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Ideal Theory in BCK/BCI-algebras in the Frame Of Hesitant Fuzzy Set Theory

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Abstract

Several generalizations and extensions of fuzzy sets have been introduced in the literature, for example, Atanassov's intuitionistic fuzzy sets, type 2 fuzzy sets and fuzzy multisets, etc. Using the Torra's hesitant fuzzy sets, the notions of Sup-hesitant fuzzy ideals in BCK/BCI-algebras are introduced, and its properties are investigated. Relations between Sup-hesitant fuzzy subalgebras and Sup-hesitant fuzzy ideals are displayed, and characterizations of Sup-hesitant fuzzy ideals are discussed.

Keywords: Sup-hesitant fuzzy subalgebra; Sup-hesitant fuzzy ideal

MSC 2010 No.: 06F35, 03G25, 08A72

1. Introduction

Uncertainty can be considered of different types such as randomness (Nillsson (1986)), fuzziness (Zadeh (1965)), indistinguishability (Pawlak (1982)), and incompleteness (McDermott and Doyle (1980)). Fuzzy set theory has been widely and successfully applied in many different areas to handle uncertainties. But it presents limitations to deal with imprecise and vague information when different sources of vagueness appear simultaneously. In order to overcome such limitations, different extensions of fuzzy sets have been introduced in the literature such as intuitionistic fuzzy sets (Atanassov (1986)), type-2 fuzzy sets (Dubois and Prade (2000)), interval-valued fuzzy sets (Bustince et al. (2013); Türksen (2013)), fuzzy multisets (Yager (1986)), and so forth.

Despite the previous extensions overcome in different ways, Torra (2010) introduced a new extension of fuzzy sets, so-called hesitant fuzzy sets, which has an important role for solving uncertainties. Torra and Narukawa (2009) and Torra (2010) discussed the relationship between hesitant fuzzy sets and intuitionistic fuzzy sets. Hesitant fuzzy sets are applied to algebraic structures (Aldhafeeri and Muhiuddin (2019); Jun (2018); Jun and Ahn (2016); Jun et al. (2017); Jun and Song (2014); Muhiuddin (2016); Muhiuddin and Aldhafeeri (2018); Jun et al. (2016); Jun et al. (2014)), decision making problems (Chen et al. (2016); Faizi et al. (2017); Krishankumar et al. (2019); Liao et al. (2014); Liu and Rodriguez (2014); Rashid et al. (2018); Rodriguez et al. (2012); Wang et al. (2014); Wei (2012); Xia and Xu (2011)), and distance and similarity measures (Liao et al. (2014); Xu and Xia (2011a); Xu and Xia (2011b)).

Using the hesitant fuzzy set theory which is introduced by Torra (2010), Muhiuddin and Jun (2019) introduced the notion of sup-hesitant fuzzy subalgebras in BCK/BCI-algebras and investigated several related properties. Also, Muhiuddin et al. studied the notion of hesitant fuzzy sets on the various aspects (for example, see Muhiuddin et al. (2016), Muhiuddin et al. (2017), Muhiuddin and Al-roqi (2018)). The ideal theory of BCK/BCI-algebras has been studied on different field (e.g., Jun et al. (2017), Muhiuddin et al. (2019), Muhiuddin and Aldhafeeri (2019), Muhiuddin et al. (2017), Tapan et al. (2019)).

Motivated by a lot of work on ideal theory in BCK/BCI-algebras, in this paper we introduce sup-hesitant fuzzy ideals in BCK/BCI-algebras and investigate related properties. We discuss relations between sup-hesitant fuzzy subalgebras and sup-hesitant fuzzy ideals. We consider characterizations of Sup-hesitant fuzzy ideals.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall a, b, c \in X)\ (((a * b) * (a * c)) * (c * b) = 0)$,

(II) $(\forall a, b \in X)\ ((a * (a * b)) * b = 0)$,
(III) $(\forall a \in X)\ (a * a = 0)$,
(IV) $(\forall a, b \in X)\ (a * b = 0, b * a = 0 \ \Rightarrow\ a = b)$.

If a BCI-algebra X satisfies the following identity,

(V) $(\forall a \in X)\ (0 * a = 0)$,

then X is called a *BCK-algebra*. A BCK-algebra X is said to be

- *positive implicative* if it satisfies

$$(\forall a, b, c \in X)\left((a * b) * c = (a * c) * (b * c)\right), \tag{1}$$

- *implicative* if it satisfies

$$(\forall a, b \in X)\left(a = a * (b * a)\right). \tag{2}$$

Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall a \in X)\left(a * 0 = a\right), \tag{3}$$
$$(\forall a, b, c \in X)\left(a \leq b \ \Rightarrow\ a * c \leq b * c,\ c * b \leq c * a\right), \tag{4}$$
$$(\forall a, b, c \in X)\left((a * b) * c = (a * c) * b\right), \tag{5}$$
$$(\forall a, b, c \in X)\left((a * c) * (b * c) \leq a * b\right), \tag{6}$$

where $a \leq b$ if and only if $a * b = 0$.

Any BCI-algebra X satisfies the following conditions:

$$(\forall a, b, c \in X)\left(0 * (0 * ((a * c) * (b * c))) = (0 * b) * (0 * a)\right), \tag{7}$$
$$(\forall a, b \in X)\left(0 * (0 * (a * b)) = (0 * b) * (0 * a)\right), \tag{8}$$
$$(\forall a \in X)\left(0 * (0 * (0 * a)) = 0 * a\right). \tag{9}$$

A subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $a * b \in S$ for all $a, b \in S$. A subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in A, \tag{10}$$
$$(\forall a \in X)\left(a * b \in A,\ b \in A \ \Rightarrow\ a \in A\right). \tag{11}$$

We refer the reader to the books by Huang (2006) and Meng and Jun (1994) for further information regarding BCK/BCI-algebras.

Torra (2010) introduced a new extension for fuzzy sets to manage those situations in which several values are possible for the definition of a membership function of a fuzzy set.

Let X be a reference set. Then we define the hesitant fuzzy set on X in terms of a function $\mathcal{H}$ that when applied to X returns a subset of $[0;1]$ (see Torra and Narukawa (2009) and Torra (2010)).

In what follows, the power set of $[0,1]$ is denoted by $\mathcal{P}([0,1])$ and

$$\mathcal{P}^*([0,1]) = \mathcal{P}([0,1]) \setminus \{\emptyset\}.$$

For any element $Q \in \mathcal{P}^*([0,1])$, the supremum of Q is denoted by $\sup Q$. For any hesitant fuzzy set $\mathcal{H}$ on X and $Q \in \mathcal{P}^*([0,1])$, consider the set

$$\mathrm{Sup}[\mathcal{H};Q] := \{a \in X \mid \sup \mathcal{H}(a) \geq \sup Q\}.$$

Definition 2.1. (Muhiuddin and Jun (2019))

Let X be a BCK/BCI-algebra. Given an element $Q \in \mathcal{P}^*([0,1])$, a hesitant fuzzy set $\mathcal{H}$ on X is called a *Sup-hesitant fuzzy subalgebra* of X related to Q (briefly, *Q-Sup-hesitant fuzzy subalgebra* of X) if the set $\mathrm{Sup}[\mathcal{H};Q]$ is a subalgebra of X. If $\mathcal{H}$ is a Q-Sup-hesitant fuzzy subalgebra of X for all $Q \in \mathcal{P}^*([0,1])$, then we say that $\mathcal{H}$ is a *Sup-hesitant fuzzy subalgebra* of X.

Lemma 2.2. (Muhiuddin and Jun (2019))

Every Sup-hesitant fuzzy subalgebra $\mathcal{H}$ of a BCK/BCI-algebra X satisfies:

$$(\forall a \in X)\,(\sup \mathcal{H}(0) \geq \sup \mathcal{H}(a)). \tag{12}$$

3. Sup-hesitant fuzzy ideals

Definition 3.1.

Let X be a BCK/BCI-algebra. Given an element $Q \in \mathcal{P}^*([0,1])$, a hesitant fuzzy set $\mathcal{H}$ on X is called a *Sup-hesitant fuzzy ideal* of X related to Q (briefly, *Q-Sup-hesitant fuzzy ideal* of X) if the set $\mathrm{Sup}[\mathcal{H};Q]$ is an ideal of X. If $\mathcal{H}$ is a Q-Sup-hesitant fuzzy ideal of X for all $Q \in \mathcal{P}^*([0,1])$, then we say that $\mathcal{H}$ is a *Sup-hesitant fuzzy ideal* of X.

Example 3.2.

(1) Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Let $\mathcal{H}$ be a hesitant fuzzy set on X defined by Table 1.

It is routine to verify that $\mathcal{H}$ is a Sup-hesitant fuzzy ideal of X.

(2) Let $(Y, *, 0)$ be a BCI-algebra and $(\mathbb{Z}, +, 0)$ an additive group of integers. Let $(\mathbb{Z}, -, 0)$ be the adjoint BCI-algebra of $(\mathbb{Z}, +, 0)$ and let $X := Y \times \mathbb{Z}$. Then $(a, \otimes, (0,0))$ is a BCI-algebra where

Table 1. Tabular representation of $\mathcal{H}$

X	0	a	b	c
$\mathcal{H}(x)$	$(0.8, 1]$	$(0.3, 0.4) \cup \{0.8\}$	$[0.5, 0.6]$	$(0.3, 0.5) \cup \{0.6\}$

the operation $\otimes$ is given by

$$(\forall (a, m), (b, n) \in X)\, ((a, m) \otimes (b, n) = (a * b, m - n))\,.$$

For a subset $A := Y \times \mathbb{N}_0$ of X where $\mathbb{N}_0$ is the set of nonnegative integers, let $\mathcal{H}$ on X be a hesitant fuzzy set on X defined by

$$\mathcal{H} : X \to \mathcal{P}([0, 1]),\ x \mapsto \begin{cases} (0.3, 0.9], & \text{if } x \in A, \\ [0.4, 0.7], & \text{otherwise.} \end{cases}$$

Then $\mathcal{H}$ is a Sup-hesitant fuzzy ideal of X.

(3) Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	b	a	0	0
d	d	d	d	d	0

Let $\mathcal{H}$ be a hesitant fuzzy set on X defined by Table 2.

Table 2. Tabular representation of $\mathcal{H}$

X	0	a	b	c	d
$\mathcal{H}(x)$	$[0.8, 0.9]$	$(0.2, 0.4)$	$(0.1, 0.3]$	$[0.4, 0.6]$	$[0, 0.1]$

If $Q_1 := [0.3, 0.5)$, then $\mathrm{Sup}[\mathcal{H}; Q_1] = \{0, c\}$ which is not an ideal of X since $b * c = 0 \in \mathrm{Sup}[\mathcal{H}; Q_1]$ but $b \notin \mathrm{Sup}[\mathcal{H}; Q_1]$. Thus $\mathcal{H}$ is not a Q_1-Sup-hesitant fuzzy ideal of X. We can easily verify that $\mathcal{H}$ on X is a Q_2-Sup-hesitant fuzzy ideal of X with $Q_2 = [0, 0.25]$.

Theorem 3.3.

A hesitant fuzzy set $\mathcal{H}$ on a BCK/BCI-algebra X is a Sup-hesitant fuzzy ideal of X if and only if it satisfies (12) and

$$(\forall a, b \in X)\, (\sup \mathcal{H}(a) \geq \min\{\sup \mathcal{H}(a * b), \sup \mathcal{H}(b)\})\,. \tag{13}$$

Proof:

Let $\mathcal{H}$ be a Sup-hesitant fuzzy ideal of X. If (12) is false, then there exists $Q \in \mathcal{P}^*([0, 1])$ and $a \in X$ such that $\sup \mathcal{H}(0) < \sup Q \leq \sup \mathcal{H}(a)$. It follows that $a \in \mathrm{Sup}[\mathcal{H}; Q]$ and $0 \notin \mathrm{Sup}[\mathcal{H}; Q]$.

This is a contradiction, and so (12) is valid. Now assume that (13) is false. Then there exist $a, b \in X$ and $K \in \mathcal{P}^*([0,1])$ such that

$$\sup \mathcal{H}(a) < \sup K \leq \min\{\sup \mathcal{H}(a * b), \sup \mathcal{H}(b)\},$$

which implies that $a * b \in \mathrm{Sup}[\mathcal{H}; K]$, $b \in \mathrm{Sup}[\mathcal{H}; K]$ but $a \notin \mathrm{Sup}[\mathcal{H}; K]$. This is a contradiction, and thus (13) holds.

Conversely, suppose that $\mathcal{H}$ satisfies two conditions (12) and (13). Let $K \in \mathcal{P}^*([0,1])$ be such that $\mathrm{Sup}[\mathcal{H}; K] \neq \emptyset$. Obviously, $0 \in \mathrm{Sup}[\mathcal{H}; K]$. Let $a, b \in X$ be such that $a * b \in \mathrm{Sup}[\mathcal{H}; K]$ and $b \in \mathrm{Sup}[\mathcal{H}; K]$. Then $\sup \mathcal{H}(a * b) \geq \sup K$ and $\sup \mathcal{H}(b) \geq \sup K$. It follows from (13) that

$$\sup \mathcal{H}(a) \geq \min\{\sup \mathcal{H}(a * b), \sup \mathcal{H}(b)\} \geq \sup K$$

and that $a \in \mathrm{Sup}[\mathcal{H}; K]$. Hence $\mathrm{Sup}[\mathcal{H}; K]$ is an ideal of X for all $K \in \mathcal{P}^*([0,1])$, and therefore $\mathcal{H}$ is an Sup-hesitant fuzzy ideal of X. ■

Theorem 3.4.

Let $\mathcal{H}$ be a hesitant fuzzy set on X defined by

$$\mathcal{H} : X \to \mathcal{P}([0,1]),\ x \mapsto \begin{cases} Q, & \text{if } x \in B, \\ D, & \text{if } x \in X \setminus B, \end{cases}$$

where B is the BCK-part of X and $Q, D \in \mathcal{P}^*([0,1])$ with $\sup Q \geq \sup D$. Then, $\mathcal{H}$ is a Sup-hesitant fuzzy ideal of X.

Proof:

Since $0 \in B$, we have $\sup \mathcal{H}(0) = \sup Q \geq \sup \mathcal{H}(a)$ for all $a \in X$. Let $a, b \in X$. If $a \in B$, then it is clear that

$$\sup \mathcal{H}(a) \geq \min\{\sup \mathcal{H}(a * b), \sup \mathcal{H}(b)\}.$$

Assume that $a \in X \setminus B$. Since B is an ideal of X, it follows that $a * b \in X \setminus B$ or $b \in X \setminus B$. Hence,

$$\sup \mathcal{H}(a) = \min\{\sup \mathcal{H}(a * b), \sup \mathcal{H}(b)\}.$$

Therefore $\mathcal{H}$ is a Sup-hesitant fuzzy ideal of X by Theorem 3.3. ■

Proposition 3.5.

Every Sup-hesitant fuzzy ideal $\mathcal{H}$ of a BCK/BCI-algebra X satisfies

$$(\forall a, b \in X)\,(a \leq b \ \Rightarrow\ \sup \mathcal{H}(a) \geq \sup \mathcal{H}(b))\,. \tag{14}$$

Proof:

Let $a, b \in X$ be such that $a \leq b$. Then $a * b = 0$, and so

$$\sup \mathcal{H}(a) \geq \min\{\sup \mathcal{H}(a * b), \sup \mathcal{H}(b)\} = \min\{\sup \mathcal{H}(0), \sup \mathcal{H}(b)\} = \sup \mathcal{H}(b),$$

by (13) and (12). ■

Theorem 3.6.

Let $\mathcal{H}$ be a hesitant fuzzy set on a BCK/BCI-algebra X which satisfies the condition (12). Then $\mathcal{H}$ is a Sup-hesitant fuzzy ideal of X if and only if the following assertion is valid,

$$(\forall a, b, c \in X)\,(a * b \leq c \;\Rightarrow\; \sup \mathcal{H}(a) \geq \min\{\sup \mathcal{H}(b), \sup \mathcal{H}(c)\})\,. \tag{15}$$

Proof:

Assume that $\mathcal{H}$ is a Sup-hesitant fuzzy ideal of X and let $a, b, c \in X$ be such that $a * b \leq c$. Then, $(a * b) * c = 0$, and thus,

$$\begin{aligned}\sup \mathcal{H}(a * b) &\geq \min\{\sup \mathcal{H}((a * b) * c), \sup \mathcal{H}(c)\}\\ &= \min\{\sup \mathcal{H}(0), \sup \mathcal{H}(c)\}\\ &= \sup \mathcal{H}(c).\end{aligned}$$

It follows that $\sup \mathcal{H}(a) \geq \min\{\sup \mathcal{H}(a * b), \sup \mathcal{H}(b)\} \geq \min\{\sup \mathcal{H}(b), \sup \mathcal{H}(c)\}$.

Conversely, suppose that the condition (15) is valid. Since $a * (a * b) \leq b$ for all $a, b \in X$, we have $\sup \mathcal{H}(a) \geq \min\{\sup \mathcal{H}(a * b), \sup \mathcal{H}(b)\}$ for all $a, b \in X$. Therefore, $\mathcal{H}$ is a Sup-hesitant fuzzy ideal of X. ■

Proposition 3.7.

For any Sup-hesitant fuzzy ideal $\mathcal{H}$ of a BCK/BCI-algebra X, the following assertions are equivalent:

(1) $(\forall a, b \in X)\ (\sup \mathcal{H}((a * b) * b) \leq \sup \mathcal{H}(a * b))$,
(2) $(\forall a, b, c \in X)\ (\sup \mathcal{H}((a * b) * c) \leq \sup \mathcal{H}((a * c) * (b * c)))$.

Proof:

Assume that (1) holds. Note that

$$((a * (b * c)) * c) * c = ((a * c) * (b * c)) * c \leq (a * b) * c,$$

for all $a, b, c \in X$. It follows from (14), (1) and (5) that

$$\begin{aligned}\sup \mathcal{H}((a * b) * c) &\leq \sup \mathcal{H}(((a * (b * c)) * c) * c)\\ &\leq \sup \mathcal{H}((a * (b * c)) * c)\\ &= \sup \mathcal{H}((a * c) * (b * c)),\end{aligned} \tag{16}$$

for all $a, b, c \in X$.

Conversely, suppose that (2) is valid and if we put $c := b$ in (2), then

$$\begin{aligned}\sup \mathcal{H}((a * b) * b) &\leq \sup \mathcal{H}((a * b) * (b * b))\\ &= \sup \mathcal{H}((a * b) * 0)\\ &= \sup \mathcal{H}(a * b)\end{aligned} \tag{17}$$

for all $a, b \in X$. ■

We consider a relation between a Sup-hesitant fuzzy ideal and a Sup-hesitant fuzzy subalgebra in BCK-algebras.

Theorem 3.8.

In a BCK-algebra X, every Sup-hesitant fuzzy ideal is a Sup-hesitant fuzzy subalgebra.

Proof:

Let $\mathcal{H}$ be a Sup-hesitant fuzzy ideal of a BCK-algebra X. Using (13), (5), (III), (V) and (12), we have

$$\begin{aligned}\sup \mathcal{H}(a*b) &\geq \min\{\sup \mathcal{H}((a*b)*a), \sup \mathcal{H}(a)\}\\ &= \min\{\sup \mathcal{H}((a*a)*b), \sup \mathcal{H}(a)\}\\ &= \min\{\sup \mathcal{H}(0*b), \sup \mathcal{H}(a)\}\\ &= \min\{\sup \mathcal{H}(0), \sup \mathcal{H}(a)\}\\ &\geq \min\{\sup \mathcal{H}(a), \sup \mathcal{H}(b)\},\end{aligned}$$

for all $a, b \in X$. Therefore, $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X. ■

The converse of Theorem 3.8 is not true in general as seen in the following example.

Example 3.9.

Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	0
c	c	c	c	0	0
d	d	c	c	a	0

Let $\mathcal{H}$ be a hesitant fuzzy set on X defined by Table 3.

Table 3. Tabular representation of $\mathcal{H}$

X	0	a	b	c	d
$\mathcal{H}(x)$	$\{0.8, 0.9\}$	$[0.2, 0.3)$	$(0.7, 0.8]$	$\{0.4\} \cup (0.5, 0.6)$	$[0.1, 0.2]$

Then $\mathcal{H}$ is a Sup-hesitant fuzzy subalgebra of X (see Muhiuddin and Jun (2019)). But, it is not a Sup-hesitant fuzzy ideal of X since

$$\sup \mathcal{H}(d) = 0.1 < 0.5 = \min\{\sup \mathcal{H}(d*b), \sup \mathcal{H}(b)\}.$$

In a BCI-algebra X, Theorem 3.8 is not true. In fact, the Sup-hesitant fuzzy ideal $\mathcal{H}$ of X in

Example 3.2(2) is not a Sup-hesitant fuzzy subalgebra of X since

$$\begin{aligned}\sup \mathcal{H}((0,0) \otimes (0,1)) &= \sup \mathcal{H}(0,-1) = 0.4 \\ &< 0.5 = \min\{\sup \mathcal{H}(0,0), \sup \mathcal{H}(0,1)\}.\end{aligned}$$

Let $\mathcal{H}$ be a hesitant fuzzy set on a BCK-algebra X. For any $a, b \in X$ and $n \in \mathbb{N}$, let

$$\mathrm{Sup}[b; a^n] := \{x \in X \mid \sup \mathcal{H}((x * b) * a^n) = \sup \mathcal{H}(0)\}$$

where $(x * b) * a^n = ((\cdots((x * b) * a) * a) * \cdots) * a$ in which a appears n-times. Obviously, $a, b, 0 \in \mathrm{Sup}[b; a^n]$.

Proposition 3.10.

Let $\mathcal{H}$ be a hesitant fuzzy set on a BCK-algebra X in which the condition (12) is valid and

$$(\forall a, b \in X)\,(\sup \mathcal{H}(a * b) = \max\{\sup \mathcal{H}(a), \sup \mathcal{H}(b)\})\,. \tag{18}$$

For any $a, b \in X$ and $n \in \mathbb{N}$, if $x \in \mathrm{Sup}[b; a^n]$ then $x * y \in \mathrm{Sup}[b; a^n]$ for all $y \in X$.

Proof:

Let $x \in \sup \mathcal{H}[b; a^n]$. Then $\sup \mathcal{H}((x * b) * a^n) = \sup \mathcal{H}(0)$, and thus

$$\begin{aligned}\sup \mathcal{H}(((x * y) * b) * a^n) &= \sup \mathcal{H}(((x * b) * y) * a^n) \\ &= \sup \mathcal{H}(((x * b) * a^n) * y) \\ &= \max\{\sup \mathcal{H}((x * b) * a^n), \sup \mathcal{H}(y)\} \\ &= \max\{\sup \mathcal{H}(0), \sup \mathcal{H}(y)\} = \sup \mathcal{H}(0),\end{aligned}$$

for all $y \in X$. Hence, $x * y \in \sup \mathcal{H}[b; a^n]$ for all $y \in X$. ∎

Proposition 3.11.

Let $\mathcal{H}$ be a hesitant fuzzy set on a BCK-algebra X. If an element $a \in X$ satisfies

$$(\forall x \in X)\,(x \le a)\,, \tag{19}$$

then $\mathrm{Sup}[b; a^n] = X = \mathrm{Sup}[a; b^n]$ for all $b \in X$ and $n \in \mathbb{N}$.

Proof:

Let $b, x \in X$ and $n \in \mathbb{N}$. Then,

$$\begin{aligned}\sup \mathcal{H}((x * b) * a^n) &= \sup \mathcal{H}(((x * b) * a) * a^{n-1}) \\ &= \sup \mathcal{H}(((x * a) * b) * a^{n-1}) \\ &= \sup \mathcal{H}((0 * b) * a^{n-1}) \\ &= \sup \mathcal{H}(0),\end{aligned}$$

by (5), (19) and (V), and so $x \in \mathrm{Inf}[b; a^n]$, which shows that $\mathrm{Sup}[b; a^n] = X$. Similarly $\mathrm{Sup}[a; b^n] = X$. ∎

Corollary 3.12.

If $\mathcal{H}$ is a hesitant fuzzy set on a bounded BCK-algebra X, then $\mathrm{Sup}[b; u^n] = X = \mathrm{Sup}[u; b^n]$ for all $b \in X$ and $n \in \mathbb{N}$ where u is the unit of X.

Proposition 3.13.

Let $\mathcal{H}$ be a Sup-hesitant fuzzy subalgebra of a BCK-algebra X satisfying the condition (14). Then the following assertion is valid:

$$(\forall a, b, c \in X)\,(\forall n \in \mathbb{N})\,(b \leq c \;\Rightarrow\; \mathrm{Sup}[b; a^n] \subseteq \mathrm{Sup}[c; a^n])\,. \tag{20}$$

Proof:

Let $b, c \in X$ be such that $b \leq c$. For any $a \in X$ and $n \in \mathbb{N}$, if $x \in \mathrm{Sup}[b; a^n]$, then,

$$\begin{aligned} \sup \mathcal{H}(0) &= \sup \mathcal{H}((x * b) * a^n) = \sup \mathcal{H}((x * a^n) * b) \\ &\leq \sup \mathcal{H}((x * a^n) * c) = \sup \mathcal{H}((x * c) * a^n), \end{aligned}$$

by (4), (5) and (14), and so $\sup \mathcal{H}((x * c) * a^n) = \sup \mathcal{H}(0)$. Thus $x \in \mathrm{Sup}[c; a^n]$, and therefore, $\mathrm{Sup}[b; a^n] \subseteq \mathrm{Sup}[c; a^n]$ for all $a \in X$ and $n \in \mathbb{N}$. ■

Corollary 3.14.

Every Sup-hesitant fuzzy ideal $\mathcal{H}$ of a BCK-algebra X satisfies the condition (20).

The following example shows that there exists a Sup-hesitant fuzzy ideal $\mathcal{H}$ of a BCK-algebra X such that the set $\mathrm{Sup}[b; a^n]$ is not an ideal of X for some $a, b \in X$ and $n \in \mathbb{N}$.

Example 3.15.

Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $\mathcal{H}$ be a hesitant fuzzy set on X defined by Table 4.

Table 4. Tabular representation of $\mathcal{H}$

X	0	a	b	c
$\mathcal{H}(x)$	$[0.5, 0.8]$	$(0.3, 0.6)$	$[0.5, 0.6]$	$[0.1, 0.2) \cup \{0.3\}$

Then $\mathcal{H}$ is a Sup-hesitant fuzzy ideal of X and

$$\mathrm{Sup}[a; c^n] = \{a \in X \mid \sup \mathcal{H}((x * a) * c^n) = \sup \mathcal{H}(0)\} = \{0, a, c\},$$

which is not an ideal of X for any $n \in \mathbb{N}$ since $b * a = a \in \mathrm{Sup}[a; c^n]$ but $b \notin \mathrm{Sup}[a; c^n]$.

We now consider conditions for a set $\mathrm{Sup}[b; a^n]$ to be an ideal of X.

Theorem 3.16.

Let $\mathcal{H}$ be a hesitant fuzzy set on a BCK-algebra X such that

$$(\forall x, y \in X)\,(\sup \mathcal{H}(x) = \sup \mathcal{H}(y) \;\Rightarrow\; x = y)\,. \tag{21}$$

If X is positive implicative, then $\mathrm{Sup}[b; a^n]$ is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$.

Proof:

Let $a, b, x, y \in X$ and $n \in \mathbb{N}$ be such that $x * y \in \mathrm{Inf}[b; a^n]$ and $y \in \mathrm{Sup}[b; a^n]$. Then $\sup \mathcal{H}((y * b) * a^n) = \sup \mathcal{H}(0)$, which implies from (21) that $(y * b) * a^n = 0$. Hence

$$\begin{aligned}
\sup \mathcal{H}(0) &= \sup \mathcal{H}(((x * y) * b) * a^n) \\
&= \sup \mathcal{H}((((x * y) * b) * a) * a^{n-1}) \\
&= \sup \mathcal{H}((((x * b) * (y * b)) * a) * a^{n-1}) \\
&= \sup \mathcal{H}(((((x * b) * a) * ((y * b) * a)) * a) * a^{n-2}) \\
&= \cdots \\
&= \sup \mathcal{H}(((x * b) * a^n) * ((y * b) * a^n)) \\
&= \sup \mathcal{H}(((x * b) * a^n) * 0) \\
&= \sup \mathcal{H}((x * b) * a^n),
\end{aligned}$$

which shows that $x \in \mathrm{Sup}[b; a^n]$. Therefore, $\mathrm{Inf}[b; a^n]$ is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$. ■

Corollary 3.17.

Let $\mathcal{H}$ be a hesitant fuzzy set on a BCK-algebra X satisfying (21). If X is implicative, then $\mathrm{Sup}[b; a^n]$ is an ideal of X for all $a, b \in X$ and $n \in \mathbb{N}$.

Proof:

It is straightforward since every implicative BCK-algebra is a positive implicative BCK-algebra. ■

Theorem 3.16 is illustrated by the following example.

Example 3.18.

Let $X = \{0, a, b, c\}$ be a set with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Then X is a positive implicative BCK-algebra. Let $\mathcal{H}$ be a hesitant fuzzy set on X defined by Table 5.

Table 5. Tabular representation of $\mathcal{H}$

X	0	a	b	c
$\mathcal{H}(x)$	$[0.5, 0.6]$	$(0.3, 0.7)$	$[0.2, 0.4]$	$\{0.1, 0.2\}$

Then $\mathcal{H}$ satisfies the condition (21), but it does not satisfy the condition (12). Hence $\mathcal{H}$ is not a Sup-hesitant fuzzy ideal of X. Note that

$\mathrm{Sup}[0 : 0^n] = \{0\}$, $\mathrm{Sup}[0; a^n] = \{0, a\}$, $\mathrm{Sup}[0; b^n] = \{0, a, b\}$, $\mathrm{Sup}[0; c^n] = \{0, c\}$,

$\mathrm{Sup}[a; 0^n] = \{0, a\}$, $\mathrm{Sup}[a; a^n] = \{0, a\}$, $\mathrm{Sup}[a; b^n] = \{0, a, b\}$, $\mathrm{Sup}[a; c^n] = \{0, a, c\}$,

$\mathrm{Sup}[b; 0^n] = \{0, a, b\}$, $\mathrm{Sup}[b; a^n] = \{0, a, b\}$, $\mathrm{Sup}[b; b^n] = \{0, a, b\}$, $\mathrm{Sup}[b; c^n] = X$,

$\mathrm{Sup}[c; 0^n] = \{0, c\}$, $\mathrm{Sup}[c; a^n] = \{0, a, c\}$, $\mathrm{Sup}[c; b^n] = X$, $\mathrm{Sup}[c; c^n] = \{0, c\}$,

and they are ideals of X.

Proposition 3.19.

Let $\mathcal{H}$ be a hesitant fuzzy set on a BCK-algebra X in which the condition (21) is valid. If J is an ideal of X, then the following assertion holds.

$$(\forall a, b \in J)\,(\forall n \in \mathbb{N})\,(\mathrm{Sup}[b; a^n] \subseteq J)\,. \tag{22}$$

Proof:

For any $a, b \in J$ and $n \in \mathbb{N}$, let $x \in \mathrm{Sup}[b; a^n]$. Then,

$$\sup \mathcal{H}(((x * b) * a^{n-1}) * a) = \sup \mathcal{H}((x * b) * a^n) = \sup \mathcal{H}(0),$$

and so $((x * b) * a^{n-1}) * a = 0 \in J$ by (21). Since J is an ideal of X, it follows from (11) that $(a * b) * a^{n-1} \in J$. Continuing this process, we have $x * b \in J$ and thus $x \in J$. Therefore, $\mathrm{Sup}[b; a^n] \subseteq J$ for all $a, b \in J$ and $n \in \mathbb{N}$. ∎

Theorem 3.20.

Let $\mathcal{H}$ be a hesitant fuzzy set on a BCK-algebra X. For any subset J of X, if the condition (22) holds, then J is an ideal of X.

Proof:

Suppose that the condition (22) is valid. Note that $0 \in \mathrm{Sup}[b; a^n] \subseteq J$. Let $x, y \in X$ be such that $x * y \in J$ and $y \in J$. Taking $b := x * y$ implies that

$$\begin{aligned}\sup \mathcal{H}((x * b) * y^n) &= \sup \mathcal{H}((x * (x * y)) * y^n)\\ &= \sup \mathcal{H}(((x * (x * y)) * y) * y^{n-1})\\ &= \sup \mathcal{H}(((x * y) * (x * y)) * y^{n-1})\\ &= \sup \mathcal{H}(0 * y^{n-1}) = \sup \mathcal{H}(0),\end{aligned}$$

and so $x \in \mathrm{Sup}[b; y^n] \subseteq J$ with $b = x * y$. Therefore, J is an ideal of X. ∎

Theorem 3.21.

If $\mathcal{H}$ is a Sup-hesitant fuzzy ideal of a BCK/BCI-algebra X, then the set

$$\mathcal{H}_a := \{x \in X \mid \sup \mathcal{H}(a) \leq \sup \mathcal{H}(x)\},$$

is an ideal of X for all $a \in X$.

Proof:

Let $x, y \in X$ be such that $x * y \in \mathcal{H}_a$ and $y \in \mathcal{H}_a$. Then, $\sup \mathcal{H}(a) \leq \sup \mathcal{H}(x * y)$ and $\sup \mathcal{H}(a) \leq \sup \mathcal{H}(y)$. It follows from (12) and (13) that

$$\sup \mathcal{H}(0) \geq \sup \mathcal{H}(x) \geq \min\{\sup \mathcal{H}(x * y), \sup \mathcal{H}(y)\} \geq \sup \mathcal{H}(a).$$

Hence $0 \in \mathcal{H}_a$ and $x \in \mathcal{H}_a$. Therefore, $\mathcal{H}_a$ is an ideal of X for all $a \in X$. ∎

Corollary 3.22.

If $\mathcal{H}$ is a Sup-hesitant fuzzy ideal of a BCK/BCI-algebra X, then the set

$$\mathcal{H}_0 := \{x \in X \mid \sup \mathcal{H}(0) = \sup \mathcal{H}(x)\},$$

is an ideal of X.

Theorem 3.23.

Let $a \in X$ and let $\mathcal{H}$ be a hesitant fuzzy set on a BCK/BCI-algebra X. Then

(1) If $\mathcal{H}_a$ is an ideal of X, then $\mathcal{H}$ satisfies

$$\sup \mathcal{H}(a) \leq \min\{\sup \mathcal{H}(x * y), \sup \mathcal{H}(y)\} \;\Rightarrow\; \sup \mathcal{H}(a) \leq \sup \mathcal{H}(x), \tag{23}$$

for all $x, y \in X$.

(2) If $\mathcal{H}$ satisfies two conditions (12) and (23), then $\mathcal{H}_a$ is an ideal of X.

Proof:

(1) Assume that $\mathcal{H}_a$ is an ideal of X and let $x, y \in X$ be such that

$$\sup \mathcal{H}(a) \leq \min\{\sup \mathcal{H}(x * y), \sup \mathcal{H}(y)\}.$$

Then $x * y \in \mathcal{H}_a$ and $y \in \mathcal{H}_a$, which imply that $x \in \mathcal{H}_a$, that is, $\sup \mathcal{H}(a) \leq \sup \mathcal{H}(x)$.

(2) Assume that a hesitant fuzzy set $\mathcal{H}$ on X satisfies two conditions (12) and (23). Then $0 \in \mathcal{H}_a$. Let $x, y \in X$ be such that $x * y \in \mathcal{H}_a$ and $y \in \mathcal{H}_a$. Then $\sup \mathcal{H}(a) \leq \sup \mathcal{H}(x * y)$ and $\sup \mathcal{H}(a) \leq \sup \mathcal{H}(y)$, and so $\sup \mathcal{H}(a) \leq \min\{\sup \mathcal{H}(x * y), \sup \mathcal{H}(y)\}$. It follows from (23) that $\sup \mathcal{H}(a) \leq \sup \mathcal{H}(x)$, that is, $x \in \mathcal{H}_a$. Therefore, $\mathcal{H}_a$ is an ideal of X. ∎

4. Conclusions

We have introduced the notion of Sup-hesitant fuzzy ideals in BCK/BCI-algebras in the framework of Torra's hesitant fuzzy set theory, and have investigated their properties. We have discussed relations between Sup-hesitant fuzzy subalgebras and Sup-hesitant fuzzy ideals. We have considered characterization of Sup-hesitant fuzzy ideal. Future research will focus on applying the notions/contents to other types of ideals in BCK/BCI-algebras and related algebraic structures.

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REFERENCES

Aldhafeeri, S. and Muhiuddin, G. (2019). Commutative ideals of BCK-algebras based on uni-hesitant fuzzy set theory, Missouri J. Math. Sci., Vol. 31, No. 1, pp. 56–65.

Atanassov, K.T. (1986). Intuitionistic fuzzy sets, Fuzzy Sets and Systems, Vol. 20, pp. 87–96.

Bustince, H., Fernandez, J., Kolesárová, A. and Mesiar, R. (2013). Generation of linear orders for intervals by means of aggregation functions, Fuzzy Sets and Systems, Vol. 220, pp. 69–77.

Chen, Z.S., Chin, K.S., Li, Y.L., and Yang, Y. (2016). Proportional hesitant fuzzy linguistic term set for multiple criteria group decision making, Inform. Sci., Vol. 357, pp. 61–87.

Dubois, D. and Prade, H. (2000). *Fundamentals of Fuzzy Sets*, Vol. 7 of The Handbooks of Fuzzy Sets, Springer, Berlin, Germany.

Faizi, S., Saabun, W., Rashid, T., Watrbski, J. and Zafar, S. (2017). Group decision-making for hesitant fuzzy sets based on characteristic objects method, Symmetry, Vol. 9, 136. doi:10.3390/sym9080136

Huang, Y. (2006). *BCI-algebra,* Science Press, Beijing, China.

Jun, Y.B. (2018). A new type of p-ideals in BCI-algebras based on hesitant fuzzy sets, Azerb. J. Math., Vol. 8, No. 1, pp. 104–115.

Jun, Y.B. and Ahn, S.S.(2016). Hesitant fuzzy set theory applied to BCK/BCI-algerbas, J. Comput. Anal. Appl., Vol. 20, No. 4, pp. 635–646.

Jun, Y. B., Ahn, S. S. and Muhiuddin, G. (2016). Hesitant fuzzy soft subalgebras and ideals in BCK/BCI-algebras, The Scientific World Journal, Volume 2014, Article ID 763929, 7 pages (2014).

Jun, Y.B., Kim, H.S. and Song, S.Z. (2017). A new type of subsemigroups and ideals of semigroups in the framework of hesitant fuzzy set theory, J. Mult.-Valued Logic Soft Comput., Vol. 29, No. 1-2, pp. 67–81.

Jun, Y. B., Muhiuddin, G., Ozturk, M. A. and Roh, E. H. (2017). Cubic soft ideals in BCK/BCI-algebras, Journal of Computational Analysis and Applications, Vol. 22, No. 5, pp. 929–940.

Jun, Y.B. and Song, S.Z. (2014). Hesitant fuzzy set theory applied to filters in MTL-algebras, Honam Mathematical J., Vol. 36, No. 4, pp. 813–830.

Jun, Y. B., Song, S. Z. and Muhiuddin, G. (2016). Hesitant fuzzy semigroups with a frontier, Journal of Intelligent and Fuzzy Systems, Vol. 30, No. 3, pp. 1613-1618.

Krishankumar, R., Subrajaa, L.S., Ravichandran, K.S., Kar, S, and Borumand, S. A. (2019). A framework for multi-attribute group decision-making using double hierarchy hesitant fuzzy linguistic term set, Int. J. Fuzzy Syst., Vol. 21, No. 4, pp. 1130–1143.

Liao, H., Xu, Z. and Zeng, X.J. (2014). Distance and similarity measures for hesitant fuzzy linguistic term sets and their application in multi-criteria decision making, Inform. Sci., Vol. 271, pp. 125–142.

Liu, H. and Rodriguez, R.M. (2014). A fuzzy envelope for hesitant fuzzy linguistic term set and its application to multicriteria decision making, Inform. Sci., Vol. 258, pp. 220–238.

McDermott, D. and Doyle, J. (1980) Non-monotonic logic I., Artif. Intell., Vol. 13, No. 1-2, pp. 41–72.

Meng, J. and Jun, Y.B. (1994). *BCK-algebras,* Kyungmoon Sa Co., Seoul, Korea.

Muhiuddin, G. (2016). Hesitant fuzzy filters and hesitant fuzzy G-filters in residuated lattices, J. Comput. Anal. Appl., Vol. 20, No. 2, pp. 394–404.

Muhiuddin, G. and Aldhafeeri, S. (2019). N-Soft p-ideal of BCI-algebras, European Journal of Pure and Applied Mathematics, Vol. 12, No. 1, pp. 79-87.

Muhiuddin, G and Aldhafeeri, S. (2018). Subalgebras and ideals in BCK/BCI-algebras based on uni-hesitant fuzzy set theory, Eur. J. Pure Appl. Math., Vol. 11, ,No. 2, pp. 417–430.

Muhiuddin, G. and Al-roqi, A. M. (2018). Regular hesitant fuzzy filters and MV-hesitant fuzzy filters of residuated lattices, Journal of Computational Analysis and Applications, Vol. 24, No. 6, pp. 1133–1144.

Muhiuddin, G. and Jun, Y.B. (2019). Sup-hesitant fuzzy subalgebras and its translations and extensions, Annals of Communications in Mathematics, Vol. 2, No. 1, pp. 48–56.

Muhiuddin, G., Kim, S. J. and Jun, Y. B. (2019). Implicative N-ideals of BCK-algebras based on neutrosophic N-structures, Discrete Mathematics Algorithms and Applications, Vol. 11, No. 01, 1950011 (17 pages).

Muhiuddin, G., Kim, H. S., Song, S. Z. and Jun, Y. B. (2017). Hesitant fuzzy translations and extensions of subalgebras and ideals in BCK/BCI-algebras, Journal of Intelligent and Fuzzy Systems, Vol. 32, No. 1, pp. 43–48.

Muhiuddin, G., Roh, E. H., Ahn, S. S. and Jun, Y. B. (2017). Hesitant fuzzy filters in lattice implication algebras, Journal of Computational Analysis and Applications, Vol. 22, No. 6, pp. 1105–1113.

Nillsson, N.J. (1986). Probabilistic logic, Artif. Intell., Vol. 28, pp. 71–87.

Pawlak, Z. (1982). Rough sets, Int. J. Comput. Inform. Sci., Vol. 11, pp. 341–356.

Rashid, T., Faizi, S., Xu, Z. and Zafar, S. (2018). ELECTRE-based outranking method for multicriteria decision making using hesitant intuitionistic fuzzy linguistic term sets, Int. J. Fuzzy Syst., Vol. 20,No. 1, pp. 78–92.

Rodriguez, R.M., Martinez, L. and Herrera, F. (2012). Hesitant fuzzy linguistic term sets for decision making, IEEE Trans. Fuzzy Syst., Vol. 20, No. 1, pp. 109–119.

Senapati, T., Jun, Y.B., Muhiuddin, G. and Shum, K. P. (2019). Cubic intuitionistic structures applied to ideals of BCI-algebras, Analele Stiintifice ale Universitatii Ovidius Constanta-Seria Matematica, Vol. 27, No. 2, pp. 213–232.

Torra, V. (2010). Hesitant fuzzy sets, Int. J. Intell. Syst., Vol. 25, pp. 529–539.

Torra V. and Narukawa, Y. (2009). On hesitant fuzzy sets and decision, in *The 18th IEEE International Conference on Fuzzy Systems*, Jeju Island, Korea, Vol. 2009, pp. 1378. 1382.

Türksen, I.B. (1986). Interval valued fuzzy sets based on normal forms, Fuzzy Sets and Systems, Vol. 20, pp. 191–210.

Wang, F.Q., Li, X. and Chen, X.H. (2014). Hesitant fuzzy soft set and its applications in multicriteria decision making, J. Appl. Math., Vol. 2014, Article ID 643785, 10 pages.

Wei, G. (2012). Hesitant fuzzy prioritized operators and their application to multiple attribute decision making, Knowledge-Based Systems, Vol. 31, pp. 176–182.

Xia, M. and Xu, Z.S. (2011). Hesitant fuzzy information aggregation in decision making, Internat. J. Approx. Reason., Vol. 52, No. 3, pp. 395–407.

Xu, Z.S. and Xia, M. (2011a). Distance and similarity measures for hesitant fuzzy sets, Inform. Sci., Vol. 181, No. 11, pp. 2128–2138.

Xu, Z.S. and Xia, M. (2011b). On distance and correlation measures of hesitant fuzzy information, Int. J. Intell. Syst., Vol. 26, No. 5, pp. 410–425.

Yager, R.R. (1986). On the theory of bags, Int. J. Generation Syst., Vol. 13, pp. 23–37.

Zadeh L. (1965). Fuzzy sets, Inform Control, Vol. 8, pp. 338–353.