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On Fejér type inequalities for convex mappings utilizing generalized fractional integrals

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Abstract

In this work, we first establish Hermite-Hadamard-Fejér type inequalities for convex function involving generalized fractional integrals with respect to another function which are generalization of some important fractional integrals such as the Riemann-Liouville fractional integrals and the Hadamard fractional integrals. Moreover, we obtain some trapezoid type inequalities for these kind of generalized fractional integrals. The results given in this paper provide generalization of several inequalities obtained in earlier studies.

Keywords: Hermite-Hadamard-Fejér inequality; generalized fractional integrals; convex functions

MSC 2010 No.: 26A51, 26A33, 26D07, 26D10, 26D15

1. Introduction

The following inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied, see (Azpeitia (1994); Bakula and Pečarič (2004); Budak and Sarikaya (2016)); (Chu et al. (2016))-(Dragomir and Agarwal (1998)); (Iqbal et al. (2018)); (Jleli and Samet (2016))-(Khan et al. (2019)); (Khurshid et al. (2018))-(Rassias and Kashuri (2019)); (Sarikaya and Budak (2016))-(Sarikaya and Ertuğral

(2019)); (Sarikaya et al. (2013); Set et al. (2014); Set et al. (2016)); (Wang et al. (2013))-(Zhang et al. (2015)).

The most well-known inequalities related to the integral mean of a convex function are the Hermite-Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities, see (Fejér (1906)). For some recent results connected with Hermite-Hadamard-Fejér type inequalities, see (Ali et al. (2017); Bombardelli and Varosaneč (2009); Chen and Katugampola (2017); Chen and Wu (2014); Fejér (1906); Işcan (2015); Khurshid et al. (2019); Sarikaya (2012); Sarikaya and Budak (2017); Sarikaya et al. (2014); Set et al. (2015); Tseng et al. (2011)).

2. Preliminaries

In the following we recall some useful known definitions and results:

Let $f \in L[a,b]$. The Riemann-Liouville fractional integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a \tag{1}$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b, \tag{2}$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

In (Sarikaya and Budak (2014)), Sarikaya et al. first proved the following important Hermite-Hadamard type utilizing Riemann-Liouville fractional integrals: if $f: [a,b] \longrightarrow \mathbb{R}$ is a positive convex function on $[a,b]$ with $0 \leq a < b$ and $f \in L[a,b]$, then for $\alpha > 0$ the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{3}$$

On the other hand, Íşcan in (Íşcan (2015)), proved the following Fejér type inequalities for Riemann-Liouville fractional integrals: if $f: [a,b] \longrightarrow \mathbb{R}$ is a positive convex function on $[a,b]$ with $0 \leq a < b$ and $f \in L[a,b]$, then for $\alpha > 0$ the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\left[J_{a^+}^{\alpha} (f \circ g)(b) + J_{b^-}^{\alpha} (f \circ g)(a)\right]}{\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]} \leq \frac{f(a)+f(b)}{2}, \tag{4}$$

where $g: [a,b] \longrightarrow \mathbb{R}$ be nonnegative, integrable and symmetric about $x = \frac{a+b}{2}$ (i.e., $g(x) = g(a+b-x)$).

Let $f \in L[a,b]$. The k-Riemann-Liouville fractional integrals $I_{a^+}^{\alpha,k} f$ and $I_{b^-}^{\alpha,k} f$ of order $\alpha > 0$ where $k > 0$ with $a \geq 0$ are defined by

$$I_{a^+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt\,, \quad x > a$$

and

$$I_{b^-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt\,, \quad x < b,$$

respectively.

Let $f \in L[a,b]$. The Hadamard fractional integrals $H_{a^+}^{\alpha} f$ and $H_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$H_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha-1} f(t) \frac{dt}{t}, \quad x > a$$

and

$$H_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\ln t - \ln x)^{\alpha-1} f(t) \frac{dt}{t}, \quad x < b,$$

respectively.

Let's define a function $\varphi: [0, +\infty[\longrightarrow [0, +\infty[$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(t)}{t} dt < +\infty,$$

$$\frac{1}{A_1} \leq \frac{\varphi(s)}{\varphi(r)} \leq A_1 \quad \text{for} \quad \frac{1}{2} \leq \frac{s}{r} \leq 2,$$

$$\frac{\varphi(r)}{r^2} \leq A_2 \frac{\varphi(s)}{s^2} \quad \text{for} \quad s \leq r,$$

$$\left| \frac{\varphi(r)}{r^2} - \frac{\varphi(s)}{s^2} \right| \leq A_3 |r-s| \frac{\varphi(r)}{r^2} \quad \text{for} \quad \frac{1}{2} \leq \frac{s}{r} \leq 2,$$

where $A_1, A_2, A_3 > 0$ are independent of $r, s > 0$. If $\varphi(r) r^{\alpha}$ is increasing for some $\alpha \geq 0$ and $\frac{\varphi(r)}{r^{\beta}}$ is decreasing for some $\beta \geq 0$, then φ satisfies the above conditions.

Sarikaya and Ertuğral in (Sarikaya Ertuğral (2019)), defined the following generalized fractional integrals:

$$I_{a^+;\varphi} f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t)dt, \qquad x > a$$

and

$$I_{b^-;\varphi} f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t)dt, \qquad x < b.$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann–Liouville fractional integral, k-Riemann–Liouville fractional integral, conformable fractional integral, Hadamard fractional integrals, etc.

Motivated by the above literatures, the paper is organized as follows: In Section 3, we will establish Hermite-Hadamard-Fejér type integral inequalities for convex function involving generalized fractional integrals with respect to another function which are generalization of some important fractional integrals such as the Riemann-Liouville fractional integrals and the Hadamard fractional integrals. Also, we will derive an identity in order to develop some trapezoid type inequalities for these kind of generalized fractional integrals. Various special cases will be given and some known results will be recaptured. In Section 4, a briefly conclusion is provided as well.

3. Main Results

In this section, we obtain some Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals.

Theorem 3.1.

Let $g: [a, b] \longrightarrow \mathbb{R}$ be nonnegative and integrable function. If $f: [a, b] \longrightarrow \mathbb{R}$ is a convex function on $[a, b]$ with $a < b$, then the following Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \frac{\left[I_{a^+;\varphi}(g \circ F)(b) + I_{b^-;\varphi}(g \circ F)(a)\right]}{\left[I_{a^+;\varphi} g(b) + I_{b^-;\varphi} g(a)\right]} \leq \frac{f(a) + f(b)}{2}, \tag{5}$$

where $F(x) = f(x) + \widetilde{f}(x)$ and $\widetilde{f}(x) = f(a + b - x)$ for all $x \in [a, b]$.

Proof:

Since f is a convex mapping on $[a, b]$, we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2},$$

for all $x, y \in [a,b]$. For $t \in [0,1]$, let $x = ta + (1-t)b$ and $y = (1-t)a + tb$. Then, we get

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(ta+(1-t)b) + f\big((1-t)a+tb\big)}{2}. \tag{6}$$

Multiplying both sides of (6) by

$$\frac{(b-a)\varphi(b-[(1-t)a+tb])}{b-[(1-t)a+tb]}\, g\big((1-t)a+tb\big)$$

and integrating the resulting inequality with respect to t over $(0,1)$, we obtain

$$(b-a)f\left(\frac{a+b}{2}\right)\int_0^1 \frac{\varphi(b-[(1-t)a+tb])}{(b-[(1-t)a+tb])}\, g\big((1-t)a+tb\big)\, dt$$

$$\leq (b-a)\int_0^1 \frac{\varphi(b-[(1-t)a+tb])}{(b-[(1-t)a+tb])}\, g\big((1-t)a+tb\big) f(ta+(1-t)b)\, dt$$

$$\leq (b-a)\int_0^1 \frac{\varphi(b-[(1-t)a+tb])}{(b-[(1-t)a+tb])}\, g\big((1-t)a+tb\big) f\big((1-t)a+tb\big) dt.$$

Using the change of variable $\tau = (1-t)a + tb$, we have

$$f\left(\frac{a+b}{2}\right) I_{a^+;\varphi} g(b) \leq \frac{1}{2} I_{a^+;\varphi}(g \circ F)(b). \tag{7}$$

Similarly, multiplying both sides of (6) by

$$\frac{(b-a)\varphi([(1-t)a+tb]-a)}{[(1-t)a+tb]-a}\, g\big((1-t)a+tb\big)$$

and integrating the resulting inequality with respect to t over $(0,1)$, we get

$$f\left(\frac{a+b}{2}\right) I_{b^-;\varphi} g(a) \leq \frac{1}{2} I_{b^-;\varphi}(g \circ F)(a). \tag{8}$$

Summing the inequalities (7) and (8), we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\frac{\left[I_{a^+;\varphi}(g \circ F)(b) + I_{b^-;\varphi}(g \circ F)(a)\right]}{\left[I_{a^+;\varphi} g(b) + I_{b^-;\varphi} g(a)\right]}.$$

So, the left-side of (5) is proved. For the proof of the right-side inequality in (5), since f is convex on $[a,b]$, we have

$$f(ta+(1-t)b)+f\big((1-t)a+tb\big)\le f(a)+f(b). \tag{9}$$

Multiplying both sides of (9) by

$$\frac{(b-a)\varphi(b-[(1-t)a+tb])}{b-[(1-t)a+tb]}g\big((1-t)a+tb\big)$$

and integrating the resulting inequality with respect to t over $(0,1)$, we get

$$(b-a)\int_0^1 \frac{\varphi(b-[(1-t)a+tb])}{(b-[(1-t)a+tb])}g\big((1-t)a+tb\big)f(ta+(1-t)b)\,dt$$

$$+(b-a)\int_0^1 \frac{\varphi(b-[(1-t)a+tb])}{(b-[(1-t)a+tb])}g\big((1-t)a+tb\big)f\big((1-t)a+tb\big)\,dt$$

$$\le (b-a)\big(f(a)+f(b)\big)\int_0^1 \frac{\varphi(b-[(1-t)a+tb])}{(b-[(1-t)a+tb])}g\big((1-t)a+tb\big)\,dt.$$

Then, we obtain

$$I_{a^+;\varphi}(g\circ F)(b)\le \big(f(a)+f(b)\big)I_{a^+;\varphi}g(b). \tag{10}$$

Similarly, multiplying both sides of (9) by

$$\frac{(b-a)\varphi([(1-t)a+tb]-a)}{[(1-t)a+tb]-a}g\big((1-t)a+tb\big)$$

and integrating the resulting inequality with respect to t over $(0,1)$, we have

$$I_{b^-;\varphi}(g\circ F)(a)\le \big(f(a)+f(b)\big)I_{b^-;\varphi}g(a). \tag{11}$$

By adding the inequalities (10) and (11), we get

$$\frac{1}{2}\frac{\big[I_{a^+;\varphi}(g\circ F)(b)+I_{b^-;\varphi}(g\circ F)(a)\big]}{\big[I_{a^+;\varphi}g(b)+I_{b^-;\varphi}g(a)\big]}\le \frac{f(a)+f(b)}{2},$$

which completes the proof of Theorem 3.1. ■

Corollary 3.1.

Under the assumptions of Theorem 3.1, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{4}\frac{\left[I_{a^+;\varphi}F(b) + I_{b^-;\varphi}F(a)\right]}{\Phi(a,b)} \leq \frac{f(a)+f(b)}{2},$$

where

$$\Phi(a,b) = \int_a^b \frac{\varphi(t-a)}{t-a}dt = \int_a^b \frac{\varphi(b-t)}{b-t}dt.$$

Proof:

If we choose $g(t) = 1$ in Theorem 3.1, we get the above Hermite-Hadamard type inequality for generalized fractional integrals.

Remark 3. 1.

Taking $\varphi(t) = t$ in Corollary 3.1, then we obtain the classical Hermite-Hadamard type inequality.

Corollary 3. 2.

Under the assumptions of Theorem 3.1, we have Hadamard fractional integral inequalities proved by Jleli and Samet in (Jleli and Samet (2016)).

Proof:

Taking $g(t) = 1$ and $\varphi(t) = \frac{1}{\Gamma(\alpha)}\frac{[\log x - \log(x-t)]^{\alpha-1}}{x-t}$, where $\alpha \in (0,1)$ in Theorem 3.1, then we get Hadamard fractional integral inequalities proved by Jleli and Samet in (Jleli and Samet (2016)).

Corollary 3. 3.

Under the assumptions of Theorem 3.1, we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\frac{\int_a^b (g\circ F)(t)dt}{\int_a^b g(t)dt} \leq \frac{f(a)+f(b)}{2}.$$

Proof:

Taking $\varphi(t) = t$ in Theorem 3.1, then we have the above inequality.

Corollary 3. 4.

Under the assumptions of Theorem 3.1, we get

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\frac{\left[J_{a^+}^{\alpha}(g\circ F)(b) + J_{b^-}^{\alpha}(g\circ F)(a)\right]}{\left[J_{a^+}^{\alpha}g(b) + J_{b^-}^{\alpha}g(a)\right]} \le \frac{f(a)+f(b)}{2}.$$

Proof:

Choosing $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$ in Theorem 3.1, then we obtain the above inequality.

Corollary 3. 5.

Under the assumptions of Theorem 3.1, we have

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\frac{\left[I_{a^+}^{\alpha,k}(g\circ F)(b) + I_{b^-}^{\alpha,k}(g\circ F)(a)\right]}{\left[I_{a^+}^{\alpha,k}g(b) + I_{b^-}^{\alpha,k}g(a)\right]} \le \frac{f(a)+f(b)}{2}.$$

Proof:

Taking $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 3.1, then we get the above inequality.

Corollary 3. 6.

Under the assumptions of Theorem 3.1, we obtain

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\frac{\left[\int_a^b[(g\circ F)(t) + (g\circ F)(a+b-t)]d_{\alpha}t\right]}{\int_a^b[g(t) + g(a+b-t)]d_{\alpha}t} \le \frac{f(a)+f(b)}{2}.$$

Proof:

Choosing $\varphi(t) = t(b-t)^{\alpha-1}$, where $\alpha \in (0,1)$ in Theorem 3.1, then we have the above inequality.

Corollary 3. 7.

Under the assumptions of Theorem 3.1, we get

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\frac{\left[M_{g\circ F}(a,b;\alpha) + N_{g\circ F}(a,b;\alpha)\right]}{\left[M_g(a,b;\alpha) + N_g(a,b;\alpha)\right]} \le \frac{f(a)+f(b)}{2},$$

where

$$M_g(a,b;\alpha) = \frac{1}{\alpha}\int_a^b exp\big(-A(b-t)\big)\, g(t)dt$$

and

$$N_g(a,b;\alpha) = \frac{1}{\alpha}\int_a^b exp(-A(t-a))\, g(t)dt.$$

Proof:

Taking $\varphi(t) = \frac{t}{\alpha} exp(-At)$, where $A = \frac{1-\alpha}{\alpha}, \alpha \in (0,1)$ in Theorem 3.1, then we obtain the above inequality.

Lemma 3. 1.

Let $g\colon [a,b] \longrightarrow \mathbb{R}$ be nonnegative and integrable function. If $f\colon [a,b] \longrightarrow \mathbb{R}$ is a differentiable mapping on (a,b) with $a < b$, then the following identity for generalized fractional integrals holds:

$$\left[\frac{f(a)+f(b)}{2}\right]\left[I_{a^+;\varphi}g(b) + I_{b^-;\varphi}g(a)\right] - \frac{1}{2}\left[I_{a^+;\varphi}(g\circ F)(b) + I_{b^-;\varphi}(g\circ F)(a)\right]$$

$$= \frac{1}{2}\int_a^b P_{g;\varphi}(t)f'(t)dt, \tag{12}$$

where the mapping $P_{g;\varphi}\colon [a,b] \longrightarrow \mathbb{R}$ is defined by

$$P_{g;\varphi}(t) = \int_{a+b-t}^{t} \frac{\varphi(s-a)}{s-a} g(s)ds + \int_{a+b-t}^{t} \frac{\varphi(b-s)}{b-s} g(s)ds$$

and $F(x) = f(x) + \widetilde{f}\,(x)$, where $\widetilde{f}\,(x) = f(a+b-x)$.

Proof:

We denote

$$T_{f,g;\varphi}(a,b) = \int_a^b P_{g;\varphi}(t)f'(t)dt. \tag{13}$$

Integrating by parts (13), we have

$$T_{f,g;\varphi}(a,b) = P_{g;\varphi}(t)f(t)\,\Big|_a^b - \int_a^b P'_{g;\varphi}\,(t)f(t)dt$$

$$= P_{g;\varphi}(b)f(b) - P_{g;\varphi}(a)f(a) - \int_a^b P'_{g;\varphi}\,(t)f(t)dt. \tag{14}$$

$$P_{g;\varphi}(b) = \int_a^b \frac{\varphi(s-a)}{s-a} g(s)ds + \int_a^b \frac{\varphi(b-s)}{b-s} g(s)ds$$

$$= I_{b^-;\varphi}g(a) + I_{a^+;\varphi}g(b) = -P_{g;\varphi}(a). \tag{15}$$

$$P'_{g;\varphi}(t) = \frac{\varphi(t-a)}{t-a}g(t) + \frac{\varphi(b-t)}{b-t}g(a+b-t)$$
$$+\frac{\varphi(b-t)}{b-t}g(t) + \frac{\varphi(t-a)}{t-a}g(a+b-t).$$

$$\int_a^b P'_{g;\varphi}(t)f(t)dt = I_{a^+;\varphi}(g\circ F)(b) + I_{b^-;\varphi}(g\circ F)(a). \tag{16}$$

Then, substituting equalities (15) and (16) in (14) we get the desired equality (12). This completes the proof of Lemma 3.1. ■

Using Lemma 3.1 we can derive the following result for function, whose the first derivative in absolute value are convex using generalized fractional integrals.

Theorem 3.2.

Let $g:[a,b]\longrightarrow\mathbb{R}$ be nonnegative and integrable function. If $f:[a,b]\longrightarrow\mathbb{R}$ is a differentiable mapping on (a,b) with $a<b$ and $|f'|$ is a convex mapping on $[a,b]$, then we have the following trapezoid type inequality for generalized fractional integrals:

$$\left|T_{f,g;\varphi}(a,b)\right| \leq \frac{\left[H_{g;\varphi}(a,a)+H_{g;\varphi}(b,b)\right]}{2(b-a)}[f'(a)+f'(b)], \tag{17}$$

where

$$H_{g;\varphi}(x,y) := \int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\frac{\varphi(y-s)}{y-s}g(s)ds\right)|x-t|dt$$
$$+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t\frac{\varphi(y-s)}{y-s}g(s)ds\right)|x-t|dt.$$

Proof:

Taking the modulus and using their properties in Lemma 3.1, we have

$$\left|T_{f,g;\varphi}(a,b)\right| \leq \frac{1}{2}\int_a^b\left|P_{g;\varphi}(t)\right||f'(t)|dt.$$

Since f is a convex mapping on $[a,b]$, we get

$$|f'(t)| = \left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right| \leq \frac{b-t}{b-a}|f'(a)| + \frac{t-a}{b-a}|f'(b)|.$$

Hence,

$$\left|T_{f,g;\varphi}(a,b)\right| \leq \frac{1}{2}\int_a^b\left|P_{g;\varphi}(t)\right|\left[\frac{b-t}{b-a}|f'(a)|+\frac{t-a}{b-a}|f'(b)|\right]dt$$

$$= \frac{|f'(a)|}{2(b-a)} \int_a^b \left|P_{g;\varphi}(t)\right|(b-t)dt + \frac{|f'(b)|}{2(b-a)} \int_a^b \left|P_{g;\varphi}(t)\right|(t-a)dt. \qquad (18)$$

Since g is nonnegative function on $[a,b]$, then $P_{g;\varphi}(t)$ is nondecreasing function on $[a,b]$. As a result, we obtain

$$\begin{cases} P_{g;\varphi}(t) \ \leq 0, & t \in \left[a, \dfrac{a+b}{2}\right], \\ P_{g;\varphi}(t) \ > 0, & t \in \left]\dfrac{a+b}{2}, b\right]. \end{cases}$$

Thus, it follows that

$$\int_a^b \left|P_{g;\varphi}(t)\right|(b-t)dt = \int_a^{\frac{a+b}{2}} \left(\int_t^{a+b-t} \frac{\varphi(s-a)}{s-a} g(s)ds\right)(b-t)dt$$
$$+ \int_{\frac{a+b}{2}}^b \left(\int_{a+b-t}^t \frac{\varphi(s-a)}{s-a} g(s)ds\right)(b-t)dt$$
$$+ \int_a^{\frac{a+b}{2}} \left(\int_t^{a+b-t} \frac{\varphi(b-s)}{b-s} g(s)ds\right)(b-t)dt$$
$$+ \int_{\frac{a+b}{2}}^b \left(\int_{a+b-t}^t \frac{\varphi(b-s)}{b-s} g(s)ds\right)(b-t)dt$$
$$= H_{g;\varphi}(b,a) + H_{g;\varphi}(b,b).$$

Using the fact that $H_{g;\varphi}(b,a) = H_{g;\varphi}(a,a)$, we have

$$\int_a^b \left|P_{g;\varphi}(t)\right|(b-t)dt = H_{g;\varphi}(a,a) + H_{g;\varphi}(b,b). \qquad (19)$$

Similarly,

$$\int_a^b \left|P_{g;\varphi}(t)\right|(t-a)dt = H_{g;\varphi}(a,a) + H_{g;\varphi}(b,b). \qquad (20)$$

If we put equalities (19) and (20) in inequality (18), we obtain the desired inequality (17). The proof of Theorem 3.2 is completed. ■

Corollary 3.8.

Under the assumptions of Theorem 3.2, we have Hadamard fractional integral inequalities proved by Jleli and Samet in (Jleli and Samet (2016)).

Proof:

Taking $g(t) = 1$ and $\varphi(t) = \frac{1}{\Gamma(\alpha)} \frac{[\log x - \log(x-t)]^{\alpha-1}}{x-t}$, where $\alpha \in (0,1)$ in Theorem 3.2, then we get Hadamard fractional integral inequalities proved by Jleli and Samet in (Jleli and Samet (2016)).

Corollary 3.9.

Under the assumptions of Theorem 3.2, we obtain

$$\left|T_{f,g;\varphi}(a,b)\right| \leq \frac{3K}{16}(b-a)^2[f'(a)+f'(b)].$$

Proof:

Let $\varphi(t) = t$ and $g(s) \leq K,\ \forall s \in [a,b]$, where K is a constant in Theorem 3.2, then we have the above inequality.

Corollary 3.10.

Under the assumptions of Theorem 3.2, we get

$$\left|T_{f,g;\varphi}(a,b)\right| \leq \frac{K[(\alpha+2)(2^{\alpha+2}-5)]}{\Gamma(\alpha+3)}\left(\frac{b-a}{2}\right)^{\alpha+2}\frac{[f'(a)+f'(b)]}{(b-a)}.$$

Proof:

If $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ and $g(s) \leq K,\ \forall s \in [a,b]$, where K is a constant in Theorem 3.2, then we obtain the above inequality.

Corollary 3.11.

Under the assumptions of Theorem 3.2, we have

$$\left|T_{f,g;\varphi}(a,b)\right| \leq \frac{K\left[(\alpha+2k)\left(2^{\frac{\alpha}{k}+2}-5\right)\right]}{k\Gamma_k(\alpha+3k)}\left(\frac{b-a}{2}\right)^{\frac{\alpha}{k}+2}\frac{[f'(a)+f'(b)]}{(b-a)}.$$

Proof:

For $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $g(s) \leq K,\ \forall s \in [a,b]$, where K is a constant in Theorem 3.2, then we get the above inequality.

Remark 3.2.

Taking $\varphi(t) = t(b-t)^{\alpha-1}$ or $\varphi(t) = \frac{t}{\alpha} exp(-At)$, where $A = \frac{1-\alpha}{\alpha}$ and $\alpha \in (0,1)$ in Theorem 3.2, then we get some new Hermite-Hadamard-Fejér type inequalities. In the special case where

$g(t) = 1$ in Theorem 3.2, then we can establish some new Hermite-Hadamard type inequalities. Applying our Theorems 3.1 and 3.2, for suitable options of convex function $f(x)$, for example $f(x) = x^r$, where $r > 1$ and $x > 0$; $f(x) = \frac{1}{x}$, $x > 0$, $f(x) = e^x$, $x \in \mathbb{R}$, etc., we can construct some new generalized conformable fractional integral inequalities. Also, we can obtain several new general fractional integral inequalities using special means (arithmetic, geometric, logarithmic, etc.). Some new bounds for the midpoint and trapezium quadrature formula using our results can be provided as well. We omit their proofs and the details are left to the interested reader.

4. Conclusion

In this work, authors established Hermite-Hadamard-Fejér type inequalities for convex function involving generalized fractional integrals with respect to another function which are generalization of some important fractional integrals such as the Riemann-Liouville fractional integrals and the Hadamard fractional integrals. Also, we obtained some trapezoid type inequalities for these kind of generalized fractional integrals. This class of convex functions, can be applied to obtain several results in convex analysis, special functions, quantum mechanics, related optimization theory, mathematical inequalities and may stimulate further research in different areas of pure and applied sciences.

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