The reduced differential transform function is

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}, \tag{4}$$

where the t-dimensional spectrum function $U_k(x)$ is the reduced differential transformed function. In this paper, the lowercase $u(x,t)$ express the original function, while the uppercase $U_k(x)$ stands for the transformed function.

Definition 2.2.

The differential inverse transform of $U_k(x)$ is determined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k. \tag{5}$$

In fact, the function $u(x,t)$ can be written in a finite series as follows,

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x)t^k, \tag{6}$$

where n is order of approximate solution.

Therefore the exact solution of the problem is given by

$$u(x,t) = \lim_{n\to\infty} \tilde{u}_n(x,t). \tag{7}$$

The basic mathematical operations performed by RDTM can be easily obtained and are listed in Table I.

3. Main results

We state the Fundamental Theorem of this paper in this section. Suppose that the functions $W_k(x)$, $U_k(x)$, and $V_k(x)$ are the reduced differential transform functions of $w(x,t)$, $u(x,t)$ and $v(x,t)$, respectively.

Theorem 3.1.

If $w(x,t) = \int_0^t\int_0^x u(\omega,\tau)\mathrm{d}\omega\mathrm{d}\tau$, then

$$W_k(x) = \frac{1}{k}\int_0^x U_{k-1}(\omega)\mathrm{d}\omega, \quad k \geq 1. \tag{8}$$

Proof:

The kth partial derivative of the function $w(x,t)$ is

$$\frac{\partial^k}{\partial t^k}w(x,t) = \int_0^x \frac{\partial^{k-1}}{\partial t^{k-1}}u(\omega,t)\mathrm{d}\omega = (k-1)!\int_0^x U_{k-1}(\omega)\mathrm{d}\omega.$$

The result can be easily deduced from Equation (4). ■

Theorem 3.2.

Let $w(x,t) = \int_0^t\int_0^x u(\omega,\tau)v(\omega,\tau)\mathrm{d}\omega\mathrm{d}\tau$. Then

$$W_k(x) = \frac{1}{k}\int_0^x \sum_{r=0}^{k-1} U_r(\omega)V_{k-r-1}(\omega)\mathrm{d}\omega, \quad k \geq 1. \tag{9}$$

Proof:

From Table I and Leibnitz formula, the kth partial derivative of $w(x,t)$ is

$$\begin{aligned}\frac{\partial^k}{\partial t^k}w(x,t) &= \int_0^x \frac{\partial^{k-1}}{\partial t^{k-1}}\{u(\omega,t)v(\omega,t)\}\mathrm{d}\omega \\ &= \int_0^x \sum_{r=0}^{k-1}\binom{k-1}{r}\frac{\partial^r}{\partial t^r}u(\omega,t)\frac{\partial^{k-r-1}}{\partial t^{k-r-1}}v(\omega,t)\mathrm{d}\omega,\end{aligned}$$

therefore,

$$\begin{aligned}\left[\frac{\partial^k}{\partial t^k}w(x,t)\right]_{t=0} &= \int_0^x \sum_{r=0}^{k-1}\binom{k-1}{r}r!(k-r-1)!U_r(\omega)V_{k-r-1}(\omega)\mathrm{d}\omega \\ &= (k-1)!\int_0^x \sum_{r=0}^{k-1} U_r(\omega)V_{k-r-1}(\omega)\mathrm{d}\omega.\end{aligned}$$

The result can be easily deduced from Equation (4). ■

Theorem 3.3.

Let $w(x,t) = h(x,t)\int_0^t\int_0^x u(\omega,\tau)\mathrm{d}\omega\mathrm{d}\tau$. Then

$$W_k(x) = \sum_{r=0}^{k-1} \frac{1}{k-r} H_r(x) \int_0^x U_{k-r-1}(\omega)\mathrm{d}\omega, \quad k \geq 1. \tag{10}$$

Proof:

The kth partial derivative of the function $w(x,t)$ with respect to t is

$$\frac{\partial^k}{\partial t^k} w(x,t) = \sum_{r=0}^{k} \binom{k}{r} \frac{\partial^r}{\partial t^r} h(x,t) \int_0^x \frac{\partial^{k-r-1}}{\partial t^{k-r-1}} u(\omega,t)\mathrm{d}\omega.$$

On the other hand, $\left[\frac{\partial^{k-r}}{\partial t^{k-r}} \int_0^t\int_0^x u(\omega,\tau)\mathrm{d}\omega\mathrm{d}\tau\right]_{t=0} = 0$ for $k = r$.

Thus,

$$\left[\frac{\partial^k}{\partial t^k} w(x,t)\right]_{t=0} = \sum_{r=0}^{k-1} \binom{k}{r} r!(k-r-1)! H_r(x) \int_0^x U_{k-r-1}(\omega)\mathrm{d}\omega.$$

The result can be easily deduced from Equation (4). ■

4. Applications

In this section, in order to illustrate the performance of the RDTM in solving the partial integro-differential equations and the efficiency of the method, the following examples are considered.

Example 4.1.

We first consider the following linear Volterra partial integro-differential equation

$$\frac{\partial u(x,t)}{\partial t} + u(x,t) = 2xe^t - \frac{1}{4}x^4 e^t + \frac{1}{4}x^4 + \int_0^t\int_0^x \omega^2 u(\omega,\tau)\mathrm{d}\omega\mathrm{d}\tau, \tag{11}$$

for $(x,t) \in [0,1] \times [0,1]$ and with initial condition $u(x,0) = x$.

According to the operations of differential transformation given in Table I and to Theorem 3.2, we have the following recurrence relation

$$(k+1)U_{k+1}(x) + U_k(x) = \frac{2}{k!}x - \frac{1}{4k!}x^4 + \frac{1}{4}x^4\delta(k) + \frac{1}{k}\int_0^x \sum_{r=0}^{k-1} \omega^2 \delta(r) U_{k-r-1}\mathrm{d}\omega. \tag{12}$$

It is easy to see that

$$U_0(x) = x. \tag{13}$$

Consequently, we find

$$U_1(x) = x, \quad U_2(x) = \frac{x}{2!}, \quad U_3(x) = \frac{x}{3!}, \quad U_4(x) = \frac{x}{4!}, \quad \ldots. \tag{14}$$

Therefore, the exact solution of the integral equation (11) is given by

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^k=xe^t. \tag{15}$$

Example 4.2.

We next consider the following linear Volterra partial integro-differential equation

$$\frac{\partial u(x,t)}{\partial t}+\frac{\partial u(x,t)}{\partial x}=-1+e^t+e^x+e^{x+t}+\int_0^t\int_0^x u(\omega,\tau)\mathrm{d}\omega\mathrm{d}\tau, \tag{16}$$

for $(x,t)\in[0,1]\times[0,1]$ and with initial condition $u(x,0)=e^x$.

Equation (16) is transformed by using Table I and Theorem 3.1 as follows,

$$(k+1)U_{k+1}(x)+\frac{\partial U_k}{\partial x}=-\delta(k)+\frac{1}{k!}+e^x\delta(k)+\frac{e^x}{k!}+\frac{1}{k}\int_0^x U_{k-1}(\omega)\mathrm{d}\omega. \tag{17}$$

It is easy to see that

$$U_0(x)=e^x. \tag{18}$$

So, elementary calculation on the last integral equation leads to

$$U_1(x)=e^x,\quad U_2(x)=\frac{1}{2!}e^x,\quad U_3(x)=\frac{1}{3!}e^x,\quad U_4(x)=\frac{1}{4!}e^x,\quad \ldots. \tag{19}$$

Therefore, the exact solution of the integral equation is

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^k=e^{x+t}. \tag{20}$$

Example 4.3.

Next, we consider the nonlinear Volterra partial integro-differential equation

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial t^2}+u(x,t)=&\frac{1}{8}x^4\sin t\cos t-\frac{1}{8}x^4 t-\frac{1}{9}x^3\sin^3 t\\&+\int_0^t\int_0^x(\omega+\cos\tau)u^2(\omega,\tau)\mathrm{d}\omega\mathrm{d}\tau,\end{aligned} \tag{21}$$

for $(x,t)\in[0,1]\times[0,1]$ and with initial condition $u(x,0)=0,\ \frac{\partial u(x,0)}{\partial t}=x$.

Taking the reduced differential transform of Equation (21), Theorem 3.2, leads to

$$\begin{aligned}&(k+1)(k+2)U_{k+2}(x)+U_k(x)=\frac{1}{8}x^4\sum_{r=0}^{k}\frac{1}{r!(k-r)!}\sin\frac{r\pi}{2}\cos\frac{(k-r)\pi}{2}\\&-\frac{1}{8}x^4\delta(k-1)-\frac{1}{9}x^3\sum_{r=0}^{k}\sum_{l=0}^{r}\frac{1}{l!(r-l)!(k-r)!}\sin\frac{l\pi}{2}\sin\frac{(r-l)\pi}{2}\sin\frac{(k-r)\pi}{2}\\&+\frac{1}{k}\int_0^x\sum_{r=0}^{k-1}\sum_{l=0}^{r}\Big(\omega\delta(l)+\frac{1}{l!}\cos\frac{l\pi}{2}\Big)U_{r-l}(\omega)U_{k-r-l}(\omega)\mathrm{d}\omega.\end{aligned} \tag{22}$$

It is clear that

$$U_0(x) = 0, \quad U_1(x) = x. \tag{23}$$

So, easy calculation on the last integral equation will produce

$$U_2(x) = 0, \quad U_3(x) = -\frac{1}{3!}x, \quad U_4(x) = 0, \quad U_5(x) = \frac{1}{5!}x, \quad \ldots. \tag{24}$$

Hence, the exact solution of this integral equation is

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k = x \sin t. \tag{25}$$

Example 4.4.

Let us consider the following Volterra partial integro-differential equation

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^2 u(x,t)}{\partial x^2} &= e^x - \frac{1}{2}te^{2x} + (1-x)te^x - \frac{1}{4}x^2t^2 + \frac{1}{3}xt^3 - \frac{1}{2}t \\ &+ \int_0^t \int_0^x (e^\omega + \tau)u(\omega,\tau)\mathrm{d}\omega\mathrm{d}\tau, \end{aligned} \tag{26}$$

for $(x,t) \in [0,1] \times [0,1]$ and with initial condition $u(x,0) = x + e^x$, $\frac{\partial u(x,0)}{\partial t} = -1$.

Applying RDTM for Equation (26), using Theorem 3.2, we obtain

$$\begin{aligned} &(k+1)(k+2)U_{k+2}(x) + \frac{\partial^2 U_k}{\partial x^2} = e^x\delta(k) - \frac{1}{2}e^{2x}\delta(k-1) + (1-x)e^x\delta(k-1) - \frac{1}{4}x^2\delta(k-2) \\ &+ \frac{1}{3}x\delta(k-3) - \frac{1}{2}\delta(k-1) + \frac{1}{k}\int_0^x \sum_{r=0}^{k-1} \Big(e^\omega\delta(r) + \delta(r-1)\Big)U_{k-r-1}(\omega)\mathrm{d}\omega. \end{aligned} \tag{27}$$

It is clear that

$$U_0(x) = x + e^x, \quad U_1(x) = -1. \tag{28}$$

So, easy calculation on the last integral equation will produce the general formula of

$$U_k(x) = 0, \qquad k \geq 2. \tag{29}$$

Therefore, the exact solution is

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k = x + e^x - t. \tag{30}$$

Example 4.5.

Lastly, we consider the system of Volterra partial integro-differential equations

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - v(x,t) = xe^t - te^{-t} - \frac{1}{15}x^5t^3 + \int_0^t\int_0^x \omega^2u^2(\omega,\tau)v^2(\omega,\tau)\mathrm{d}\omega\mathrm{d}\tau, \\ u(x,t) + \frac{\partial v(x,t)}{\partial t} = xe^t + e^{-t} - te^{-t} - \frac{1}{8}x^2t^4 + \int_0^t\int_0^x \tau^2u(\omega,\tau)v(\omega,\tau)\mathrm{d}\omega\mathrm{d}\tau, \end{cases} \tag{31}$$

for $(x,t) \in [0,1] \times [0,1]$ and with initial condition $u(x,0) = x, \;\; v(x,0) = 0$.

Applying RDTM on both sides of equation (31) and using Theorem 3.2, for the first equation of system we obtain

$$(k+1)U_{k+1}(x) - V_k(x) = \frac{x}{k!} - \sum_{r=0}^{k} \delta(r-1)\frac{(-1)^{k-r}}{(k-r)!} - \frac{1}{15}x^5\delta(k-3)$$
$$+ \frac{1}{k}\int_0^x \sum_{r=0}^{k-1}\sum_{q=0}^{r}\sum_{p=0}^{q}\sum_{l=0}^{p} \omega^2\delta(l)U_{p-l}(\omega)U_{q-p}(\omega)V_{r-q}(\omega)V_{k-r-1}(\omega)\mathrm{d}\omega, \tag{32}$$

and for the second equation

$$U_k(x) + (k+1)V_{k+1}(x) = \frac{x}{k!} + \frac{(-1)^k}{k!} - \sum_{r=0}^{k} \delta(r-1)\frac{(-1)^{k-r}}{(k-r)!} - \frac{1}{8}x^2\delta(k-4)$$
$$+ \frac{1}{k}\int_0^x \sum_{r=0}^{k-1}\sum_{l=0}^{r} \delta(l-2)U_{r-l}(\omega)V_{k-r-1}(\omega)\mathrm{d}\omega. \tag{33}$$

It is easy to see that

$$U_0(x) = x, \quad V_0(x) = 0, \tag{34}$$

also,

$$U_1(x) = x, \quad U_2(x) = \frac{x}{2!}, \quad U_3(x) = \frac{x}{3!}, \quad U_4(x) = \frac{x}{4!}, \quad \ldots, \tag{35}$$
$$V_1(x) = 1, \quad V_2(x) = -1, \quad V_3(x) = \frac{1}{2!}, \quad V_4(x) = -\frac{1}{3!}, \quad \ldots. \tag{36}$$

Therefore, the exact solution is

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k = xe^t, \tag{37}$$
$$v(x,t) = \sum_{k=0}^{\infty} V_k(x)t^k = te^{-t}. \tag{38}$$

It is worth pointing out that, the results in Examples 1-5 are exactly the same as the results of DTM (Moghadam and Saeedi (2010); Tari and Shahmorad (2011)).

Also, results for Examples 1-5 are reported in Figures 1-5 and Tables 1-5, respectively. The terms u_E, u_n and $e(u) = |u_E - u_n|$ stand for exact solution, nth order approximate solution of RDTM, and their absolute error, respectively.

5. Conclusion

In this work, we analyzed the applicability of the reduced differential transform method for solving two-dimensional linear and nonlinear Volterra partial integro-differential equations. The results indicate the efficiency and reliability of the method and furthermore the comparison of the methods with other analytical methods available in the literature shows that although the results of these methods are the same, RDTM is much easier, more convenient and efficient than them and is a powerful technique to handle many linear and nonlinear two-dimensional Volterra integro-differential equations without linearization, discretization and perturbation.

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Table I. Reduced differential transformation

Functional Form	Transformed Form
$u(x,t)$	$U_k(x) = \frac{1}{k!}[\frac{\partial^k}{\partial t^k} u(x,t)]_{t=0}$
$w(x,t) = \alpha u(x,t) \pm \beta v(x,t)$	$W_k(x) = \alpha U_k(x) \pm \beta V_k(x)$ (α and β are constants)
$w(x,t) = x^m t^r$	$W_k(x) = x^m \delta(k-r), \qquad \delta(k) = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}$
$w(x,t) = x^m t^r u(x,t)$	$W_k(x) = x^m U_{k-r}(x)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} V_r(x) U_{k-r}(x) = \sum_{r=0}^{k} U_r(x) V_{k-r}(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r} u(x,t)$	$W_k(x) = (k+1)\ldots(k+r) U_{k+r}(x) = \frac{(k+r)!}{k!} U_{k+r}(x)$
$w(x,t) = \frac{\partial^{m+r}}{\partial x^m \partial t^r} u(x,t)$	$W_k(x) = \frac{(k+r)!}{k!} \frac{\partial^m}{\partial x^m} U_{k+r}(x)$
$w(x,t) = sin(\alpha x + \omega t)$	$W_k(x) = \frac{\omega^k}{k!} sin(\frac{\pi k}{2!} + \alpha x)$
$w(x,t) = cos(\alpha x + \omega t)$	$W_k(x) = \frac{\omega^k}{k!} cos(\frac{\pi k}{2!} + \alpha x)$
$w(x,t) = e^{\alpha t}$	$W_k(x) = \frac{\alpha^k}{k!}$

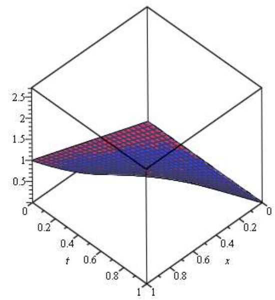


Figure 1. Comparison of the exact solution (blue) and the approximate solutions (red) of Example 1

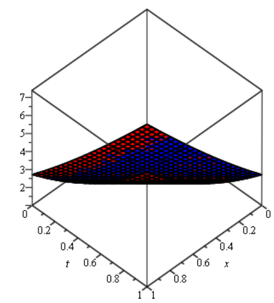


Figure 2. Comparison of the exact solution (blue) and the approximate solutions (red) of Example 2

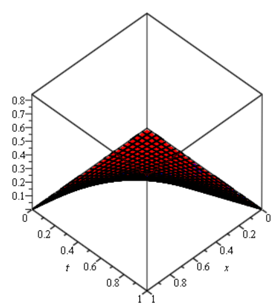


Figure 3. Comparison of the exact solution (blue) and the approximate solutions (red) of Example 3

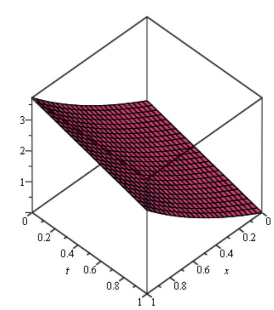
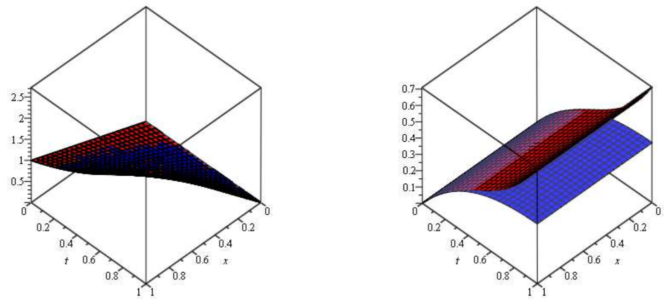


Figure 4. Comparison of the exact solution (blue) and the approximate solutions (red) of Example 4



(a) The solution of $u_n(x,t)$ (b) The solution of $v_n(x,t)$

Figure 5. Comparison of the exact solution (blue) and the approximate solutions (red) of Example 5

Table 1. The absolute error, between the exact solution and the numerical solution for example 1.

(x,t)	$e(u), n=5$	$e(u), n=7$	$e(u), n=10$
(0.5,0.1)	6.6667033×10^{-10}	$3.769474932\times10^{-11}$	$3.782014891\times10^{-11}$
(0.5,0.2)	4.5666737×10^{-8}	4.7548710×10^{-11}	$8.001431846\times10^{-11}$
(0.5,0.3)	5.29×10^{-7}	1.05357143×10^{-9}	$2.120212068\times10^{-10}$
(0.5,0.4)	3.01566713×10^{-6}	8.6830035×10^{-9}	$1.803721077\times10^{-10}$
(0.5,0.5)	1.16771637×10^{-5}	5.14196568×10^{-8}	$1.533549222\times10^{-10}$

Table 2. The absolute error, between the exact solution and the numerical solution for example 2.

(x,t)	$e(u), n=5$	$e(u), n=7$	$e(u), n=10$
(0.5,0.1)	1.6009978×10^{-9}	$7.216055776\times10^{-10}$	$7.220190754\times10^{-10}$
(0.5,0.2)	1.50009546×10^{-7}	7.30684525×10^{-10}	$8.377380035\times10^{-10}$
(0.5,0.3)	1.74274327×10^{-6}	1.87025642×10^{-9}	$9.047070874\times10^{-10}$
(0.5,0.4)	9.9427653×10^{-6}	2.74079170×10^{-8}	$6.290214768\times10^{-10}$
(0.5,0.5)	3.85033055×10^{-5}	1.68082496×10^{-7}	$9.651888236\times10^{-10}$

Table 3. The absolute error, between the exact solution and the numerical solution for example 3.

(x,t)	$e(u), n=5$	$e(u), n=7$	$e(u), n=9$
(0.5,0.1)	1.332667×10^{-11}	3.40603508×10^{-12}	$3.407412946\times10^{-12}$
(0.5,0.2)	1.266633×10^{-9}	3.20827×10^{-12}	$2.502802628\times10^{-12}$
(0.5,0.3)	2.160×10^{-8}	9.642857×10^{-11}	$6.930803429\times10^{-11}$
(0.5,0.4)	1.6213267×10^{-7}	4.070125×10^{-10}	4.58132055×10^{-11}
(0.5,0.5)	7.723633×10^{-7}	2.6863031×10^{-9}	4.841355×10^{-12}

Table 4. The absolute error, between the exact solution and the numerical solution for example 4.

(x,t)	$e(u)$
(0.5,0.1)	0
(0.5,0.2)	0
(0.5,0.3)	0
(0.5,0.4)	0
(0.5,0.5)	0

Table 5. The absolute error, between the exact solution and the numerical solution for example 5.

(x, t)	$e(u), n = 5$	$e(u), n = 7$	$e(u), n = 10$
(0.5,0.1)	6.6667033×10^{-10}	$3.769474932\times10^{-11}$	$3.782014891\times10^{-11}$
(0.5,0.2)	4.5666737×10^{-8}	4.7548710×10^{-11}	$8.001431846\times10^{-11}$
(0.5,0.3)	5.29×10^{-7}	1.05357143×10^{-9}	$2.120212068\times10^{-10}$
(0.5,0.4)	3.01566713×10^{-6}	8.6830035×10^{-9}	$1.803721077\times10^{-10}$
(0.5,0.5)	1.16771637×10^{-5}	5.14196568×10^{-8}	$1.533549222\times10^{-10}$
(x, t)	$e(v), n = 5$	$e(v), n = 7$	$e(v), n = 10$
(0.5,0.1)	8.1999967×10^{-9}	5.5522559×10^{-12}	3.59265493×10^{-12}
(0.5,0.2)	5.1606663×10^{-7}	5.1107448×10^{-10}	$1.555419785\times10^{-11}$
(0.5,0.3)	5.7838×10^{-6}	1.255×10^{-8}	4.04018137×10^{-12}
(0.5,0.4)	3.19815987×10^{-5}	1.23820926×10^{-7}	1.8078654×10^{-12}
(0.5,0.5)	1.20086863×10^{-4}	7.2922415×10^{-7}	7.5997355×10^{-11}