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Complexity Dynamics of Gumowski-Mira Map

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Abstract

In the context of nonlinear dynamics, interesting dynamic behavior of Gumowski-Mira Map has been noted under various feasible circumstances. Evolutionary phenomena are discussed through the study of bifurcation analysis leading to period-doubling and chaos. The appearance of chaos in the method is identified by plotting Lyapunov characteristic exponents (LCE) and Topological Entropy within certain parameter range. Dynamic Lyapunov Indicator (DLI) has been procured for further identification of regular and chaotic motions of the Gumowski-Mira Map. The numerical results through the indicator DLI clearly demonstrate the behavior of our map. The correlation dimension has been calculated numerically for the dimension of the chaotic attractor.

Keywords: Bifurcation; Chaos indicators; Topological entropy; Correlation dimension

MSC 2010 No.: 37G15, 37D45

1. Introduction

Nonlinear systems are generally complex in nature. The complex systems are constituted of interacting components, and interactions in these systems are in nonlinear manner. There are several examples where the self organized evolution of the system is neither regular completely nor completely chaotic. Nonlinear properties of these systems are defined by the parameters involved in the system. Presence of complexity in a system observed by sudden increase in topological entropy and so the measure of complexity provided by increase of topological entropy. The bifurcation diagrams of systems obtained by varying one of the system parameters show the appearance of chaos from regularity. The indicators for regularity and chaos, such as time series graph, phase plot, Poincaré map, power spectrum, etc., though efficient, may not provide appropriate measure of chaos. In this regard, some recent articles by Grassberger et al. (1983), Osinga et al. (2011), Tsaneva-Atanasova et al. (2010), Saha and Tehri (2010), Saha and Prasad (2018), Ansari et al. (2018) and Ansari et al. (2020) etc., explain details of chaos indicators for regularity and chaos in dynamical systems.

For evolutionary behavior, we need to find LCEs and topological entropy and by applying certain statistical measure, such as correlation dimension we look to get the complete knowledge of the behavior of a system. For complex systems, LCEs, topological entropy and correlation dimensions are considered as perfect measure of complexity and chaos. The positivity of LCEs indicates the presence of chaos in the system, topological entropy provides measure of complexity in a system. Showing higher topological entropy signifies more complexity in the system and correlation dimension provides the dimension of the chaotic set.

The objective of this work is to study complex behavior of G M Map by theoretically as well as numerically. The G M Map chosen here, is a two dimensional map Gumowski and Mira (1980) to detect the appearance of ordered and chaotic motions. The work of Gumowski and Mira (1980) can be taken as guidance for important revelation regarding the measure of complexity in the proposed discrete systems. Our aim in this study is to find bifurcation scenario for this model and then to calculate numerically the LCEs, topological entropy and correlation dimension of chaotic attractor. The display of graphics of bifurcations, LCEs, topological entropy and correlation dimensions for regular and chaotic cases provide very significant information of evolution. For numerical computation we have used some Mathematica codes, such as those of Martelli (1999).

We have also applied indicator DLI to discuss the regular and chaotic behavior of the G M Map. We have calculated these numerically and represented graphically. Then, we have proceeded to obtain correlation dimension for the chaotic set appearing in the bifurcation diagram.

The paper is organized as follows. The overview of the literature review is presented in Section 1. Gumowski-Mira Map is performed in Section 2 with application. The correlation dimension for the system is performed in Section 3. Finally, the paper completed with conclusion in Section 4.

2. Gumowski-Mira Map

Different nonlinear analytical models present rich and complex characteristics existing in the literature. Some of them may present aesthetic probabilities and are therefore attractive to review in the connection of artistic creation. Among these models, one of the most important is the Gumowski-Mira model on account of its exceptionally high sensitivity to the parameters. This model has been developed for modeling and analyzing accelerated particles trajectories at CERN in 1980. The Gumowski-Mira map is a two dimensional non-linear discrete dynamic system, that produces a broad variety of phase space plots resembling fractal designs of nature, which is defined by the following recurrent formula:

$$\begin{aligned} x_{n+1} &= y_n + a\,(1 - b\,y_n^2)\,y_n + G(x_n), \\ y_{n+1} &= -x_n + G(x_{n+1}), \end{aligned} \tag{1}$$

where $G(x) = \mu\,x + \dfrac{2(1-\mu)\,x^2}{(1+x^2)}$.

The starting point (x_1, y_1) can be chosen as an arbitrary point where as a, b and μ are real parameters.

2.1. Bifurcation Analysis

Bifurcation in the ordinary sense is splitting into two. In a dynamical system it is a sudden change in behavior due to the sudden change of a set of parameter values according to a certain rule. The point where such changes occur is known as the bifurcation point. Emergence of chaos can easily be visualized by observing bifurcation diagrams. Within this diagram (Figure 1) certain periodic windows appearing, having very specific significance for nonlinear systems emerging to chaos. Here, $a = -1.1$, $b = -0.2$ are fixed and μ varying from $\mu = 0$ to $\mu = -2.0$. We observe the appearance of one cycle up to the value $\mu = -1.0$. Then, one cycle changes into two cycles when parameter μ decreases from $\mu = -1.0$ to $\mu = -1.5$ and then around $\mu = -1.6$, it shows chaotic. Appearance of periodic windows indicates the behavior of intermittency. Such behavior is observed in a wide variety of non-linear systems. As we move away from the window, in parameter space, the intervals of periodic motion become gradually shorter and more infrequent. Eventually, they cease altogether. Likewise, as we move towards the window, the intervals of periodic motion become gradually longer and more frequent. Ultimately, the whole motion becomes periodic. We can also observe a periodic window for the parameter value $-1.9 < \mu < -2$.

2.2. Lyapunov Characteristic Exponent (LCE)

The Lyapunov exponents are dynamic measurements capable of characterizing deterministic chaos in systems that facilitate highly sensitive dependencies on initial conditions. Actually it means the

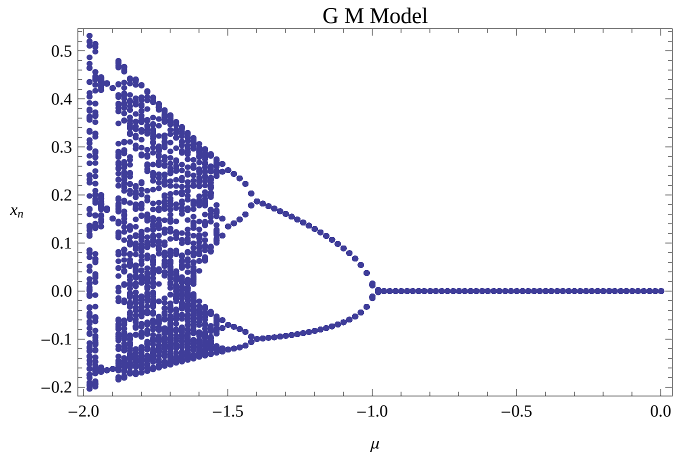
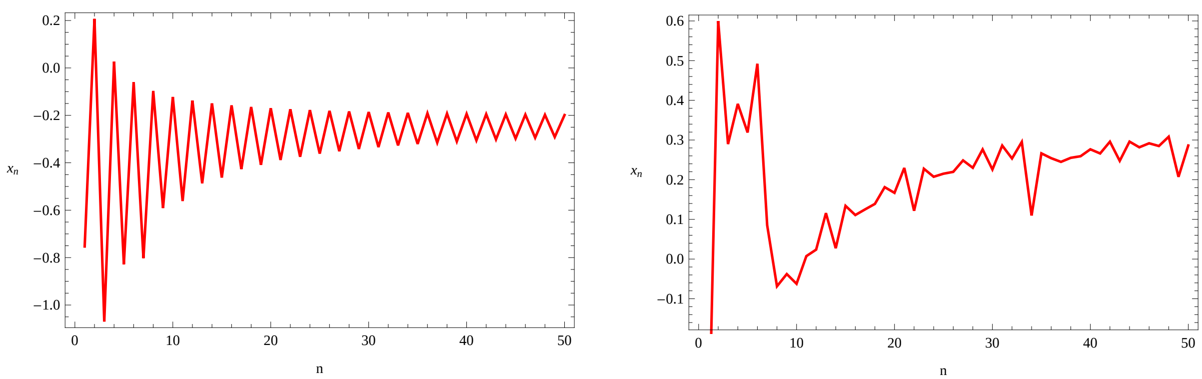


Figure 1. Bifurcation diagram: μ vs x_n determined from Eq. (1) with $a = -1.1$ and $b = -0.2$



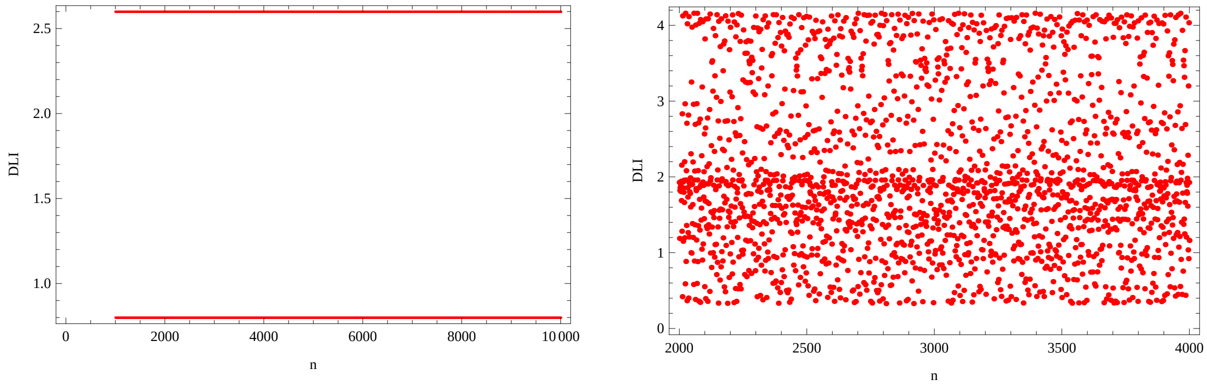
(a) For the evolution Regular case
$a = -1.1, b = -0.2, \mu = -1.5$

(b) For the evolution Chaotic case
$a = -1.1, b = -0.2, \mu = -2.0$

Figure 2. Lyapunov Characteristic Exponents of Gumowski-Mira Map

exponential divergence of orbits arises closely, with very small differences in initial conditions. Identifying and qualifying chaos in a dynamical system is a crucial matter that is solved by measuring the largest Lyapunov exponent. These numbers describe the average exponential rate of convergence or divergence of nearby orbits in the phase space of the considered dynamical system.

It is well known that the Lyapunov exponents (LCEs) are positive for chaotic evolutions and negative for regular evolution (we refer to Benettin et al. (1980), Abarbanel et al. (1992) and Andrecut and Kauffman (2007)). For this map, because of its evolutionary situations, LCEs achieved are important. The plots for chaotic and regular cases are shown in Figures 2(a) and 2(b) respectively. As shown in bifurcation diagram, Figure 1, negative and positive values of LCEs may appear on regular basis as we decrease μ from value zero.



(a) Regular case $a = -1.1$, $b = -0.2$, $\mu = -1.5$

(b) Chaotic case $a = -1.1$, $b = -0.2$, $\mu = -2.0$

Figure 3. DLI plots of Gumowski-Mira Map a regular case and a chaotic case

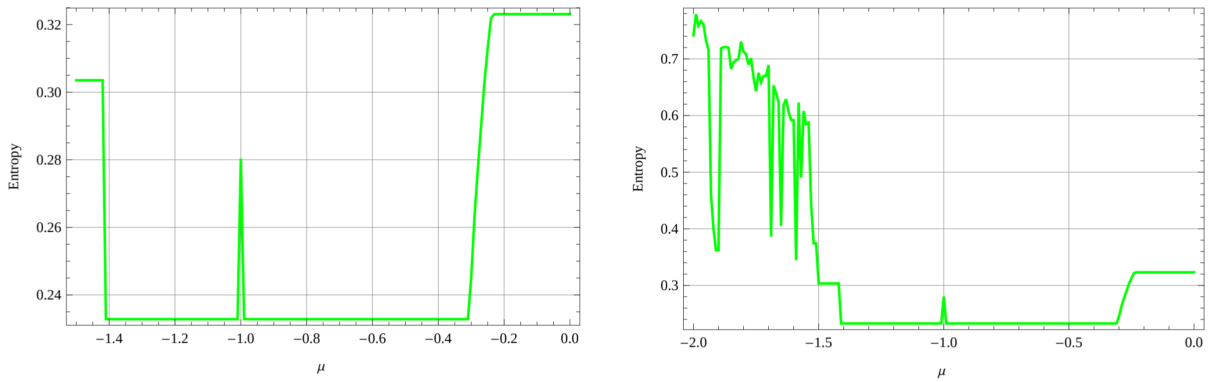
2.3. Application of Indicator DLI

Though Lyapunov exponents, topological entropies, and correlation dimensions are for chaos measure, these can also be used as indicators to distinguish chaotic and regular motions. The difficulty arises when we have a system of higher dimension. To overcome this problem, we have used the Dynamic Lyapunov Indicator (DLI) by which we proved to be more efficient to detect chaotic motions especially in higher dynamical systems. The dynamic Lyapunov indicator (DLI) is defined by the largest value estimated from all eigenvalues λ_i of the Jacobian matrix J such that $|J - \lambda_i I|$ = 0; i = 1, 2, . . ., n (for n-dimensional map) of the examined map for all discrete times. If these eigenvalues form a definite pattern, then the motion is regular and if they are distributed randomly, (with no definite pattern), then the motion is chaotic (Saha and Budhraja (2007)).

For parameter values $a = -1.1$, $b = -0.2$ and $\mu = -2.0$, we get the attractor of this map. Such a chaotic motion gets controlled and display regular behavior for $a = -1.1$, $b = -0.2$ and $\mu = -1.5$. This can be observed through the phase plots given in Figure 5. For the map DLI is computed with the parameter values mentioned above with the initial conditions as $(0.1, 0.1)$ and the results are displayed through the plots of regularity and chaos in Figure 3.

2.4. Topological Entropy

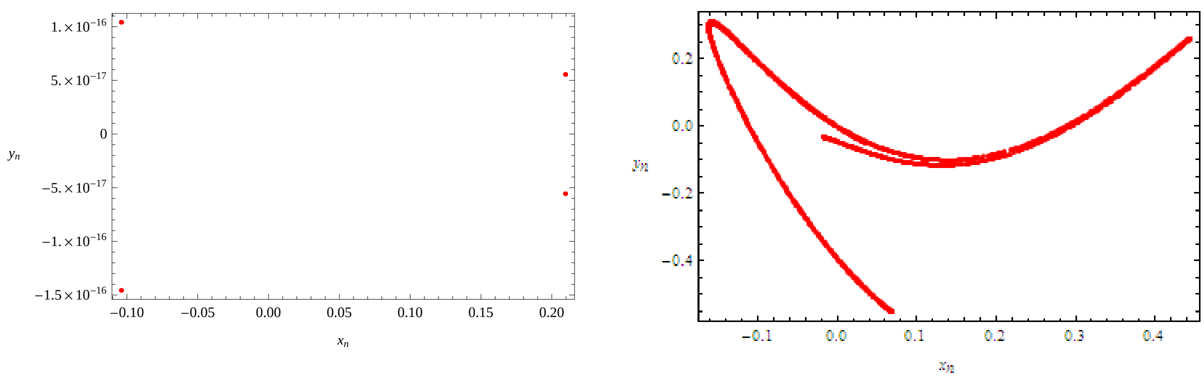
Topological entropy, a positive number, presents the measure of complexity in the system (see Adler et al. (1965), Bowen (1973), Baldwin and Slamink (1997), Manson (2001), Walby (2007) and Nagashima and Baba (2005)) to examine chaotic behavior in a broad variety of system for a better admissible indicator. The measure of complexity is provided by topological entropy. More topological entropy in a system signify the system is more complex (Beddington, et al. (1975), Kaitala and Heino (1996) and Xiao et al. (2002)). Complexity in the system does not suggest that it is chaotic and vice versa. Definition and mathematical formulation of topological entropy be addressed in Nagashima and Baba (2005) and within the recent article by Saha and Kumari (2013). For system (1), topological entropies have been obtained for different values of μ and



(a) For the evolution Regular case $a = -1.1, b = -0.2, \mu = -1.5$

(b) For the evolution Chaotic case $a = -1.1, b = -0.2, \mu = -2.0$

Figure 4. Plots of topological entropies of Gumowski-Mira Map



(a) For the evolution Regular case $a = -1.1, b = -0.2, \mu = -1.5$

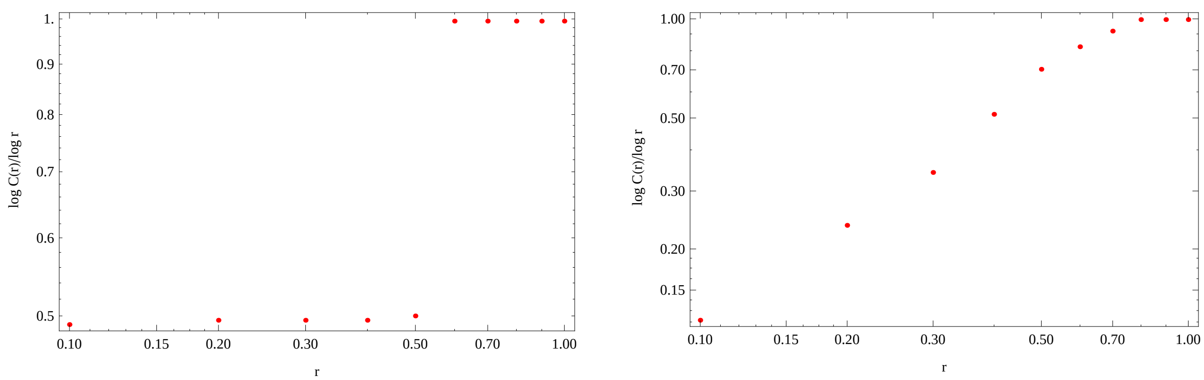
(b) For the evolution Chaotic case $a = -1.1, b = -0.2, \mu = -2.0$

Figure 5. Attractors of Gumowski-Mira Map

shown in Figure 4. From the Topological Entropy plot, we can see the increase in topological entropy in the zones of complexity when the parameter value of $(-1.5 \leq \mu \leq -1.7)$. Also, from Figures 4(a) and 4(b), we can observe significant growth at the parameter value μ = - 1.4, - 1, - 0.3, etc. where the evolution is regular and sometimes minimum when evolution is chaotic. So, if a system is chaotic it does not imply complexity within the system and vice versa. To check the complexity of the system we have plotted the chaotic attractors for the two values of parameter μ (= - 1.5 and - 2.0), for which we have calculated LCE and correlation dimension as well in Figure 5.

3. Correlation Dimension

The correlation dimension actually gives a measure of complexity for the underlying attractor of the system. To determine correlation dimension we use statistical method. It is a very practical and efficient method than other methods, like box-counting, etc. We use the procedure described in Martelli (1999) to obtain correlation dimension. Correlation dimension provide the dimensionality



(a) Regular case $a = -1.1$, $b = -0.2$, $\mu = -1.5$

(b) Chaotic case $a = -1.1$, $b = -0.2$, $\mu = -2.0$

Figure 6. Plots of correlation integral curves

of the system. Here below we have two plots for correlation integral data for the system for a chaotic case when $\mu = -2.0$ and for a regular case when $\mu = -1.5$, while as keeping other parameters fixed as $a = -1.1$ and $b = -0.2$ in figure (6). The plot 6(a) of the regular case shows zero slope of the curve and zero intercept to y-axis; thus the correlation dimension is zero in this case. However, plot 6(b) of the chaotic case is different. By using least square linear fit to the data of correlation curves.This provides the equation of a straight lines, one for regular and one for chaotic cases, respectively, are given by

$$y = -0.0352669 + 0.4025\,x \text{ and } y = 0.184232 + 0.982662\,x.$$

The intercept of this straight line with the y-axis is equal to 0.184232. Thus, the correlation dimension for chaotic attractor form in this case is approximately equal to -0.0352669.

4. Conclusion

In G M Map, we observe specific nature of evolution represented through bifurcation diagram in Figure 1, where we have seen period doubling and chaotic scenario for variation of certain parameter for their certain ranges. Lyapunov characteristic exponents (LCE) and Topological Entropy have been calculated for the above map to see the parameter ranges showing regularity and chaotic evolution. The negative value of LCE indicates the regular regions whereas the positive LCE indicates the chaotic regions of evolution which can be seen in Figure 2(a) and 2(b). Indicator DLI is very consistent in identifying regular and chaotic behavior of dynamical systems. In Figure 3, we have graphically represented GM map with respect to the DLI indicator which clearly shows the behavior of our map. Thus, we have assumed limited chance of occurrence of numerical error of chaos. We have also used meaningful statistical measures to justify the results obtained through this study. Topological entropy behave similar to those of LCEs and it may be observed in Figures 4(a) and 4(b). In Figure 5, we have plotted the attractors for our map that is appearing through bifurcation diagrams bear non-integer dimension and show self similarity or fractal property. To this end, the correlation dimensions for GM map is shown through Figures 6(a) and 6(b). For numerical calculations we have used the well-known software Mathematica, where the possibility of the occurring of round off error is minimum.

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