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# The Impact of Nonlinear Harvesting on a Ratio-dependent Holling-Tanner Predator-prey System and Optimum Harvesting

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## Abstract

In this paper, a Holling-Tanner predator-prey model with ratio-dependent functional response and non-linear prey harvesting is analyzed. The mathematical analysis of the model includes existence, uniqueness and boundedness of positive solutions. It also includes the permanence, local stability and bifurcation analysis of the model. The ratio-dependent model always has complex dynamics in the vicinity of the origin; the dynamical behaviors of the system in the vicinity of the origin have been studied by means of blow up transformation. The parametric conditions under which bionomic equilibrium point exist have been derived. Further, an optimal harvesting policy has been discussed by using Pontryagin maximum principle. The numerical simulations have been presented in support of the analytical findings.

**Keywords:** Ratio-dependent; Bifurcation; Harvesting; Bionomic equilibria; Optimal harvesting policy

**MSC 2010 No.:** 92B05, 35B35, 34C23 , 37G10

# 1. Introduction

The predator-prey interaction is one of the most important parts of the ecosystem. These interactions can be seen everywhere in the biosphere and effects the human life in various ways. The commercial exploitation of the renewable natural biological resources like fishery and forestry is a global challenge for maintaining the ecosystem. The harvesting of species is a way of commercial utilization of the species. The harvesting of species impact much on the ecosystem as over harvesting of the species may collapse the ecosystem. The first mathematical model to study the interactions between predators and their preys was proposed by Lotka (1925) and Volterra (1926), independently. This model consists of a system of two first order nonlinear ordinary differential equations equipped with a linear function (functional response: the function that describes the number of prey consumed by a predator per unit time). Thereafter, a number of functional response have been developed by many researchers: Holling type (Freedman (1980); Gaie and Zhang (2006); Hwang and Xiao (2004); Sugie et al. (1997)), ratio-dependent type (Arditi and Ginzburg (1989); Arditi and Saiah (1992); Hsu et al. (2001)), Beddington-DeAngelis type (Cantrell and Cosner (2001); Fan and Wang (2004); Holling (2011)), Hassell-Varley type (Hsu et al. (2005); Wu and Lin (2009)), etc. The results of various field and laboratory experiments confirm that the ratio-dependent predator-prey models are more appropriate for predator-prey interactions, when the predators involve serious hunting processes (Arditi et al. (1991); Berryman (1992); Reeve (1997)). The merits of ratio-dependent versus prey-dependent models are discussed by Lundberg and Fryxell (1995).

The Holling-Tanner predator-prey model was proposed by May (1973). Hsu and Hwang (1995) derived the conditions under which the model is globally stable. Saez and Oliares (1999) studied the bifurcation of the model. A Holling-Tanner predator-prey model with ratio-dependent functional response was proposed by Liang and Pan (2007). They studied the global stability of the unique positive interior equilibrium point of the model by constructing Lyapunov function. Saha and Chakrabarti (2009) studied a delayed ratio-dependent Holling-Tanner predator-prey model and used blow up transformations to study the qualitative behavior of the origin.

The first mathematical model to study the impact of harvesting on a predator-prey model was proposed by Clark (1976), in which he studied the problem of combined harvesting of two ecologically independent fish species. A predator-prey model with constant harvesting in predator species has been studied by Brauer and Soudack (1979). The Bogdanov-Takens bifurcation of this model was studied by Xiao and Ruan (1999). Beddington and May (1980) analyzed the Leslie-Gower predator-prey model when both the species were harvested at linear rate. Xiao and Jennings (2005) studied a ratio-dependent predator-prey model with constant harvesting in prey species while Xiao et al. (2006) studied the same model for constant predator harvesting and obtained different dynamics. Ragozin and Brown (2006) studied an optimal policy for a predator-prey system in which the predator is selectively harvested and prey has no commercial value. Kar (2004) proposed a ratio-dependent predator-prey model in which taxation is used as a controlling instrument and only prey species are subjected to harvest, he also discussed the optimal tax policy using the Pontryagin's maximal principle. Kar et al. (2006) studied a ratio-dependent predator-prey system with combined harvesting and solved the problem of optimal harvesting using Pontryagin's maximal principle. Lenzini and Rebaza (2010) studied a ratio-dependent predator-prey model for linear harvesting and nonlinear harvesting in prey species. Zhu and Lan (2010) proposed Leslie-Gower predator-prey model in which the prey species are harvested at constant rate and discussed the saddle-node bifurcation and the supercritical and subcritical hopf bifurcations. The Bogdanov-Takens bifurcation for this model is discussed by Gong and Huang (2014). Gupta et al. (2012) analysed a Leslie-Gower predator-prey model with nonlinear harvesting in prey. They derived the conditions for existence of bionomic equilibrium and discussed optimal harvesting policy using Pontryagin's maximal principle. Singh et al. (2016) analyzed a Leslie-Gower predator-prey model with Michaelis-Menten type predator harvesting.

The aim of this work is to study the impact of nonlinear harvesting on ratio-dependent predator-prey system proposed by Liang and Pan (2007), and determine how much one can harvest without altering dangerously the harvested species. We also find an optimal harvesting policy which maximizes the total discount net revenue derived from exploitation of natural renewable resources.

## 2. Model Equations

A Leslie-Gower type predator-prey model with ratio-dependent functional response is given by

$$\begin{cases} \frac{dN}{dT} = Nf(N) - g\Big(\frac{N}{P}\Big)P, \\ \frac{dP}{dT} = s(1-\frac{P}{bN})P, \end{cases} \tag{1}$$

with the initial conditions $N(0) > 0, P(0) > 0$, where $N(T)$ and $P(T)$ denote the densities of prey and predator at time $T$, respectively; $f(N)$ is the per capita growth rate of prey; $g(\frac{N}{P})$ is the predator functional response to prey; $s$ the intrinsic growth rate of predator; $\frac{1}{b}$ is the amount of prey required to support a predator at equilibrium.

The per capita growth rate $f(N)$ and the functional response $g\big(\frac{N}{P}\big)$ are considered as

$$f(N) = r\Big(1-\frac{N}{K}\Big); \qquad g\Big(\frac{N}{P}\Big) = \frac{mN}{AP+N}, \tag{2}$$

where the positive parameters $r$ denotes the intrinsic growth rate of prey, $K$ represents the carrying capacity of prey in the absence of predator, $m$ represents the maximal predator per capita consumption rate and $A$ is the number of prey necessary to achieve one half of the maximum rate $m$. Thus, the model (1) becomes

$$\begin{cases} \frac{dN}{dT} = r\Big(1-\frac{N}{K}\Big)N - \frac{mNP}{AP+N}, \\ \frac{dP}{dT} = s\Big(1-\frac{P}{bN}\Big)P, \end{cases} \tag{3}$$

with the initial conditions $N(0) > 0, P(0) > 0$. The system (3) is a ratio-dependent Holling-Tanner predator-prey model. The system (3) is not well defined at origin and to avoid this problem, we redefine it as

$$\begin{cases} \frac{dN}{dT} = r\Big(1 - \frac{N}{K}\Big)N - \frac{mNP}{AP+N}, \\ \frac{dP}{dT} = s(1 - \frac{P}{bN})P, \\ \frac{dN}{dT} = \frac{dP}{dT} = 0 \quad at \quad (0,0), \end{cases} \tag{4}$$

with the initial conditions $N(0) > 0, P(0) > 0$.

Next, we consider the prey species are harvesting with Michalis-Menten type function:

$$H(N) = \frac{qEN}{m_1E + m_2N}, \tag{5}$$

where $q, E > 0$ denote the catchability coefficient, effort applied to harvest individuals, respectively and $m_1, m_2$ are suitable positive constants.

Thus, the model (4) becomes

$$\begin{cases} \frac{dN}{dT} = r\Big(1 - \frac{N}{K}\Big)N - \frac{mNP}{AP+N} - \frac{qEN}{m_1E+m_2N}, \\ \frac{dP}{dT} = s(1 - \frac{P}{bN})P, \\ \frac{dN}{dT} = \frac{dP}{dT} = 0 \quad at \quad (0,0), \end{cases} \tag{6}$$

with the initial conditions $N(0) > 0, P(0) > 0$.

Let $N = Kx, \;\; P = \frac{Ky}{A}, \;\; T = \frac{t}{r}$, the system (6) takes the form

$$\begin{cases} \frac{dx}{dt} = x(1 - x) - \frac{\alpha xy}{x+y} - \frac{hx}{c+x} \equiv xf_1(x,y), \\ \frac{dy}{dt} = \rho(\beta - \frac{y}{x})y \equiv yf_2(x,y), \\ \frac{dx}{dt} = \frac{dy}{dt} = 0 \quad at \quad (0,0), \end{cases} \tag{7}$$

with the initial conditions: $x(0) > 0, y(0) > 0$, where $\alpha = \frac{m}{rA}$, $\beta = Ab$, $h = \frac{qE}{rKm_2}$, $C = \frac{m_1E}{m_2K}$, $\rho = \frac{s}{Abr}$. The set $R^2_+ = \{(x,y) \in R^2 : x,y \geq 0\}$ represents the closed first quadrant. In the following, it is demonstrated that the system (7) is "well behaved" in $R^2_+$.

**Proposition 2.1.**

$a)$ There exists a unique continuous solution of the system (7) in the interior of $R^2_+$.
$b)$ The set $R^2_+$ is invariant for the system (7).
$c)$ The nonnegative solutions $(x(t), y(t))$ of the system (7) are bounded for all $t \geq 0$.
$d)$ The system (7) is permanent whenever the condition $\alpha + \frac{h}{c} < 1$ holds.

***Proof:***

$a)$ Let

$$F(x,y) = \begin{cases} x(1 - x - \frac{\alpha y}{x+y} - \frac{h}{c+x}), & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0), \end{cases} ; \quad G(x,y) = \begin{cases} \rho(\beta - \frac{y}{x})y, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Then the system (7) reduces to

$$\frac{dx}{dt} = F(x, y), \qquad \frac{dy}{dt} = G(x, y).$$

It is easy to observe that the functions $F(x, y)$, $G(x, y)$ and their first order partial derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial G}{\partial x}$, $\frac{\partial G}{\partial y}$ are continuous in $R^2_+$. The existence and uniqueness theorem (Simmons (2017)) confirms that there exist a unique solution for the system (7) through any point $(x(0), y(0))$ of $R^2_+$, which is well defined and is continuous for all $t \geq 0$.

$b)$ The integration of the system (7) yields

$$x(t) = x(0) \exp\left(\int_0^t f_1(x(s), y(s))ds\right), \tag{8}$$

$$y(t) = y(0) \exp\left(\int_0^t f_2(x(s), y(s))ds\right). \tag{9}$$

Equations (8) and (9) with initial conditions of the system (7) confirms that $x(t) > 0, y(t) > 0$. Moreover, $x(t) = 0$ for $t > 0$ whenever $x(0) = 0$, and so, the $y$-axis $\{(x, y) : x(t) = 0, y(t) \geq 0\}$ is positively invariant. Similarly, the $x$-axis $\{(x, y) : y(t) = 0, x(t) \geq 0\}$ is positive invariant. Hence, the result.

$c)$ Using condition (b), from the first equation of system (7), we get

$$\frac{dx}{dt} \leq x(1 - x). \tag{10}$$

On simplifying (10), we get

$$x(t) \leq \frac{1}{1 + C_0 e^{-t}}, \tag{11}$$

where $C_0 = \frac{1-x(0)}{x(0)}$. Now, consider the following two cases:

i) If $x(0) \leq 1$, then $C_0 \geq 0$. It is evident from equation (11) that $x(t) \leq 1$ for all $t \geq 0$.
ii) If $x(0) > 1$, then $C_0 < 0$, hence equation (11) leads to $x(t) \geq 1$. From equation (10), we get $\frac{dx}{dt} \leq 0$. This implies that $x(t)$ is monotonic decreasing. Hence, $\max\limits_{t\geq 0}\{x(t)\} = x(0)$.

The above said cases yield: $x(t) \leq \max\{x(0), 1\} \equiv M_1$.

The second equation of system (7) with the boundedness of $x(t)$ leads to

$$\frac{dy}{dt} \leq \rho y\Big(\beta - \frac{y}{M_1}\Big). \tag{12}$$

Similar computation gives

$$y(t) \leq \max\{y(0), \beta M_1\} \equiv M_2. \tag{13}$$

$d)$ From first equation of the system (7), we have

$$\frac{dx}{dt} \geq x\left(1-\alpha-\frac{h}{c}-x\right). \tag{14}$$

If $x_0$ is the root of the equation $1-\alpha-\frac{h}{c}-x=0$, then by standard comparison argument we get $\lim_{t\to\infty}\inf x(t) \geq x_0$. Hence, we have $x(t) \geq x_0$ for large $t$.

Further, from the second equation of the system (7), we have

$$\frac{dy}{dt} \geq \rho y\left(\beta-\frac{y}{x_0}\right). \tag{15}$$

If $y_0$ is the root of the equation $\beta-\frac{y}{x_0}=0$, we obtain $\lim_{t\to\infty}\inf y(t) \geq y_0$. It is to be noticed that $x_0>0$ whenever $\alpha+\frac{h}{c}<1$.

Thus, from Equation (11), we have

$$\lim_{t\to\infty}\sup x(t) \leq 1.$$

Similarly,

$$\lim_{t\to\infty}\sup y(t) \leq M_1\beta.$$

This completes the proof. ■

## 3. Equilibrium points

The equilibrium points of the system (7) are the nonnegative real solutions of zero growth isoclines

$$xf_1(x,y)=0, \tag{16}$$

$$yf_2(x,y)=0, \tag{17}$$

where Equation (16) is the prey zero growth isocline and Equation (17) is the predator zero growth isocline of the system. The following three types of equilibrium points for the system (7) exist.

$(a)$ *Trivial equilibrium point:* The trivial equilibrium point of the system (7) is $E_0=(0,0)$.

$(b)$ *Axial equilibrium points:* The axial equilibrium points of the system (7) are only the points of intersection of the curves $y=0$ and $f_1(x,y)=0$. The Abscissa of these equilibrium points are the roots of the quadratic equation:

$$x^2-(1-c)x+h-c=0. \tag{18}$$

The quadratic equation (18) has two distinct positive roots

$$x_k=\frac{1-c+(-1)^k\sqrt{(1-c)^2-4(h-c)}}{2},\ k=1,2,$$

if $c < 1, c < h < \left(\frac{1+c}{2}\right)^2$; one positive root $x_2$ if $h < c$; a double positive root

$$x_1 = x_2 = \frac{1-c}{2},$$

if $c < 1, h = (\frac{1+c}{2})^2$, and a positive root

$$x_2 = 1 - c,$$

if $h = c < 1$.

$(c)$ *Interior equilibrium points:* The interior equilibrium points of the system (7) are the intersection points of the curves $f_1(x,y) = 0$ and $f_2(x,y) = 0$ and the abscissa of these equilibrium points are given by the solution of the quadratic equation

$$(1+\beta)x^2 - (1 - c + \beta - c\beta - \alpha\beta)x + h - c + h\beta - c\beta + \alpha\beta c = 0, \tag{19}$$

while the ordinates are given by $y_k^* = \beta x_k^*, \;\; k = 1, 2$.

The quadratic equation (19) has two distinct positive roots

$$x_k^* = \frac{1 - c + \beta - c\beta - \alpha\beta + (-1)^k\sqrt{(1 - c + \beta - c\beta - \alpha\beta)^2 - 4(1+\beta)(h - c + h\beta - c\beta + \alpha\beta c)}}{2(1+\beta)}, \; k = 1, 2,$$

if $c < 1 - \frac{\alpha\beta}{1+\beta}, \frac{c(1+\beta-\alpha\beta)}{1+\beta} < h < \frac{1}{4}\left(1 + c - \frac{\alpha\beta}{1+\beta}\right)^2$; one positive root $x_2^*$, if $h < \frac{c(1+\beta-\alpha\beta)}{1+\beta}$; a double root

$$\overline{x} = \frac{1 - c + \beta - c\beta - \alpha\beta}{2(1+\beta)},$$

if $c < 1 - \frac{\alpha\beta}{1+\beta}, h = \frac{1}{4}\left(1 + c - \frac{\alpha\beta}{1+\beta}\right)^2$, and a positive root

$$x^* = \frac{1 - c + \beta - c\beta - \alpha\beta}{1+\beta},$$

if $c < 1 - \frac{\alpha\beta}{1+\beta}, h = \frac{c(1+\beta-\alpha\beta)}{1+\beta}$.

The above discussion is summarized in Table (1).

## 4. Qualitative analysis of equilibrium points

In this section, the dynamical behaviors of system (7) have been discussed in the neighborhood of each biological feasible equilibrium point.

### 4.1. Trivial equilibrium point

The functions $f_1$ and $f_2$ are not differentiable at the origin, to analyze the behavior of this point we use blow up transformation: $x = x, \; y = vx$ (Jost et al. (1999)), which transforms the system (7)

**Table 1.** Table presenting the equilibrium solutions admitted by the system (7) in various regions

| Range of $c$ | Conditions | Equilibrium Points |
|---|---|---|
| $1 \leq c$ | $c < h$ | $E_0(0,0)$ |
| | $0 < h < \frac{c(1+\beta-\alpha\beta)}{1+\beta}$ | $E_0(0,0), E_2(x_2,0), E_2^*(x_2^*, y_2^*,)$ |
| $\frac{1+\beta-\alpha\beta}{1+\beta} < c < 1$ | $h > \left(\frac{1+c}{2}\right)^2$ | $E_0(0,0)$ |
| | $h = \left(\frac{1+c}{2}\right)^2$ | $E_0(0,0), E_1(x_1,0) = E_2(x_2,0)$ |
| | $c < h < \left(\frac{1+c}{2}\right)^2$ | $E_0(0,0), E_1(x_1,0), E_2(x_2,0)$ |
| | $\frac{c(1+\beta-\alpha\beta)}{1+\beta} \leq h \leq c$ | $E_0(0,0), E_2(x_2,0)$ |
| | $0 < h < \frac{c(1+\beta-\alpha\beta)}{1+\beta}$ | $E_0(0,0), E_2(x_2,0), E_2^*(x_2^*, y_2^*,)$ |
| $c < \frac{1+\beta-\alpha\beta}{1+\beta}$ | $h > \left(\frac{1+c}{2}\right)^2$ | $E_0(0,0)$ |
| | $h = \left(\frac{1+c}{2}\right)^2$ | $E_0(0,0), E_1(x_1,0) = E_2(x_2,0)$ |
| | $c < h < \left(\frac{1+c}{2}\right)^2$ | $E_0(0,0), E_1(x_1,0), E_2(x_2,0)$ |
| | $\frac{1}{4}\left(1+c-\frac{\alpha\beta}{1+\beta}\right)^2 \leq h \leq c$ | $E_0(0,0,)E_2(x_2,0)$ |
| | $h = \frac{1}{4}\left(1+c-\frac{\alpha\beta}{1+\beta}\right)^2$ | $E_1(x_1,0), E_2(x_2,0), \overline{E}(\overline{x}, \overline{y})$ |
| | $c < h < \frac{1}{4}\left(1+c-\frac{\alpha\beta}{1+\beta}\right)^2$ | $E_0(0,0), E_1(x_1,0), E_2(x_2,0), E_1^*(x_1^*, y_1^*),$ $E_2^*(x_2^*, y_2^*)$ |
| | $h = c < \frac{1}{4}\left(1+c-\frac{\alpha\beta}{1+\beta}\right)^2$ | $E_0(0,0), E_2(x_2,0), E_1^*(x_1^*, y_1^*), E_2^*(x_2^*, y_2^*)$ |
| | $\frac{c(1+\beta-\alpha\beta)}{1+\beta} < h < c$ | $E_0(0,0), E_2(x_2,0), E_1^*(x_1^*, y_1^*), E_2^*(x_2^*, y_2^*)$ |
| | $h = \frac{c(1+\beta-\alpha\beta)}{1+\beta}$ | $E_0(0,0), E_2(x_2,0), E^*(x^*, y^*)$ |
| | $0 < h < \frac{c(1+\beta-\alpha\beta)}{1+\beta}$ | $E_0(0,0), E_2(x_2,0), E_2^*(x_2^*, y_2^*)$ |

into

$$\begin{cases} \frac{dx}{dt} = x\left(1 - x - \frac{\alpha v}{1+v} - \frac{h}{c+x}\right), \\ \frac{dv}{dt} = v\left(\rho(\beta - v) - 1 + x + \frac{\alpha v}{1+v} + \frac{h}{c+x}\right). \end{cases} \tag{20}$$

The above system has either two or three equilibrium points at positive $v$-axis, namely, $E_{00} = (0,0)$, $E_{01} = (0, v_1)$ and $E_{02} = (0, v_2)$, where $v_1$ and $v_2$ $(0 \leq v_1 < v_2)$ are two distinct real roots of the following quadratic equation

$$c\rho v^2 + (c - h + c\rho - c\alpha - c\rho\beta)v + (c - h - c\rho\beta) = 0, \tag{21}$$

that is, $v_k = \frac{-\zeta + (-1)^k \Delta}{2c\rho}, k = 1, 2$, where $\zeta = c - h + c\rho - c\alpha - c\rho\beta,\ \Delta^2 = (c - h + c\rho - c\alpha - c\rho\beta)^2 - 4c\rho(c - h - c\rho\beta)$.

The Jacobian matrix of the system (20) at the equilibrium point $E_{00}$ is

$$J_{E_{00}} = \begin{bmatrix} \frac{c-h}{c} & 0 \\ 0 & \frac{-c+h+c\rho\beta}{c} \end{bmatrix},$$

and at the equilibrium points $E_{0k} = (0, v_k), k = 1, 2,$

$$J_{E_{0k}} = \begin{bmatrix} \rho(\beta - v_k) & 0 \\ v_k(1 - \frac{h}{c^2}) & v_k(\frac{\alpha}{(1+v_k)^2} - \rho) \end{bmatrix}.$$

There are only three possibilities.

1) If $c - h - c\rho\beta < 0$, then the system (20) has two equilibrium points $E_{00} = (0, 0)$ and $E_{02} = (0, v_2)$, where $v_2 = \frac{-\zeta+\Delta}{2c\rho}$ on the positive $v$-axis. The eigenvalues of the Jacobian matrix $J_{E_{00}}$ are $\lambda_1 = \frac{c-h}{c}$ and $\lambda_2 = \frac{-c+h+c\rho\beta}{c} > 0$. Thus, the equilibrium point $E_{00}$ is an unstable node for $c > h$ and is a saddle point for $c < h$.

   The eigenvalues of the Jacobian matrix of the system (20) at the equilibrium point $E_{02} = (0, v_2)$ are $\lambda_1 = \rho(\beta - v_2) = \frac{2c\rho\beta+\zeta-\Delta}{2c}$ and $\lambda_2 = v_2\Big(\frac{\alpha}{(1+v_2)^2} - \rho\Big)$. From the quadratic equation (21), we have $\sqrt{\frac{\alpha}{\rho}} - 1 < v_2$, and so, $\lambda_2 < 0$. Thus, the equilibrium point $E_{02}$ is a saddle point for $2c\rho\beta + \zeta - \Delta > 0$ and a stable point for $2c\rho\beta + \zeta - \Delta < 0$. Hence, by inverse blow up transformation the trivial equilibrium point $E_0$ of the system (7) is asymptotically stable if $2c\rho\beta + \zeta - \Delta < 0$.

2) If $c-h-c\rho\beta > 0$, then the system (20) has three equilibrium points $E_{00} = (0, 0)$, $E_{01} = (0, v_1)$ and $E_{02} = (0, v_2)$ on the positive $v$-axis provided $\zeta < 0$ and $\Delta^2 > 0$, where $v_1 = \frac{-\zeta-\Delta}{2c\rho}$ and $v_2 = \frac{-\zeta+\Delta}{2c\rho}$. The eigenvalues of the Jacobian matrix $J_{E_{00}}$ are $\lambda_1 = \frac{c-h}{c}$ and $\lambda_2 = \frac{-c+h+c\rho\beta}{c} < 0$. Thus, the equilibrium point $E_{00}$ is a saddle point if $c > h$.

   The eigenvalues of the Jacobian matrix of the system (20) at the equilibrium point $E_{01} = (0, v_1)$ are $\lambda_1 = \rho(\beta - v_1) = \frac{2c\rho\beta+\zeta+\Delta}{2c}$ and $\lambda_2 = v_1\Big(\frac{\alpha}{(1+v_1)^2} - \rho\Big)$. From the quadratic equation (21), we have $0 < v_1 < \sqrt{\frac{\alpha}{\rho}} - 1$, and so, $\lambda_2 > 0$. This shows that the equilibrium point $E_{01}$ of the system (20) is a saddle point if $2c\rho\beta+\zeta+\Delta < 0$ and an unstable node if $2c\rho\beta+\zeta+\Delta > 0$.

   The eigenvalues of the Jacobian matrix of the system (20) at the equilibrium point $E_{02} = (0, v_2)$ are $\lambda_1 = \frac{2c\rho\beta+\zeta-\Delta}{2c}$ and $\lambda_2 = v_2\Big(\frac{\alpha}{(1+v_1)^2} - \rho\Big)$. From quadratic equation (21), we have $0 < \sqrt{\frac{\alpha}{\rho}} - 1 < v_{02}$, and so, $\lambda_2 < 0$. This shows that the equilibrium point $E_{02}$ of the system (20) is a saddle point if $2c\rho\beta + \zeta - \Delta > 0$ and asymptotically stable if $2c\rho\beta + \zeta - \Delta < 0$. Hence, by inverse blow up transformation the trivial equilibrium point $E_0$ of the system (7) is asymptotically stable if $2c\rho\beta + \zeta - \Delta < 0$.

3) If $c - h - c\rho\beta = 0$, then the system (20) has two equilibrium points $E_{00} = (0, 0)$ and $E_{03} = (0, v_3)$ on the positive $v$-axis, where $v_3 = \frac{\alpha-\rho}{\rho}$. The eigenvalues of the Jacobian matrix

of the system (20) at the equilibrium point $E_{00}$ are $\lambda_1 = \rho\beta$ and $\lambda_2 = 0$. So, we cannot study the equilibrium point $E_{00}$ using linearization technique.

Now, we use the technique given in Zhang et al. (1991). Using the condition $c - h - c\rho\beta = 0$, the system (20) reduces to

$$\begin{cases} \frac{dx}{dt} = x + P_2(x, v), \\ \frac{dv}{dt} = Q_2(x, v), \end{cases} \tag{22}$$

where

$$P_2(x, v) = \frac{1}{\rho\beta}((-1 + \frac{h}{c^2}x^2)x^2 - \alpha vx + \cdots); Q_2(x, v) = \frac{1}{\rho\beta}((\alpha - \rho)v^2 + (1 - \frac{h}{c^2})vx). \tag{23}$$

From $x + P_2(x, v) = 0$, we obtain

$$x(v) = 0, \qquad \psi = \frac{\alpha - \rho}{\rho\beta}v^2 + [v]_3, \tag{24}$$

which implies that $m = 2$ and $a_m = \frac{\alpha-\rho}{\rho\beta} > 0$ as $\alpha > \rho$. Hence, the trivial equilibrium point $E_{00}$ is a saddle-node, and the parabolic sector is on the right half-plane.

The eigenvalues of the Jacobian matrix of the system (20) at the equilibrium point $E_{03}$ are $\lambda_1 = \rho\beta - \alpha + \rho$ and $\lambda_2 = v_3\left(\frac{\alpha}{(1+v_2)^2} - \rho\right) = \frac{(\alpha-\rho)(\rho-\alpha)}{\rho}$, and so, $\lambda_2 < 0$. This shows that the equilibrium point $E_{03}$ of the system (20) is a saddle point if $\rho\beta - \alpha + \rho > 0$ and is asymptotically stable if $\rho\beta - \alpha + \rho < 0$. By using inverse blow up transformation the trivial equilibrium point $E_0$ of the system (7) is asymptotically stable if $\rho\beta - \alpha + \rho < 0$.

The above discussions can be summarized as follows.

**Theorem 4.1.**

The trivial equilibrium point $E_0 = (0, 0)$ of the system (7) is locally asymptotically stable if any one of the following conditions is satisfied

$a)$ $c - h - c\rho\beta < 0$, and $2c\rho\beta + \zeta - \Delta < 0$. (see figure 2a)
$b)$ $c - h - c\rho\beta > 0, \zeta < 0, \Delta > 0$ and $2c\rho\beta + \zeta - \Delta < 0$. (see figure 2b)
$c)$ $c - h - c\rho\beta = 0$ and $\rho\beta - \alpha + \rho < 0$. (see figure 2c)

### 4.2. Axial equilibrium point

In Section (3), we have obtained the parametric conditions for the existence of axial equilibrium points. Now, we will discuss the nature of these axial equilibrium points.

**Theorem 4.2.**

$a)$ The axial equilibrium points $E_1$ is an unstable point.
$b)$ The axial equilibrium points $E_2$ is a saddle point.

***Proof:***

$a)$ The Jacobian matrix of the system (7) at the axial equilibrium point $E_1$ is given by
$J_{E_1} = \begin{bmatrix} \frac{x_1}{c+x_1}\sqrt{(1-c)^2-4(h-c)} & -\alpha \\ 0 & \rho\beta \end{bmatrix}.$
The above matrix confirms that the axial equilibrium point $E_1$ is an unstable point.

$b)$ The Jacobian matrix of the system (7) at the axial equilibrium point $E_2$ is given by
$J_{E_2} = \begin{bmatrix} -\frac{x_2}{c+x_2}\sqrt{(1-c)^2-4(h-c)} & -\alpha \\ 0 & \rho\beta \end{bmatrix}.$
The above matrix confirms that the axial equilibrium point $E_2$ is a saddle point. ∎

### 4.3. Interior equilibrium point

In Section (3), parametric conditions for the existence of interior equilibrium points have been obtained. We have also shown that the number of interior equilibrium points varies from two to zero. Now we discuss the nature of these interior equilibrium points.

**Theorem 4.3.**

$a)$ The interior equilibrium point $E_1^*$ of the system (7) is always a saddle point.

$b)$ The interior equilibrium point $E_2^*$ of the system (7) is asymptotically stable point if $x_2^*\Big(-1+\frac{h}{(c+x_2^*)^2}\Big)+\frac{\alpha\beta}{(1+\beta)^2}-\rho\beta<0$, is an unstable hyperbolic node if $x_2^*\Big(-1+\frac{h}{(c+x_2^*)^2}\Big)+\frac{\alpha\beta}{(1+\beta)^2}-\rho\beta>0$ and is a weak focus or a center if $x_2^*\Big(-1+\frac{h}{(c+x_2^*)^2}\Big)+\frac{\alpha\beta}{(1+\beta)^2}-\rho\beta=0$.

$c)$ The interior equilibrium point $E^*$ of the system (7), if exist, is asymptotically stable point if $x^*(-1+\frac{h}{(c+x^*)^2})+\frac{\alpha\beta}{(1+\beta)^2}-\rho\beta<0$, is weak focus or center if $x^*(-1+\frac{h}{(c+x^*)^2})+\frac{\alpha\beta}{(1+\beta)^2}-\rho\beta=0$ and is an unstable hyperbolic saddle node if $x^*(-1+\frac{h}{(c+x^*)^2})+\frac{\alpha\beta}{(1+\beta)^2}-\rho\beta>0$.

***Proof:***

$a)$ The Jacobian matrix of the system (7) at the equilibrium point $E_1^*$ is given by

$$J_{E_1^*} = \begin{bmatrix} x_1^*\Big(-1+\frac{h}{(c+x_1^*)^2}\Big)+\frac{\alpha\beta}{(1+\beta)^2} & \frac{-\alpha}{(1+\beta)^2} \\ \rho\beta^2 & -\rho\beta \end{bmatrix}.$$

The determinant of the Jacobian matrix $J_{E_1^*}$ is $\det(J_{E_1^*}) = \rho\beta x_1^*(1-\frac{h}{(c+x_1^*)^2})$. Using the value of $x_1^*$ from (19), we have

$$\det(J_{E_1^*}) = -\frac{\rho\beta x_1^*}{(1+\beta)(c+x_1^*)}\sqrt{(1-c+\beta-c\beta-\alpha\beta)^2-4(1+\beta)(h-c+h\beta-c\beta+c\alpha\beta)} < 0,$$

which confirms that the equilibrium point $E_1^*$ is an unstable hyperbolic saddle.

$b)$ The Jacobian matrix of the system (7) at the equilibrium point $E_2^*$ is

$$J_{E_2^*} = \begin{bmatrix} x_2^*\Big(-1+\frac{h}{(c+x_2^*)^2}\Big)+\frac{\alpha\beta}{(1+\beta)^2} & \frac{-\alpha}{(1+\beta)^2} \\ \rho\beta^2 & -\rho\beta \end{bmatrix}.$$

The determinant of the Jacobian matrix $J_{E_2^*}$ is $\det(J_{E_2^*}) = \rho\beta x_2^*(1-\frac{h}{(c+x_2^*)^2})$. Using the value of $x_2^*$ from (19), we have

$\det(J_{E_2^*}) = \frac{\rho\beta x_2^*}{(1+\beta)(c+x_2^*)}\sqrt{(1-c+\beta-c\beta-\alpha\beta)^2-4(1+\beta)(h-c+h\beta-c\beta+c\alpha\beta)} > 0,$

and the trace of $J_{E_2^*}$ is $tr(J_{E_2^*}) = x_2^*\Big(-1+\frac{h}{(c+x_2^*)^2}\Big)+\frac{\alpha\beta}{(1+\beta)^2}-\rho\beta.$

The Routh-Hurwitz criterion confirms the result.

$c)$ The Jacobian matrix of the system (7) at the equilibrium point $E^*$ is

$$J_{E^*} = \begin{bmatrix} x^*\Big(-1+\frac{h}{(c+x^*)^2}\Big)+\frac{\alpha\beta}{(1+\beta)^2} & \frac{-\alpha}{(1+\beta)^2} \\ \rho\beta^2 & -\rho\beta \end{bmatrix}.$$

The determinant of the Jacobian matrix $J_{E^*}$ is $\det(J_{E^*}) = \rho\beta x^*(1-\frac{h}{(c+x^*)^2})$. Using the value of $x^*$ from (19), we have

$\det(J_{E^*}) = \frac{\rho\beta x^*}{(1+\beta)(c+x^*)}\sqrt{(1-c+\beta-c\beta-\alpha\beta)^2-4(1+\beta)(h-c+h\beta-c\beta+c\alpha\beta)} > 0,$

and trace of the Jacobian matrix $J_{E*}$ is $tr(J_{E^*}) = x^*\Big(-1+\frac{h}{(c+x^*)^2}\Big)+\frac{\alpha\beta}{(1+\beta)^2}-\rho\beta.$

The Routh-Hurwitz criterion confirms the result. ■

The Jacobian matrix of the system (7) at the equilibrium point $\overline{E}$ is

$$J_{\overline{E}} = \begin{bmatrix} \overline{x}\Big(-1+\frac{h}{(c+\overline{x})^2}\Big)+\frac{\alpha\beta}{(1+\beta)^2} & \frac{-\alpha}{(1+\beta)^2} \\ \rho\beta^2 & -\rho\beta \end{bmatrix}.$$

Since $\frac{h}{(c+\overline{x})^2} = 1$, determinant of $J_{\overline{E}} = 0$. Thus, $\overline{E}$ is a degenerate singularity and may have complex nature. The following theorem depicts the nature of this equilibrium point.

**Theorem 4.4.**

If $\overline{E}$ exist, it is a saddle node if $\rho \neq \frac{\alpha}{(1+\beta)^2}$ and is a cusp of codimension 2 if $\rho = \frac{\alpha}{(1+\beta)^2}$.

***Proof:***

By using the transformation $\hat{x} = x - \overline{x}, \hat{y} = y - \overline{y}$ we shift the equilibrium point $\overline{E}$ of the system (7) to the origin and still denote $\hat{x}$ as $x$ and $\hat{y}$ as $y$, the system (7) can be rewritten as

$$\begin{cases} \frac{dx}{dt} = \frac{\alpha\beta}{(1+\beta)^2}x - \frac{\alpha}{(1+\beta)^2}y + \xi_1 x^2 + \xi_2 xy + \xi_3 y^2 + o^3(x,y), \\ \frac{dy}{dt} = \rho\beta^2 x - \rho\beta y - \frac{\rho\beta^2}{\overline{x}}x^2 + \frac{2\rho\beta}{\overline{x}}xy - \frac{\rho}{\overline{x}}y^2 + o^3(x,y), \end{cases} \tag{25}$$

where $\xi_1 = -1 + \frac{ch}{(c+\overline{x})^3} + \frac{\alpha\beta^2}{\overline{x}(1+\beta)^3}, \xi_2 = -\frac{2\alpha\beta}{\overline{x}(1+\beta)^3}, \xi_3 = \frac{\alpha}{\overline{x}(1+\beta)^3}$.

If $\rho \neq \frac{\alpha}{(1+\beta)^2}$, the trace of the Jacobian matrix of the system (7) at the equilibrium point $\overline{E}$ is nonzero while the determinant is zero. Hence, $\overline{E}$ is a saddle node.

If $\rho = \frac{\alpha}{(1+\beta)^2}$, the trace of the Jacobian matrix of the system (7) at the equilibrium point $\overline{E}$ is zero, hence both the eigenvalue of the Jacobian matrix of the system (7) is zero. The system (25) can be written as

$$\begin{cases} \frac{dx}{dt} = \frac{\alpha\beta}{(1+\beta)^2}x - \frac{\alpha}{(1+\beta)^2}y + \xi_1 x^2 + \xi_2 xy + \xi_3 y^2 + o^3(x,y), \\ \frac{dy}{dt} = \frac{\alpha\beta^2}{(1+\beta)^2}x - \frac{\alpha\beta}{(1+\beta)^2}y - \frac{\alpha\beta^2}{\overline{x}(1+\beta)^2}x^2 + \frac{2\alpha\beta}{\overline{x}(1+\beta)^2}xy - \frac{\alpha}{\overline{x}(1+\beta)^2}y^2 + o^3(x,y). \end{cases} \tag{26}$$

Let $T = \frac{\alpha\beta}{(1+\beta)^2}t$, then the system (26) reduces to the following system (after transformation $T$ is taken as $t$)

$$\begin{cases} \frac{dx}{dt} = x - \frac{1}{\beta}y + \overline{\xi_1}x^2 + \overline{\xi_2}xy + \overline{\xi_3}y^2 + o^3(x,y), \\ \frac{dy}{dt} = \beta x - y - \frac{\beta}{\overline{x}}x^2 + \frac{2}{\overline{x}}xy - \frac{1}{\overline{x}\beta}y^2 + o^3(x,y), \end{cases} \tag{27}$$

where $\overline{\xi_1} = \frac{(1+\beta)^2}{\alpha\beta}\big(-1 + \frac{ch}{(c+\overline{x})^3} + \frac{\alpha\beta^2}{\overline{x}(1+\beta)^3}\big), \overline{\xi_2} = -\frac{2}{\overline{x}(1+\beta)}, \overline{\xi_3} = \frac{1}{\beta(1+\beta)\overline{x}}$.

On using the transformation $x_0 = x, y_0 = x - \frac{1}{\beta}y$, then the system (27) reduces to the following system

$$\begin{cases} \frac{dx_0}{dt} = y_0 + \hat{\xi_1}x_0^2 + \hat{\xi_3}y_0^2 + o^3(x,y), \\ \frac{dy_0}{dt} = \hat{\xi_1}x_0^2 + (\hat{\xi_3} + \frac{1}{\overline{x}})y_0^2 + o^3(x,y), \end{cases} \tag{28}$$

where $\hat{\xi_1} = -\frac{\overline{x}(1+\beta)^2}{\alpha\beta(c+\overline{x})}, \hat{\xi_3} = \frac{\beta}{(1+\beta)\overline{x}}$.

On using the transformation $x_1 = x_0, y_1 = y_0 + \hat{\xi_3}y_0^2$, the system (28) reduces to

$$\begin{cases} \frac{dx_1}{dt} = y_1 + \hat{\xi_1}x_1^2 + o^3(x,y), \\ \frac{dy_1}{dt} = \hat{\xi_1}x_1^2 + (\hat{\xi_3} + \frac{1}{\overline{x}})y_1^2 + o^3(x,y). \end{cases} \tag{29}$$

On using the transformation $x_2 = x_1, y_2 = y_1 - (\hat{\xi_3} + \frac{1}{\bar{x}})x_1y_1$, the system (29) reduces to

$$\begin{cases} \frac{dx_2}{dt} = y_2 + \hat{\xi_1}x_2^2 + (\hat{\xi_3} + \frac{1}{\bar{x}})x_2y_2 + o^3(x,y), \\ \\ \frac{dy_2}{dt} = \hat{\xi_1}x_2^2 + o^3(x,y). \end{cases} \tag{30}$$

Finally, using the transformation $X = x_2 - \frac{1}{2}(\hat{\xi_3} + \frac{1}{\bar{x}})x_2^2, Y = y_2 + \hat{\xi_1}x_2^2 + o^3(x,y)$, the system (30) reduces to

$$\begin{cases} \frac{dX}{dt} = Y \\ \\ \frac{dY}{dt} = \hat{\xi_1}X_2^2 + 2\hat{\xi_1}XY + o^3(X,Y). \end{cases} \tag{31}$$

Since $\hat{\xi_1} \neq 0$, therefore the system (31) confirms that origin in $XY$ plane, that is, $\overline{E}$ in $xy$-plane is a cusp of codimension 2 (Perko (1996)). ■

## 5. Bifurcation Analysis

In this section, we investigate the bifurcations that take place in system (7) with the original parameters varying.

### 5.1. Hopf bifurcation

In Theorem 4.3$(a)$, it is proved that if both interior equilibrium points $E_1^*$ and $E_2^*$ of the system (7) exist, then $E_1^*$ is always a saddle point while $E_2^*$ is a weak focus or a center provided $x_2^*\Big(-1 + \frac{h}{(c+x_2^*)^2}\Big) + \frac{\alpha\beta}{(1+\beta)^2} - \rho\beta = 0$. Now, we show that system (7) undergoes to a hopf bifurcation.

If $\rho$ is considered as a bifurcation parameter, then the threshold magnitude is $\rho = \rho^{[hf]} = \frac{x_2^*}{\beta}\Big(-1 + \frac{h}{(c+x_2^*)^2}\Big) + \frac{\alpha}{(1+\beta)^2}$, and for $\rho = \rho^{[hf]}$, we have

$(a)$ $\det(J_{E_2^*}) > 0$; $(b)$ $tr(J_{E_2^*}) = 0$; and $(c)$ $\Big[\frac{\partial}{\partial\rho}(TrJ_{E_2^*})\Big] = -\beta \neq 0.$

Thus, the transversality condition for the hopf bifurcation is satisfied (condition $(c)$). This guarantees the existence of hopf bifurcation, that is, a limit cycle exists around the interior equilibrium point $E_2^*(x_2^*, y_2^*)$.

Next, stability of limit cycle is discussed by computing the first Lyapunov number (Perko (1996)) $\sigma$ at interior equilibrium point $E_2^*(x_2^*, y_2^*)$. Using the transformation $x = u - x_2^*,\ y = v - y_2^*$, the system (7), in the vicinity of origin can be written as

$$\frac{du}{dt} = a_{10}u + a_{01}v + a_{20}u^2 + a_{11}uv + a_{02}v^2 + a_{30}u^3 + a_{21}u^2v + a_{12}uv^2 + a_{03}v^3 + P(u,v),$$
$$\frac{dv}{dt} = b_{10}u + b_{01}v + b_{20}u^2 + b_{11}uv + b_{02}v^2 + b_{30}u^3 + b_{21}u^2v + b_{12}uv^2 + b_{03}v^3 + Q(u,v),$$

where $a_{10} = x_2^*\Big(-1+\frac{h}{(c+x_2^*)^2}\Big)+\frac{\alpha\beta}{(1+\beta)^2}$, $a_{01} = -\frac{\alpha}{(1+\beta)^2}$, $a_{20} = -1+\frac{hc}{(c+x_2^*)^3}+\frac{\alpha\beta^2}{(1+\beta)^3 x_2^*}$, $a_{11} = -\frac{2\alpha\beta}{(1+\beta)^3 x_2^*}$, $a_{02} = \frac{\alpha}{(1+\beta)^3 x_2^*}$, $a_{30} = -\frac{\alpha\beta^2}{(1+\beta)^4 x_2^{*2}}-\frac{hc}{(c+x_2^*)^4}$, $a_{21} = \frac{2\alpha\beta-\alpha\beta^2}{(1+\beta)^4 x_2^{*2}}$, $a_{12} = \frac{2\alpha\beta-\alpha}{(1+\beta)^4 x_2^{*2}}$, $a_{03} = -\frac{\alpha}{(1+\beta)^4 x_2^{*2}}$, $b_{10} = \rho\beta^2$, $b_{01} = -\rho\beta$, $b_{20} = -\frac{\rho\beta^2}{x_2^*}$, $b_{11} = \frac{2\rho\beta}{x_2^*}$, $b_{02} = -\frac{\rho}{x_2^*}$, $b_{30} = \frac{\rho\beta^2}{x_2^{*2}}$, $b_{21} = -\frac{2\rho\beta}{x_2^{*2}}$, $b_{12} = \frac{\rho}{x_2^{*2}}$, $b_{03} = 0$, $P(u,v) = \sum_{i+j=4}^{\infty} a_{ij}u^i v^j$ and $Q(u,v) = \sum_{i+j=4}^{\infty} b_{ij}u^i v^j$.

Hence, the first Lyapunov number $\sigma$ for the planer system is

$$\sigma = -\frac{3\pi}{2a_{01}\Delta^{3/2}}\Big\{\big[a_{10}b_{10}(a_{11}^2+a_{11}b_{02}+a_{02}b_{11})+a_{10}a_{01}(b_{11}^2+a_{20}b_{11}+a_{11}b_{02})$$
$$+b_{10}^2(a_{11}a_{02}+2a_{02}b_{02})-2a_{10}b_{10}(b_{02}^2-a_{20}a_{02})-2a_{10}a_{01}(a_{20}^2-b_{20}b_{02})$$
$$-a_{01}^2(2a_{20}b_{20}+b_{11}b_{20})+(a_{01}b_{10}-2a_{10}^2)(b_{11}b_{02}-a_{11}a_{20})\big]$$
$$-(a_{10}^2+a_{01}b_{10})[3(b_{10}b_{03}-a_{01}a_{30})+2a_{10}(a_{21}+b_{12})+(b_{10}a_{12}-a_{01}b_{21})]\Big\},$$

where $\Delta = \frac{\rho\beta x_1^*}{(1+\beta)(c+x_1^*)}\sqrt{(1-c+\beta-c\beta-\alpha\beta)^2-4(1+\beta)(h-c+h\beta-c\beta+c\alpha\beta)}$.

The above discussions can be summarized as follows

**Theorem 5.1.**

The system (7) undergoes a hopf bifurcation at the point $E_2^*$ whenever $x_2^*\Big(-1+\frac{h}{(c+x_2^*)^2}\Big)+\frac{\alpha\beta}{(1+\beta)^2}-\rho\beta = 0$. Moreover, an unstable (stable) limit cycle arises around the equilibrium point $E_2^*$ as $\sigma > 0 (\sigma < 0)$.

## 5.2. Saddle-node bifurcation

It has been shown that the system (7) has two axial equilibrium points $E_1$ and $E_2$, if $\frac{1+\beta-\alpha\beta}{1+\beta} < c < 1$ and $h < \Big(\frac{1+c}{2}\Big)^2$. These two axial equilibrium points coincide, if $h = \Big(\frac{1+c}{2}\Big)^2$ and no axial equilibrium point exists, if $h > \Big(\frac{1+c}{2}\Big)^2$. Thus, the number of axial equilibrium points vary from two to zero as the harvesting parameter $h$ crosses the critical value $h = \Big(\frac{1+c}{2}\Big)^2$ from left to right, and so,

$$SN_1 = \Big\{(\alpha,\beta,c,h,\rho) : \frac{1+\beta-\alpha\beta}{1+\beta} < c < 1, h = \Big(\frac{1+c}{2}\Big)^2\Big\}$$

be a saddle-node bifurcation surface.

It is proved in Theorem (4.4) that if $\overline{E}$ exist, then it is a saddle-node when $\rho \neq \frac{\alpha}{(1+\beta)^2}$. It has also been shown that the system (7) has two interior equilibrium point $E_1^*$ and $E_2^*$, whenever $c < \frac{1+\beta-\alpha\beta}{1+\beta}$ and $c \leq h < \frac{1}{4}\Big(1+c-\frac{\alpha\beta}{1+\beta}\Big)^2$. These two interior equilibrium points coincide to a unique interior equilibrium point $\overline{E} = (\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{1 - c + \beta - c\beta - \alpha\beta}{2}, \ \ \overline{y} = \beta\overline{x},$$

whenever $c < h = \frac{1}{4}\Big(1 + c - \frac{\alpha\beta}{1+\beta}\Big)^2$ (this value of $h$ is known as critical harvesting rate and written as $h^{[SN]}$) and no interior equilibrium point exists, whenever $h > \frac{1}{4}\Big(1 + c - \frac{\alpha\beta}{1+\beta}\Big)^2$. Thus, the number of interior equilibrium points vary from two to zero as the harvesting parameter $h$ crosses the critical value $h^{[SN]}$ from left to right, and so,

$$SN_2 = \Big\{(\alpha, \beta, c, h, \rho) : c < \frac{1 + \beta - \alpha\beta}{1 + \beta}, c < h = \frac{1}{4}\Big(1 + c - \frac{\alpha\beta}{1+\beta}\Big)^2, \rho \neq \frac{\alpha}{(1+\beta)^2}\Big\},$$

be another saddle-node bifurcation surface.

Considering harvesting parameter $h$ as bifurcation parameter, Sotomayor's theorem is used in the following to guarantee that the system (7) undergoes a saddle-node bifurcation.

**Theorem 5.2.**

The system (7) undergoes a saddle-node bifurcation with respect to the bifurcation parameter $h$ around the equilibrium point $\overline{E} = (\overline{x}, \overline{y})$ if $c < \frac{1+\beta-\alpha\beta}{1+\beta}, c < h = \frac{1}{4}\Big(1 + c - \frac{\alpha\beta}{1+\beta}\Big)^2, \rho > \frac{\alpha}{(1+\beta)^2}$.

***Proof:***

The Jacobian matrix of the system (7) at the equilibrium point $\overline{E}(\overline{x}, \overline{y})$ is given by

$$J_{\overline{E}} = DG(\overline{x}, \overline{y}) = \begin{bmatrix} \frac{\alpha\beta}{(1+\beta)^2} & -\frac{\alpha}{(1+\beta)^2} \\ \rho\beta^2 & -\rho\beta \end{bmatrix},$$

where $G(x, y, h) = (g_1, g_2)^T, \ g_1 = xf_1(x, y), \ g_2 = yf_2(x, y)$, and $f_1, f_2$ are defined in equation (7). The determinant of $J_{\overline{E}}$ is zero and trace is $\frac{\alpha\beta}{(1+\beta)^2} - \rho\beta < 0$ as $\rho > \frac{\alpha}{(1+\beta)^2}$, and so, one of the eigenvalues is zero and other has negative real part. Hence, the parametric conditions of the Sotomayor's theorem is satisfied. Let $V$ and $W$ be the eigenvectors corresponding to zero eigenvalue for $J_{\overline{E}}$ and the transpose $(J_{\overline{E}})^T$, respectively. A simple computation yields

$$V = \begin{bmatrix} 1 \\ \beta \end{bmatrix} \text{ and } W = \begin{bmatrix} 1 \\ -\frac{\alpha}{\rho\beta(1+\beta)^2} \end{bmatrix}.$$

On computing, we get

$$G_h(\overline{x}, \overline{y}, h^{[SN]}) = \begin{bmatrix} -\frac{\overline{x}}{c+\overline{x}} \\ 0 \end{bmatrix},$$

$$D^2G_h(\overline{x}, \overline{y}, h^{[SN]})(V, V) = \begin{bmatrix} -2 + \frac{2\alpha\beta}{(1+\beta)^2\overline{x}} - \frac{2\alpha\beta}{(1+\beta)^3\overline{x}} + \frac{2h}{(c+\overline{x})^2} - \frac{2h\overline{x}}{(c+\overline{x})^3} - \frac{2\alpha\beta^2}{(1+\beta)^3\overline{x}} \\ 0 \end{bmatrix}.$$

Noticed that $\frac{h}{(c+\overline{x})^2} = 1$, we have

$$W^T.G_h(\overline{x}, \overline{x}, h^{[SN]}) = -\frac{\overline{x}}{c+\overline{x}} \neq 0,$$

$$W^T.D^2G(\overline{x}, \overline{x}, h^{[SN]})(V, V) = -2(1 - \frac{hc}{(c+\overline{x})^3}) = -\frac{2\overline{x}}{c+\overline{x}} \neq 0.$$

Thus, the transversality conditions for the saddle-node bifurcation are satisfied. This guarantees the existence saddle-node bifurcation at the equilibrium point $\overline{E} = (\overline{x}, \overline{y})$. ■

### 5.3. Bogdanov-Takens bifurcation

The occurrence of codimension one bifurcations for the system (7) have been discussed so far. In Theorem 4.4, it is shown that if $\overline{E}$ exist, then it is a cusp of codimension 2 whenever $\rho = \frac{\alpha}{(1+\beta)^2}$. Therefore, codimension 2 bifurcation (Bogdanov-Takens bifurcation of codimension 2) occurs for the system. Below the normal form of the Bogdanov-Takens bifurcation has been derived by using a series of nontrivial transformations (Xiao and Ruan (1999)).

**Theorem 5.3.**

The system (7) undergoes a Bogdanov-Takens bifurcation with respect to the bifurcation parameters $h$ and $\rho$ around the equilibrium point $\overline{E} = (\overline{x}, \overline{y})$, whenever $c < \frac{1+\beta-\alpha\beta}{1+\beta}, c < h = \frac{1}{4}\left(1 + c - \frac{\alpha\beta}{1+\beta}\right)^2, \rho = \frac{\alpha}{(1+\beta)^2}$. Also in the small neighbourhood of the point $\overline{E}$, the system (7) is topologically equivalent to the following model

$$\begin{cases} \frac{dZ_1}{dt} = Z_2, \\ \\ \frac{dZ_2}{dt} = \mu_1(\lambda_1, \lambda_2) + \mu_2(\lambda_1, \lambda_2)Z_2 + Z_1^2 + Z_1Z_2. \end{cases} \tag{32}$$

Moreover, the following bifurcation curves divides the bifurcation plane into four regions.

Saddle-node curve: $SN = \{(\lambda_1, \lambda_2) : \mu_1(\lambda_1, \lambda_2) = 0\}$,

Hopf bifurcation curve: $H = \{(\lambda_1, \lambda_2) : \mu_2(\lambda_1, \lambda_2) = \frac{\gamma_{11}}{\sqrt{\gamma_{20}}}\sqrt{-\mu_1(\lambda_1, \lambda_2)}, \mu_2(\lambda_1, \lambda_2) < 0\}$,

Homoclinic bifurcation curve: $HL = \{(\lambda_1, \lambda_2) : \mu_2(\lambda_1, \lambda_2) = \frac{5\gamma_{11}}{7\sqrt{\gamma_{20}}}\sqrt{-\mu_1(\lambda_1, \lambda_2)}, \mu_2(\lambda_1, \lambda_2) < 0\}$.

***Proof:***

Let the parameters $h$ and $\rho$ vary in a small neighbourhood of Bogdanov-Taken point (in brief, BT-point) $(h_0, \rho_0)$, where $h_0$ and $\rho_0$ are the threshold magnitude of bifurcation parameters $h$ and $\rho$ respectively such that $[\det(J_{\overline{E}})]_{(h_0,\rho_0)} = 0$ and $[tr(J_{\overline{E}})]_{(h_0,\rho_0)} = 0$. Also suppose $(h_0 + \lambda_1, \rho_0 + \lambda_2)$ be a point of the neighbourhood of the BT-point $(h_0, \rho_0)$, where $\lambda_1, \lambda_2$ are small. Thus, the system

(7) can be written as follows

$$\begin{cases} \frac{dx}{dt} = x(1-x) - \frac{\alpha xy}{x+y} - \frac{h_0 x}{c+x} - \frac{\lambda_1 x}{c+x}, \\ \\ \frac{dy}{dt} = (\rho_0 + \lambda_2)(\beta - \frac{y}{x})y. \end{cases} \tag{33}$$

Noticed that the system (33) is $C^\infty$ smooth with respect to the variables $x, y$ in a small neighborhood of $(h_0, \rho_0)$.

Using the transformation $x_1 = x - \overline{x}, x_2 = y - \overline{y}$, the system (33) gets modified to

$$\begin{cases} \frac{dx_1}{dt} = a_{00}(\lambda) + a_{10}(\lambda)x_1 + a(\lambda)x_1 + b(\lambda)x_2 + \frac{1}{2}p_{20}(\lambda)x_1^2 + p_{11}(\lambda)x_1x_2 + \frac{1}{2}p_{02}(\lambda)x_2^2 + R_1, \\ \\ \frac{dx_2}{dt} = b_{00}(\lambda) + b_{10}(\lambda)x_1 + b_{01}(\lambda)x_2 + \frac{1}{2}q_{20}(\lambda)x_1^2 + q_{11}(\lambda)x_1x_2 + \frac{1}{2}q_{02}(\lambda)x_2^2 + R_2, \end{cases} \tag{34}$$

where $a_{00}(\lambda) = -\frac{\lambda_1\overline{x}}{c+\overline{x}}$, $a_{10}(\lambda) = -\frac{\lambda_1 c}{(c+\overline{x})^2}$, $a(\lambda) = \frac{\alpha\beta}{(1+\beta)^2}$, $b(\lambda) = -\frac{\alpha}{(1+\beta)^2}$, $p_{20}(\lambda) = \frac{2\alpha\beta^2}{(1+\beta)^3\overline{x}} + \frac{2\lambda_1 c}{(c+\overline{x})^3} - \frac{2\overline{x}}{c+\overline{x}}$, $p_{11}(\lambda) = -\frac{2\alpha\beta}{(1+\beta)^3\overline{x}}$, $p_{02}(\lambda) = \frac{2\alpha}{(1+\beta)^3\overline{x}}$, $b_{00}(\lambda) = 0$, $b_{10}(\lambda) = \beta^2(\rho+\lambda_2)$, $b_{01}(\lambda) = -\beta(\rho+\lambda_2)$, $q_{20}(\lambda) = -\frac{2(\rho+\lambda_2)\overline{y}^2}{\overline{x}^3}$, $q_{11}(\lambda) = \frac{2(\rho+\lambda_2)\overline{y}}{\overline{x}^2}$, $q_{02}(\lambda) = -\frac{2(\rho+\lambda_2)}{\overline{x}}$, and $R_1 = R_1(x_1, x_2), R_2 = R_2(x_1, x_2)$ are the power series in $(x_1, x_2)$ with powers $x_1^i x_2^j$ satisfying $i + j \geq 3$.

Now, on introducing the affine transformation $y_1 = x_1,\ y_2 = ax_1 + bx_2$ in the system (34), we get

$$\begin{cases} \frac{dy_1}{dt} = \alpha_{00}(\lambda) + \alpha_{10}(\lambda)y_1 + y_2 + \frac{1}{2}\alpha_{20}(\lambda)y_1^2 + \frac{1}{2}\alpha_{02}(\lambda)y_2^2 + \overline{R}_1(y_1, y_2), \\ \\ \frac{dy_2}{dt} = \beta_{00}(\lambda) + \beta_{10}(\lambda)y_1 + \beta_{01}(\lambda)y_2 + \frac{1}{2}\beta_{20}(\lambda)y_1^2 + \frac{1}{2}\beta_{02}(\lambda)y_2^2 + \overline{R}_2(y_1, y_2), \end{cases} \tag{35}$$

where $\alpha_{00}(\lambda) = -\frac{\lambda_1\overline{x}}{c+\overline{x}}$, $\alpha_{10}(\lambda) = \frac{\alpha\beta}{(1+\beta)^2}$, $\alpha_{20}(\lambda) = \frac{2\lambda_1 c}{(c+\overline{x})^3} - \frac{2\overline{x}}{c+\overline{x}}$, $\alpha_{02}(\lambda) = \frac{2(1+\beta)}{\alpha\overline{x}}$, $\beta_{00}(\lambda) = -\frac{a\lambda_1\overline{x}}{c+\overline{x}}$, $\beta_{10}(\lambda) = \frac{\alpha^2\beta^2}{(1+\beta)^4}$, $\beta_{01}(\lambda) = \frac{\alpha\beta}{(1+\beta)^2} - \beta(\rho + \lambda_2)$, $\beta_{20}(\lambda) = \frac{2\alpha\beta}{(1+\beta)^2(c+\overline{x})}\left(\frac{\lambda_1 c}{(c+\overline{x})^2} - \overline{x}\right)$, $\beta_{02}(\lambda) = -\frac{2}{b\overline{x}}(\rho + \lambda_2 + \frac{\alpha\beta}{(1+\beta)^3})$, and $\overline{R_1}, \overline{R_2}$ are the power series in $(y_1, y_2)$ with powers $y_1^i y_2^j$ satisfying $i + j \geq 3$.

Consider the $C^\infty$ change of coordinates in the small neighbourhood of $(0, 0)$ : $z_1 = y_1 - \frac{1}{2}\beta_{02}y_1^2,\ z_2 = y_2 + \frac{1}{2}\alpha_{20}y_1^2 + \frac{1}{2}\alpha_{02}y_2^2$, which transforms the system (35) into

$$\begin{cases} \frac{dz_1}{dt} = r_{00}(\lambda) + r_{10}(\lambda)z_1 + z_2 + r_{20}(\lambda)z_1^2 + r_{11}(\lambda)z_1z_2 + \overline{\overline{R_1}}(z_1, z_2), \\ \\ \frac{dz_2}{dt} = s_{00}(\lambda) + s_{10}(\lambda)z_1 + s_{01}(\lambda)z_2 + s_{20}(\lambda)z_1^2 + s_{11}(\lambda)z_1z_2 + s_{02}(\lambda)z_2^2 + \overline{\overline{R_2}}(z_1, z_2), \end{cases} \tag{36}$$

where $r_{00} = \alpha_{00}$, $r_{10} = \alpha_{10} - \alpha_{00}\beta_{02}$, $r_{20} = -\frac{1}{2}\beta_{02}(\beta_{02}\alpha_{00} + \alpha_{10})$, $r_{11} = -\beta_{02}$, $s_{00} = \beta_{00}$, $s_{10} = \beta_{10} + \alpha_{00}\alpha_{20}$, $s_{01} = \beta_{01} + \alpha_{02}\beta_{00}$, $s_{20} = \frac{1}{2}(\beta_{02}\beta_{10} + \alpha_{00}\alpha_{20}\beta_{02} - \alpha_{20}\beta_{01} - \alpha_{20}\beta_{00}\alpha_{02} + \beta_{20} + 2\alpha_{10}\alpha_{20})$, $s_{11} = \alpha_{20} + \alpha_{02}\beta_{10}$, $s_{02} = \frac{1}{2}(\beta_{02} + \alpha_{02}\beta_{01} - \alpha_{02}^2\beta_{00})$ and $\overline{\overline{R_1}}$, $\overline{\overline{R_2}}$ are the power series in $(z_1, z_2)$ with powers $z_1^i z_2^j$ satisfying $i + j \geq 3$.

Next, consider $C^\infty$ change of coordinates in the small neighbourhood of $(0, 0)$: $u_1 = z_1 - \frac{1}{2}(r_{11} +$

$s_{02})z_1^2,\;\; u_2 = z_2 + r_{20}z_1^2 - s_{02}z_1z_2$. Then, the system (36) becomes

$$\begin{cases} \frac{du_1}{dt} = \xi_{00} + \xi_{10}u_1 + u_2 + \xi_{20}u_1^2 + \hat{R}_1(u_1, u_2), \\ \\ \frac{du_2}{dt} = \eta_{00} + \eta_{10}u_1 + \eta_{01}u_2 + \eta_{20}u_1^2 + \eta_{11}u_1u_2 + \hat{R}_2(u_1, u_2), \end{cases} \tag{37}$$

where $\xi_{00} = r_{00},\;\; \xi_{10} = r_{10} - r_{00}(r_{11} + s_{02}),\;\; \xi_{20} = \frac{1}{2}(r_{11} + s_{02})(r_{10} - r_{00}(r_{11} + s_{02})) - r_{10}(r_{11} + s_{02})$, $\eta_{00} = s_{00},\;\; \eta_{10} = s_{10} + 2r_{20}r_{00} - s_{02}s_{00},\;\; \eta_{01} = s_{01} - s_{02}r_{00}, \eta_{20} = \frac{1}{2}(r_{11} + s_{02})(s_{10} + 2r_{20}r_{00} - s_{02}s_{00}) - r_{20}(s_{01} - s_{02}r_{00}) + (s_{20} + 2r_{20}r_{10} - s_{02}s_{10}), \eta_{11} = s_{11} + 2r_{20} - s_{01}s_{02} - s_{02}r_{10} + s_{02}(s_{01} - s_{02}r_{00})$, and $\hat{R}_1$, $\hat{R}_2$ are the power series in $(u_1, u_2)$ with powers $u_1^i u_2^j$ satisfying $i + j \geq 3$.

Again consider $C^\infty$ change of coordinates in the small neighbourhood of $(0, 0) : v_1 = u_1,\;\; v_2 = \xi_{00} + \xi_{10}u_1 + u_2 + \xi_{20}u_1^2$ which transformed the system (37) into

$$\begin{cases} \frac{dv_1}{dt} = v_2 + s_1(v_1, v_2), \\ \\ \frac{dv_2}{dt} = \gamma_{00} + \gamma_{10}v_1 + \gamma_{01}v_2 + \gamma_{20}v_1^2 + \gamma_{11}v_1v_2 + s_2(v_1, v_2), \end{cases} \tag{38}$$

where $\gamma_{00} = \eta_{00} - \eta_{01}\xi_{00},\;\;\; \gamma_{10} = \eta_{10} - \eta_{01}\xi_{10} - \xi_{00}\eta_{11},\;\;\; \gamma_{01} = \xi_{10} + \eta_{01}, \gamma_{20} = \eta_{20} - \eta_{01}\xi_{20} - \xi_{10}\eta_{11},\;\;\; \gamma_{11} = \eta_{11} + 2\xi_{20}$ and $s_1(v_1, v_2), s_2(v_1, v_2)$ are the power series in $(v_1, v_2)$ with powers $v_1^i v_2^j$ satisfying $i + j \geq 3$.

Next, we consider $C^\infty$ change of coordinates in the small neighbourhood of $(0, 0) : w_1 = v_1,\; w_2 = v_2 + s_1(v_1, v_2)$ which transforms the system (38) into

$$\begin{cases} \frac{dw_1}{dt} = w_2, \\ \\ \frac{dw_2}{dt} = \gamma_{00} + \gamma_{10}w_1 + \gamma_{01}w_2 + \gamma_{20}w_1^2 + \gamma_{11}w_1w_2 \\ \qquad +F_1(w_1) + w_2F_2(w_1) + w_2^2F_3(w_1, w_2), \end{cases} \tag{39}$$

where $F_1, F_2$ and $F_3$ are the power series in $w_1$ and $(w_1, w_2)$ with powers $w_1^{k_1}, w_1^{k_2}$ and $w_1^i w_2^j$ satisfying $k_1 \geq 3, k_2 \geq 2$ and $i + j \geq 1$, respectively.

It is cumbersome to obtain the sign of $\gamma_{20}(0)$ analytically. Therefore, we use numerical simulation. We take $\alpha = 0.9,\;\; \beta = 0.5,\;\; \rho = 0.4,\;\; h = 0.16,\;\; c = 0.1$. It is easy to verify that for these values system 7 has a unique interior equilibrium point which is a cusp of codimension 2. Also

$\gamma_{00} = (-0.15\lambda_1 + 1.75\lambda_1^2 + 5.20833\lambda_1^3) + (-0.375\lambda_1 + 3.125\lambda_1^2)\lambda_2,\;\;\; \gamma_{10} = (0.04 - 18.0556\lambda_1^2 - 34.7222\lambda_1^3) + (0.1 + 0.833333\lambda_1 - 59.0278\lambda_1^2 - 86.8056\lambda_1^3)\lambda_2 + (2.08333\lambda_1 - 52.0833\lambda_1^2)\lambda_2^2,\;\;\; \gamma_{01} = (0.2 + 5\lambda_1 + 13.8889\lambda_1^2) + (-0.5 + 8.33333\lambda_1)\lambda_2,\;\;\; \gamma_{20} = (-0.15 - 3.4375\lambda_1 + 294.271\lambda_1^2 - 769.596\lambda_1^3 + 160.751\lambda_1^4 + -669.796\lambda_1^5) + (-0.486111 + 4.36921\lambda_1 + 1552.13\lambda_1^2 - 2780.99\lambda_1^3 + 803.755\lambda_1^4)\lambda_2 + (-0.277778 + 76.3889\lambda_1 + 2883.87\lambda_1^2 - 2652.39\lambda_1^3)\lambda_2^2 + (109.954\lambda_1 + 1880.79\lambda_1^2)\lambda_2^3,\;\;\; \gamma_{11} = (-2.83333 + 88.3102\lambda_1 - 123.457\lambda_1^2 + 128.601\lambda_1^3) + (-2.22222 + 370.37\lambda_1 - 308.642\lambda_1^2)\lambda_2 + 393.519\lambda_1\lambda_1^2$.

We have $\gamma_{20}(0) = -0.15 < 0$. To make it positive we consider $Z_1 = -w_1,\;\; Z_2 = w_2,\;\; T = -t$.

The system (39) can be written as

$$\begin{cases} \frac{dZ_1}{dT} = Z_2, \\ \frac{dZ_2}{dT} = -\gamma_{00} + \gamma_{10} Z_1 - \gamma_{20} Z_1^2 + R_1(Z_1) - \gamma_{01} Z_2 + \gamma_{11} Z_1 Z_2 \\ \qquad +Z_2 R_2(Z_1) + Z_2^2 R_3(Z_1, Z_2), \end{cases} \tag{40}$$

where $R_1, R_2$ and $R_3$ are the power series in $Z_1$ and $(Z_1, Z_2)$ with powers $Z_1^{k_1}, Z_1^{k_2}$ and $Z_1^i Z_2^j$ satisfying $k_1 \geq 3, k_2 \geq 2$ and $i + j \geq 1$, respectively.

Applying the Malgrange preparation theorem (Chow and Hale (1983)) on

$$-\gamma_{00} + \gamma_{10} Z_1 - \gamma_{20} Z_1^2 + R_1(Z_1),$$

we have

$$-\gamma_{00} + \gamma_{10} Z_1 - \gamma_{20} Z_1^2 + R_1(Z_1) = \Big(Z_1^2 - \frac{\gamma_{10}}{\gamma_{20}} Z_1 + \frac{\gamma_{00}}{\gamma_{20}}\Big) B_1(w_1, \lambda),$$

where $B_1(0, \lambda) = -\gamma_{20}$ and $B_1$ is a power series of $Z_1$ whose coefficients depend on parameters $(\lambda_1, \lambda_2)$.

Let $X_1 = Z_1,\ X_2 = \frac{Z_2}{\sqrt{-\gamma_{20}}}$, and $d\tau = \sqrt{-\gamma_{20}} dT$, then the system (40) gets modified to

$$\begin{cases} \frac{dX_1}{d\tau} = X_2, \\ \frac{dX_2}{d\tau} = \frac{\gamma_{00}}{\gamma_{20}} - \frac{\gamma_{10}}{\gamma_{20}} X_1 - \frac{\gamma_{01}}{\sqrt{-\gamma_{20}}} X_2 + X_1^2 + \frac{\gamma_{11}}{\sqrt{-\gamma_{20}}} X_1 X_2 + \overline{S}(X_1, X_2, \lambda), \end{cases} \tag{41}$$

where $\overline{S}(X_1, X_2, 0)$ is a power series in $(X_1, X_2)$ with powers $X_1^i X_2^j$ satisfying $i + j \geq 3$ with $j \geq 2$.

Applying the parameter dependent affine transformation $Y_1 = X_1 - \frac{\gamma_{10}}{2\gamma_{20}},\ Y_2 = X_2$ in the system (41) and using Taylor series expansion, we get

$$\begin{cases} \frac{dY_1}{d\tau} = Y_2, \\ \frac{dY_2}{d\tau} = \mu_1(\lambda_1, \lambda_2) + \mu_2(\lambda_1, \lambda_2) Y_2 + Y_1^2 - 7.31564 Y_1 Y_2 + \overline{\overline{S}}(Y_1, Y_2, \mu), \end{cases} \tag{42}$$

where $\mu_1(\lambda_1, \lambda_2) = \frac{\gamma_{00}}{\gamma_{20}} - \frac{\gamma_{10}^2}{4\gamma_{20}^2}, \quad \mu_2(\lambda_1, \lambda_2) = -\frac{\gamma_{01}}{\sqrt{-\gamma_{20}}} + \frac{\gamma_{11}\gamma_{00}}{2(-\gamma_{20})^{\frac{3}{2}}}$ and $\overline{\overline{S}}(X_1, X_2, 0)$ is a power series in $(Y_1, Y_2)$ with powers $Y_1^i Y_2^j$ satisfying $i + j \geq 3$ with $j \geq 2$.

The system (42) is strongly topologically equivalent to the normal form of the Bogdanov-Takens bifurcation as given below

$$\begin{cases} \frac{dZ_1}{dt} = Z_2, \\ \frac{dZ_2}{dt} = \mu_1(\lambda_1, \lambda_2) + \mu_2(\lambda_1, \lambda_2) Z_2 + Z_1^2 + Z_1 Z_2. \end{cases} \tag{43}$$

The determinant of the matrix $\begin{bmatrix} \frac{\partial \mu_1}{\partial \lambda_1} & \frac{\partial \mu_1}{\partial \lambda_2} \\ \frac{\partial \mu_2}{\partial \lambda_1} & \frac{\partial \mu_2}{\partial \lambda_2} \end{bmatrix} = 2.0757$. Thus, system (7) undergoes to Bogdanov-Takens bifurcation. There exist bifurcation curves which divides the bifurcation plane into four regions (Perko (1996)). The local representations of the bifurcation curves in the $\lambda_1 \lambda_2$ plane are

Saddle-node curve: $SN = \{(\lambda_1, \lambda_2) : \mu_1(\lambda_1, \lambda_2) = 0\}$,

Hopf bifurcation curve: $H = \{(\lambda_1, \lambda_2) : \mu_2(\lambda_1, \lambda_2) = -7.31564\sqrt{-\mu_1(\lambda_1, \lambda_2)}, \mu_2(\lambda_1, \lambda_2) < 0\}$,

Homoclinic bifurcation curve:
$HL = \{(\lambda_1, \lambda_2) : \mu_2(\lambda_1, \lambda_2) = -5.22546\sqrt{-\mu_1(\lambda_1, \lambda_2)}, \mu_2(\lambda_1, \lambda_2) < 0\}$. ■

These bifurcation curves have been depicted in the figure 3a and the possible phase portrait diagrams in the small neighbourhood of the interior equilibrium point $\overline{E}$ are shown in Figure 3 (b,c,d,e,f).

## 6. Bionomic equilibria

An equilibrium point of the system (6) which is biological equilibrium point as well as economic equilibrium point is referred to as bionomic equilibrium point of the system (6), that is, the intersection point of the zero growth isoclines and zero profit line (a curve at which total revenue obtained by selling the harvested biomass equals the total cost for the effort devoted to harvesting) is referred to as bionomic equilibrium point.

Let $p$ be the cost price per unit biomass of the prey species, $c$ be the constant harvesting cost per unit effort, then the economic rent is given by

$$\pi(N, P, E) = \Big(\frac{pqN}{m_1E + m_2N} - c\Big)E, \tag{44}$$

where $N$ represents the density of prey, $P$ represents the density of predator and $E$ represents the effort applied to harvest. The bionomic equilibrium point $(N_\infty, P_\infty, E_\infty)$ is the positive solution of the curves $\frac{dN}{dT} = 0$, $\frac{dP}{dT} = 0$ and $\pi = 0$, that is, the positive solution of the system

$$\begin{cases} r(1 - \frac{N}{K}) - \frac{mP}{AP+N} - \frac{qE}{m_1E+m_2N} = 0, \\ 1 - \frac{P}{bN} = 0, \\ \frac{pqN}{m_1E+m_2N} - c = 0. \end{cases} \tag{45}$$

The positive solutions of (45) are

$$(N_\infty, P_\infty, E_\infty) = \left(\frac{k\left[(Ab+1)(rm_1P + cm_2 - pq) - bmm_1p\right]}{(Ab+1)rm_1p}, bN_\infty, \frac{pq - cm_2}{cm_1}N_\infty\right), \tag{46}$$

provided $\frac{cm_2}{p} < q < \frac{(Ab+1)(rm_1p+cm_2)-bmm_1P}{(Ab+1)p}$.

## 7. Optimal Harvesting

The objective of the harvesting agency is to find an optimal harvesting policy which maximizes the total discount net revenue derived from exploitation of renewable resources. Using the same procedure as give in (Gupta et al. (2012)), we find a relation between optimum prey value and optimum harvesting effort.

The present value of continuous time stream of revenues is given by

$$J(N,P,E,T)=\int_0^{\infty}\pi(N,P,E,T)e^{-\delta T}dT, \tag{47}$$

where $\pi(N,P,E)=\Big(\frac{pqN}{m_1E+m_2N}-c\Big)E$ and $\delta$ is the continuous annual discount rate.

Now, the problem is to

$$\left.\begin{aligned}&\text{Max.}(\pi(N,P,E))\\ &\text{subject to}\\ &\frac{dN}{dT}=\Big(r\Big(1-\frac{N}{k}\Big)-\frac{mP}{AP+N}-\frac{qE}{m_1E+m_2N}\Big)N,\\ &\frac{dP}{dT}=s\Big(1-\frac{P}{bN}\Big)P,\\ &0\le E(T)\le E_{\max},\quad (N,P)\neq(0,0),\ N(0)=N_0,\ P(0)=P_0,\end{aligned}\right\}$$

by invoking Pontryagin's maximum principle (Pontryagin et al. (1962)). Here $E_{\max}$ is a feasible upper limit for the harvesting effort.

To use the Pontryagin's maximum principle (Pontryagin et al. (1962)), we define the Hamiltonian $H$ for the above maximization problem as,

$$\begin{aligned}H=&\Big(\frac{pqN}{m_1E+m_2N}-c\Big)Ee^{-\delta T}+\lambda_1\left(r\left(1-\frac{N}{k}\right)-\frac{mP}{AP+N}-\frac{qE}{m_1E+m_2N}\right)N\\ &+\lambda_2 s\left(1-\frac{P}{bN}\right)P,\end{aligned} \tag{48}$$

where $\lambda_1$ and $\lambda_2$ are the adjoint variables. The maximization condition of $H$ yields

$$\lambda_1e^{\delta T}=p-\frac{c(m_1E+m_2N)^2}{qm_2N^2}, \tag{49}$$

where the term $\lambda_1e^{\delta T}$ is known as shadow price.

The adjoint equations $\frac{d\lambda_1}{dT}=-\frac{\partial H}{\partial N}$, $\frac{d\lambda_2}{dT}=-\frac{\partial H}{\partial P}$ are

$$\begin{cases}\dfrac{d\lambda_1}{dT}=-\dfrac{pqE^2m_1}{(m_1E+m_2N)^2}e^{-\delta T}-\lambda_1\Big[-\dfrac{r}{k}+\dfrac{mP}{(AP+N)^2}+\dfrac{qEm_2}{(m_1E+m_2N)^2}\Big]N\\ \qquad -\lambda_2s\Big(\dfrac{P^2}{bN^2}\Big)-\lambda_1\Big[r\Big(1-\dfrac{N}{k}\Big)-\dfrac{mP}{AP+N}-\dfrac{qE}{m_1E+m_2N}\Big],\\ \dfrac{d\lambda_2}{dT}=\dfrac{mN^2}{(AP+N)^2}\lambda_1-s\Big[1-\dfrac{2P}{bN}\Big]\lambda_2.\end{cases} \tag{50}$$

Our aim is to find an equilibrium solution $(N^*,P^*)$ which optimize the control variable $E$, and so, the optimal values of control variable $E^*$ and $N^*,P^*$ treated as constant. Now, using equation

(45), the adjoint equations in (50) can be written as

$$\frac{d\lambda_1}{dT} = -\frac{pqE^{*2}m_1}{(m_1E^* + m_2N^*)^2}e^{-\delta T} + \lambda_1\Big[\Big(\frac{r}{k} - \frac{qEm_2}{(m_1E^* + m_2N^*)^2}\Big)N^* - \frac{mb}{(Ab+1)^2}\Big] - \lambda_2 sb, \tag{51}$$

$$\frac{d\lambda_2}{dT} = \frac{m}{(Ab+1)^2}\lambda_1 + s\lambda_2. \tag{52}$$

Due to presence of $e^{-\delta T}$ in Equation (51), the steady state is not possible. We consider the following transformation to eliminate $e^{-\delta T}$

$$\lambda_i(T) = \mu_i(T)e^{-\delta T}, i = 1, 2 \tag{53}$$

where $\mu_i(T)$ represents the present value of the adjoint variables $\lambda_i, i = 1, 2$. From equations (49), (52) and (53), we have

$$\frac{d\mu_2}{dT} - (\delta + s)\mu_2 = -R(N^*) = -\frac{m}{(Ab+1)^2}\Big(\frac{c(m_1E^* + m_2N^*)^2}{qm_2N^{*2}} - p\Big). \tag{54}$$

The solution of the differential equation (54) satisfying the transversality condition at infinity (i.e. $\lim_{t\to\infty}\lambda_i(T) = 0$, for $i = 1, 2$) is

$$\mu_2(T) = \frac{R(N^*)}{\delta + s}. \tag{55}$$

Again from Equations (51), (52), and (53), we have

$$\frac{d\mu_1}{dT} - (\delta + Q_1(N^*))\mu = -Q_2(N^*), \tag{56}$$

where

$$Q_1(N^*) = \Big(\frac{r}{k} - \frac{qE^*m_2}{(m_1E^* + m_2N^*)^2}\Big)N^* - \frac{mb}{(Ab+1)^2}, \tag{57}$$

$$Q_2(N^*) = \frac{bsR(N^*)}{\delta + s} + \frac{pqE^2m_1}{(m_1E^* + m_2N^*)^2}. \tag{58}$$

The solution of the differential equation (56) satisfying the transversality condition at infinity is

$$\mu_1(T) = \frac{Q_2(N^*)}{\delta + Q_1(N^*)}. \tag{59}$$

From Equations (49), (53) and (59), we obtain

$$\frac{Q_2(N^*)}{\delta + Q_1(N^*)} = p - \frac{c(m_1E + m_2N)^2}{qm_2N^2}, \tag{60}$$

which is the required path and can be written as the following cubic equation

$$A_3N^3 + A_2N^2 + A_1N + A_0 = 0, \tag{61}$$

where

$A_3 = 2kqpr^3(1 + Ab)^4(\delta + s)$,

$A_2 = (1 + Ab)^2k^2qr^2m_1(p(-(5r - \delta)(\delta + s) + Ab^2(\delta + s)(5m + A(-5r + \delta)) + b(-2A(5r - \delta)(\delta + s) + m(\delta + 2s)))m_1 + (1 + Ab)^2(\delta + s)(pq - cm_2)),$

$A_1 = 2((1 + Ab)k^3qr(q + Abq - (-bm + r + Abr)m_1)(p(-(rq - \delta)(\delta + s) + Ab^2(\delta + s)(2m + A(-2r + \delta)) + b(-2A(2r - \delta)(\delta + s) + m(\delta + 2s)))m_1 + (1 + Ab)^2(\delta + s)(pq - cm_2))),$

$A_0 = -k^4q(-(-bm+r+Abr)^2p(-(r-\delta)(\delta+s)+Ab^2(\delta+s)(m+A(-r+\delta))+b(ms-2A(r-\delta)(\delta + s)))m_1^2 - (1 + Ab)(mb - (1 + Ab)r)m_1(2pq(-(r - \delta)(\delta + s) + Ab^2(\delta + s)(m + A(-r + \delta)) + b(ms - 2A(r - \delta)(\delta + s) + (1 + Ab)c(-bm + r + Abr)(\delta + s)m_2) + (1 + Ab)^2q(-qp((-r + \delta)(\delta + s) + Ab^2(\delta + s)(m + A(-r + \delta)) + b(ms - 2A(r - \delta)(\delta + s))) + (Abms + c(\delta + s)(-r + \delta + Ab(-2r + 2\delta + b(m + A(-r + \delta)))))m_2)).$

Clearly $A_3 > 0$. Therefore, equation (61) has at least one positive real root for $N^*$ whenever $A_0 < 0$.

On substituting the value of $N^*$ in equation (45), we obtain $P^*$ and $E^*$. Thus, the required singular equilibrium solution

$$(N^*, P^*, E^*) = \Big(N^*, bN^*, \frac{N^*((N^* - k)r - b(N^*Ar + k(m - Ar)))m_2}{(1 + Ab)kq + ((-k + N^*)r + b(km - Akr + AN^*r)m_1}\Big).$$

Thus, the maximum $J(N, P, E, T)$ is

$$J(N^*, P^*, E^*, T) = \int_0^\infty \Big(\frac{pqN^*}{m_1E^* + m_2N^*} - c\Big)E^*e^{-\delta T}dT.$$

On integrating above, we get

$$J(N^*, P^*, E^*, T) = \frac{E^*}{\delta}\Big(\frac{pqN^*}{m_1E^* + m_2N^*} - c\Big) = \frac{e^{\delta T}}{\delta}H(N^*, P^*, E^*, 0).$$

## 8. Numerical Simulation

1) If $\alpha = 0.9,\ \beta = 0.7,\ h = 0.1,\ c = 0.01$. Then, the system (7) has two interior equilibrium points $E_1^*(x_1^*, y_1^*) = (0.2627, 0.1839)$, $E_2^*(x_2^*, y_2^*) = (0.3567, 0.2497)$. The threshold value of $\rho$ is $\rho^{[hf]} = 0.1807$ and the first Lyapunov number $\sigma = 656.371\pi > 0$, and hence, an unstable limit cycle is created around $E_2^*$ and the other equilibrium point $E_1^*$ is a saddle point, see figure 4a. If $\rho = 0.2 > \rho^{[hf]}$, the equilibrium point $E_2^*$ is a stable point and the equilibrium point $E_1^*$ is a saddle point, see figure 4b. If $\rho = 0.15 < \rho^{[hf]}$, the equilibrium point $E_2^*$ is an unstable point and the equilibrium point $E_1^*$ is a saddle point, see figure 4c. If $\rho = 0.2117091$, then the limit cycle collide with the saddle point $E_1^*$, and hence, a homoclinic loop is created around $E_2^*$, this cyclic loop is unstable because of $\sigma = 480.543\pi > 0$, see figure 4d. Thus, interior equilibrium point $E_2^*(x_2^*, y_2^*)$ losses its stability as the bifurcation parameter $\rho$ passes through the threshold value $\rho^{[hf]}$ from right to left.

2) If $\alpha = 0.6,\ \beta = 0.5,\ \rho = 0.6,\ c = 0.1$, then we obtain $h = h^{[SN]} = 0.202499$. If $h = 0.16 < h^{[SN]}$, then there exits two interior equilibrium points in which one is a saddle point while the other is a stable, see figures 1a and 5a. If $h = 0.202499$, then these two

interior equilibrium points coincide to each other and a unique instantaneous equilibrium point is obtained which is stable from right side of the separatrix and unstable from left side of the separatix, see figures 1b and 5b. Figures 5c and 5d are the saddle-node bifurcation diagram. When $h > 0.202499$, then no interior equilibrium point exist, see figure 1c.

3 If $\alpha = 0.7,\ \beta = 0.6,\ c = 0.5, h = 0.36875$. Then, the system (7) has a unique interior equilibrium point $E^*(x^*, y^*) = (0.2375, 0.1425)$, and the threshold value of $\rho$ is $\rho^{[hf]} = 0.145966$. Thus, a stable limit cycle is created around the equilibrium point $E^*$, see figure 6a as the first Lyapunov number is $\sigma = -236.907\pi < 0$. If $\rho = 0.2$, then the unique equilibrium point $E^*$ is a stable point, see figure 6b.

4 If $r = 5, k = 500, A = 1, m = 1, p = 4, s = 2, b = 3, m_1 = 1, m_2 = 2, c = 1$, the bionomic equilibrium is $(275, 825, 1650)$.

5 If $r = 4, k = 500, A = 1, m = 0.3, p = 5, s = 2, b = 1, m_1 = 0.1, m_2 = 0.2, c = 1$. The solution of the cubic equation in the following three cases has been shown in figure 7.

**Case I:** $h - c + h\beta - c\beta + c\alpha\beta > 0$. In this case the system (6) has only one optimum solution for every discount rate $\delta$ which is shown in Table 2 for $q = 2$.

**Table 2.** Optimum solution for different discount rate $\delta$ with $q = 2$

| $\delta$ | $N_1$ | $N_2$ | $N_3$ | $E_1$ | $E_2$ | $E_3$ |
|---|---|---|---|---|---|---|
| 0.00 | 240.915 | −1769.77 | −2243.02 | 51.2468 | −32001.6 | 54494.1 |
| 0.02 | 239.715 | −1769.85 | −2242.94 | 51.2734 | −32014.4 | 54509.3 |
| 0.04 | 238.513 | −1769.93 | −2242.86 | 51.2975 | −32027.3 | 54524.6 |
| 0.06 | 237.311 | −1770.01 | −2242.79 | 51.3190 | −32040.2 | 54539.9 |
| 0.08 | 236.108 | −1770.09 | −2242.71 | 51.3380 | −32053.2 | 54555.2 |
| 0.10 | 234.904 | −1770.18 | −2242.63 | 51.3545 | −32066.1 | 54570.6 |

**Case II:** $h - c + h\beta - c\beta + c\alpha\beta < 0$. In this case the system (6) has only one optimum solution for every discount rate $\delta$ which is shown in Table 3 for $q = 0.1$.

**Table 3.** Optimum solution for different discount rate $\delta$ with $q = 0.1$

| $\delta$ | $N_1$ | $N_2$ | $N_3$ | $E_1$ | $E_2$ | $E_3$ |
|---|---|---|---|---|---|---|
| 0.00 | 436.878 | complex | complex | 480.860 | complex | complex |
| 0.02 | 436.869 | complex | complex | 480.998 | complex | complex |
| 0.04 | 436.860 | complex | complex | 481.175 | complex | complex |
| 0.06 | 436.851 | complex | complex | 481.851 | complex | complex |
| 0.08 | 436.843 | complex | complex | 481.404 | complex | complex |
| 0.10 | 436.835 | complex | complex | 481.537 | complex | complex |

**Case III:** $h - c + h\beta - c\beta + c\alpha\beta = 0$. In this case the system (6) has only one optimum solution for every discount rate $\delta$ which is shown in Table 4 for $q = 0.385$.

**Table 4.** Optimum solution for different discount rate $\delta$ with $q = 0.385$

| $\delta$ | $N_1$ | $N_2$ | $N_3$ | $E_1$ | $E_2$ | $E_3$ |
|---|---|---|---|---|---|---|
| 0.00 | 265.625 | complex | complex | 431.250 | complex | complex |
| 0.02 | 264.834 | complex | complex | 432.831 | complex | complex |
| 0.04 | 264.048 | complex | complex | 434.404 | complex | complex |
| 0.06 | 263.266 | complex | complex | 435.968 | complex | complex |
| 0.08 | 262.489 | complex | complex | 437.523 | complex | complex |
| 0.10 | 261.715 | complex | complex | 439.069 | complex | complex |

The predators are computed by the following relation

$$P_k^* = AbN_k^*, \quad k = 1, 2, 3.$$

It is observed from Tables 2, 3 and 4 that in either case, for every discount rate $\delta$ there exists at least one optimum equilibrium point $(N_k^*, P_k^*, E_k^*)$ such that the density of the prey is inversely proportional to the effort applied to harvest.

## 9. Conclusion

In this paper, the authors have analyzed a Holling-Tanner predator-prey model with ratio-dependent functional response and Michaelis-Menten type prey harvesting. The qualitative analysis of the proposed system shows that the harvesting rate $h$ significantly affects the system. It is shown that the origin is a non-hyperbolic equilibrium point and its stability has been discussed by blow up transformation. It is clear from Figures 2 and 3c that in the absence of the interior equilibrium points, the trivial equilibrium point is globally asymptotically stable. The stability of other equilibrium points have been discussed by linearization technique. The proposed system shows bistability under certain parametric conditions which concludes that the solutions are highly sensitive for initial values.

It is shown that the proposed system undergoes Hopf bifurcation. The stability of the limit cycle have been discussed by computing the first Lyapunov number. By using Sotomayor's theorem, it is shown that the system undergoes saddle-node bifurcation where the harvesting parameter $h$ is taken as the bifurcation parameter. This bifurcation gives the maximum sustainable yield (maximum harvesting compatibility with survival), see Clark (1976). The existence of the homoclinic loop has been shown and also by calculating the first Lyapunov number, which shows that homoclinic loop is unstable. The system (reffinalmod) is reduced to the normal form of the Bogdanov-Takens bifurcation by means of a series of transformations. This ensures that the predator and prey coexist in the form of a positive equilibrium or a periodic orbit for different initial values, respectively.

The condition under which bionomic equilibrium exists, has been derived. An optimal harvesting problem has been solved by using Pontryagin's maximum principle. It is clear from figure 8 that at very low prey population the profit is very low. As population of prey increases the profit starts increasing in all the three cases. The increase is the fastest in first case but become constant after a stage while in the third case the profit increases gradually. Hence, we conclude that; i)for very low prey population, no harvesting should occur; ii) once the prey population starts increasing, the first case gives maximum profit; iii) for large prey population the third case gives maximum profit. In the last, it is shown that the number of optimal prey is inversely proportional to the optimal harvesting effort.

## REFERENCES


Arditi, R. and Ginzburg, L.R. (1989). Coupling in predator-prey dyanamics: ratio-dependence, J. Theor. Biol., Vol. 139, pp. 311-326.

Arditi, R., Ginzburg, L.R. and Akcakaya, H.R. (1991). Variation in plankton densities among lakes: A case for ratio-dependent models, Amer. Naturalist, Vol. 138, pp. 1287-1296.

Arditi, R. and Saiah, H. (1992). Impirical evidence of the role of heterogeneity in ratio-dependent consumption, Ecology, Vol. 73, pp. 1544-1551.

Beddington, J.R. and May, R.M. (1980). Maximum sustainable yields in systems subject to harvesting at more than one trophic level, Math. Biosci. Vol. 51, pp. 261-281.

Berryman, A.A. (1992). The origins and evolution of predator-prey theory, Ecology, Vol. 73, pp. 1530-1535.

Brauer, F. and Soudack, A.C. (1979). Stability regions and transition phenomena for harvested predator-prey system, J. Math. Biol. Vol. 7, pp. 319-337.

Cantrell, R.S. and Cosner, (2001). On the dynamics of predator-prey models with Beddington-DeAngelis functional response, J. Math. Anal. Appl., Vol. 257, pp. 206-220.

Chow, S.N. and Hale, J.K. (1983). *Methods of Bifurcation Theory, Tim Clutton-Brock Grundlehren Mathematischen Wissenschaften*, Springer-Verlag, New York, Berlin.

Clark, C.W. (1976), *Mathematical Bionomics: The Optimal Management of Renewable Resources*, Wiley, New York.

Fan, M. and Wang, Y. (2004). Dynamics of a nonautonomous predator-prey system with the Beddington DeAngelis functional response, J. Math. Anal. Appl., Vol. 295, pp. 15-39.

Freedman, H.I. (1980). *Deterministic mathematical models in population Ecology*, Dekker, New York.

Gaie, P. and Zhang, H. (2006). Qualitative analysis of a prey-predator system with Holling I functional response, J. Jilin. Univ. Sci. Vol. 44, No. 6, pp. 373-376.

Gong, Y. and Huang, J. (2014). Bogdanov-Takens bifurcation in a Leslie-Gower predator-prey model with prey harvesting, Acta Math. Appl. Sin. English series., Vol. 30, No. 1, pp. 239-244.

Gupta, R. P. Banarjee, M. and Chandra, P. (2012). Bifurcation analysis and control of Leslie-Gower predator-prey model with Michaelies-Menten type prey harvesting, Diff. Equ. Dyn. Sys., Vol. 20, pp. 339-366.

Holling, C.S. (2011). On the dynamics of predator-prey system with Beddington DeAngelis functional response, Asian-Eur. J. Math., Vol. 4, pp. 35-48.

Hsu, B.S. and Hwang, T.W. (1995). Global stability for a class of predator-prey system, SIAM J.Appl.Math. Vol. 55, pp. 763-783.

Hsu, S.B., Hwang, T.W. and Kuang, Y. (2001). Global analysis of the Michaelies-Menten type ratio-dependent predator-prey system, Math. Biol., Vol. 42, pp. 489-506.

Hsu, S.B., Hwang, T.W. and Kuang, Y. (2005). Global dynamics of a predator-prey model with Hassell-Varley type functional response, Discrete Contin. Dyn. Syst. Ser. B, Vol. 10, No. 4, pp. 857-875.

Hwang, J. and Xiao, D. (2004). Analysis of bifurcation and stability in a predator-prey system with Holling type-IV functional response, Acta Math. Appl. Sin., Vol. 20, pp. 167-178.

Jost, C., Arino, O. and Arditi, R. (1999). About deterministic extinction ratio-dependent predator-prey models, Bull. Math. Biol., Vol. 61, pp. 19-32.

Kar, T.K. (2004). Managemant of a fishery based on continuous fishing effort, Nonlinear Analysis: Real World Applications, Vol. 5, pp. 629-644.

Kar, T.K., Mishra, S. and Mukhopadhayay, B. (2006). A bionomic model of a ratio-dependent predator-prey system and optimal harvesting, J. Appl. Math. Comp., Vol. 22, No. 1-2, pp. 387-401.

Lenzini, P. and Rebaza, J. (2010). Nonconstant predator harvesting on ratio-dependent predator-prey models, Appl. Math. Sces., Vol. 4, No. 16, pp. 791-803.

Liang, Z. and Pan, H. (2007). Qualitative analysis of a ratio-dependent Holling-Tanner model, J. Math. Anal. Appl. Vol. 334, pp. 954-964.

Lundberg, P. and Fryxell, J.M. (1995). Expected population density versus productivity in ratio-dependent and prey-dependent models, Vol. 146, pp. 153-161.

May, R.M. (1973). *Stability and Complexity in Model Ecosystems*, Princeton University Press, Princeton, NJ.

Perko, L. (1996). *Differential Equations and Dynamical Systems*, Springer, New York.

Pontryagin, L.S., Boltyonsku, V.G., Gamkrelidre, R.V. and Mishchenko E.F. (1962). *The Mathematical Theory of Optimal Process*, Wiley, New York.

Ragozin, D.L. and Brown, J.G. (1985). Harvest policies and non-market valuations in a predator-prey system, J. Environ. Econ. Managm., Vol. 12, pp. 155-168.

Reeve, A.D. (1997). Predation and bark beetle dynamics, Oecologia, Vol. 112, pp. 48-55.

Saez, E. and Oliares, E.G. (1999). Dynamics of predator-prey model, SIAM J. Appl. Math., Vol. 59, pp. 1867-1878.

Saha, T. and Chakrabarti, (2009). Dyanamical analysis of a delayed ratio-dependent Holling-Tanner predator-prey model, J. Math. Anal. Appl. Vol. 358, pp. 389-402.

Simmons, G.F. (2017). *Differential Equations with Applications and Historical Notes*, CRC Press, New York.

Singh, M.K., Bhadauria, B.S. and Singh, B.K. (2016). Qualitative analysis of a Leslie-Gower predator-prey system with nonlinear harvesting in predator, International Journal of Engineering Mathematics, Vol. 2016, Article ID 2741891, 15 pages.

Sugie, J., Kohno, R.and Miyazaki, R. (1997). On a predator-prey system of Holling type, Proc. Am. Math. Soc., Vol. 125, No. 7, pp. 2041-2050.

Wu, R and Lin, L. (2009). Permanence and global attractivity of discrete predator-prey system with Hassell Varley type functional response, Discrete Dyn. Nat. Soc., pp. 1-17.

Xiao, D. and Jennings, L.S. (2005). Bifurcation of a ratio-dependent ratio-dependent predator-prey system with constant rate harvesting, SIAM J. Appl. Math., Vol. 65, No. 3, pp. 737-753.

Xiao, D. and Jennings, L.S. (2006). Dynamics in a ratio-dependent predator-prey model with predator harvesting, J. Math. Anal. Appl., Vol. 324, pp. 14-29.

Xiao, D. and Ruan, S. (1999). Bodgdanov-Takens bifurcations in predator-prey system with constant rate harvesting, Fields Inst. Commun. Vol. 21, pp. 493-506.

Zhang, Z., Ding, T., Huang, W. and Dong, Z. (1999). *Qualitative Theory of Differential Equations*, Amer. Math. Soc. Peking Univ. Press.

Zhu, C.R. and Lan, K.Q. (2010). Phase portraits, Hopf bifurcation and limit cycles of Leslie-Gower predator-prey systems with harvesting rates, Dis. Con. Dyn. Sys. Series B, Vol. 14, No. 1, pp. 389-306.

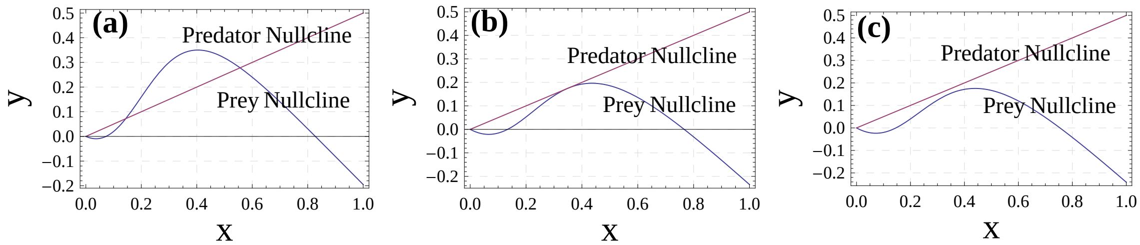


**Figure 1.** These diagrams shows the number of interior equilibrium points of the system (7) when the parameter $h$ varies through critical value $h^{[SN]}$ and all other parameters are fixed. $\alpha = 0.6, \;\; \beta = 0.5, \;\; c = 0.1, \;\; \rho = 0.6 \;\; (a)h = 0.16 \;\; (b)h = 0.2025 \;\; (c)h = 0.21$.

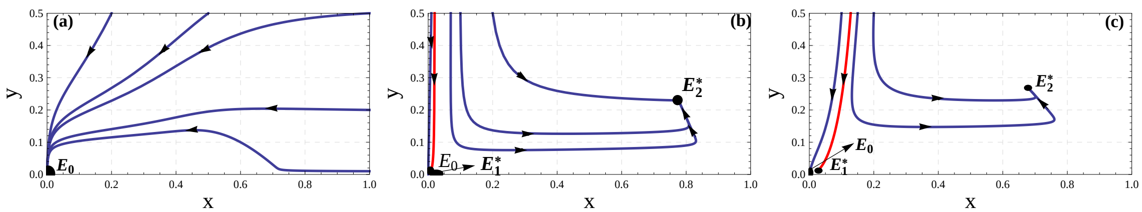


**Figure 2.** The phase portrait diagram of the system (7). (a) $\alpha = 0.6, \;\; \beta = 0.3, \;\; h = 0.2, \;\; c = 0.02, \;\; \rho = 0.2$ satisfies the conditions of the theorem 4.1 a. It shows that $E_0$ is globally stable. (b) $\alpha = 0.4, \;\; \beta = 0.3, \;\; h = 0.12, \;\; \rho = 0.2, \;\; c = 0.13$ satisfies the conditions of the theorem 4.1 b.The equilibrium points $E_0$ and $E_2^*$ are locally asymptotically stable points. (c) $\alpha = 0.7, \;\; \beta = 0.4, \;\; c = 0.1, \;\; \rho = 0.1, \;\; h = 0.096$ satisfies the conditions of the theorem 4.1 c. It shows that $E_0$ and $E_2^*$ are locally asymptotically stable. The red trajectory is a separatrix.

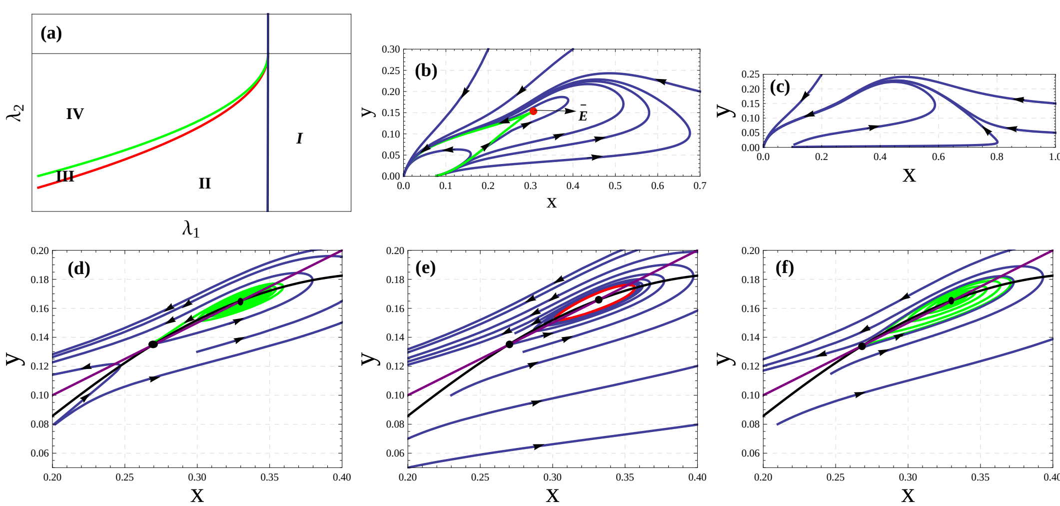


**Figure 3.** (a) The blue curve is saddle-node bifurcation curve, red curve is hopf bifurcation curve and green curve is homoclinic bifurcation curve. (b) When $\lambda_1 = \lambda_2 = 0$, the unique equilibrium point $\overline{E}$ is a cusp of codimention 2. (c) When $\lambda_1 = 0.001, \lambda_2 = 0.001$ lies in the region $I$, then the system (7) has no interior equilibrium point and the origin is globally asymptotically stable. (d) When $\lambda_1 = -0.001, \lambda_2 = -0.1$ lies in the region $II$, then the system (7) has two interior equilibrium point. One point is unstable and other is a saddle point. (e) When $\lambda_1 = -0.001, \lambda_2 = -0.09$ lies in the region $III$, then the system (7) has two interior equilibrium point. One point is enclosed by an unstable limit cycle and other is a saddle point. (d) When $\lambda_1 = -0.001, \lambda_2 = -0.07$ lies in the region $IV$, then the system (7) has two interior equilibrium point. One point is asymptotically stable and other is a saddle point. $\alpha = 0.9, \;\; \beta = 0.5, \;\; h = 0.16, \;\; c = 0.1, \;\; \rho = 0.4$.

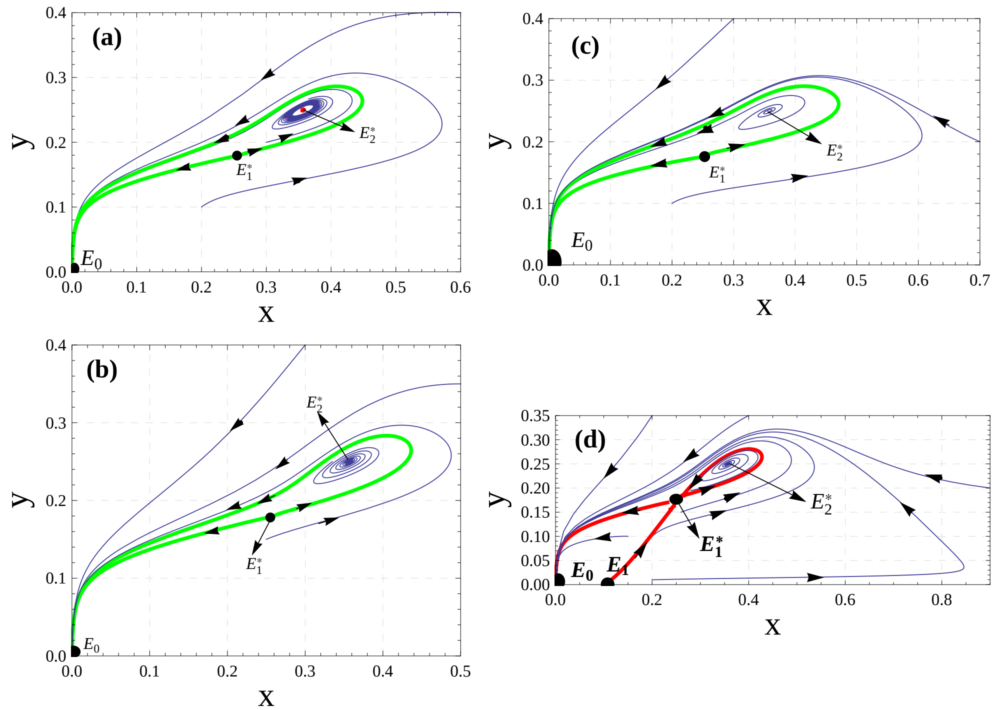

**Figure 4.** The phase portrait diagram of the system (7) $E_1^*$ is always a saddle point. $\alpha = 0.9, \ \beta = 0.7, \ h = 0.1, \ c = 0.01$. (a) $\rho = 0.18071$, an unstable limit cycle bifurcates around the interior point $E_2^*$ (b) $\rho = 0.2$, $E_2^*$ is a stable point. (c) $\rho = 0.15$, $E_2^*$ is an unstable point. (d) $\rho = 0.2117091$, homoclinic loop is created around the interior point $E_2^*$,

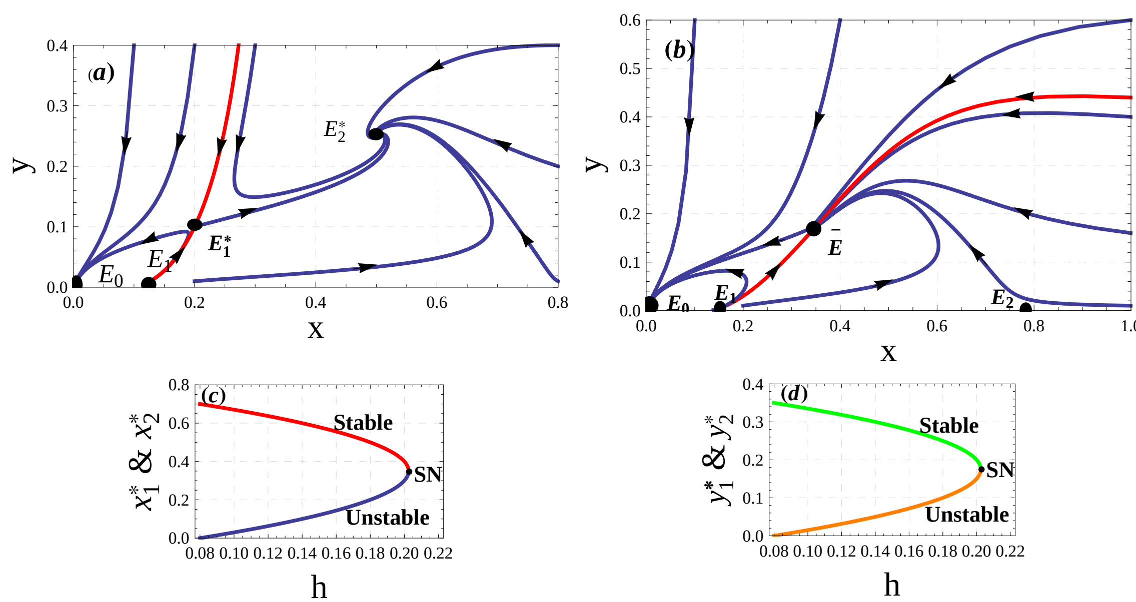

**Figure 5.** The phase portrait diagram of the system (7) for $\alpha = 0.6, \ \beta = 0.5, \ c = 0.1, \ \rho = 0.6$. The red trajectories are the separatrix. (a) $h = 0.16$, $E_0$ and $E_2^*$ are two locally asymptotically stable points and $E_1^*$ is a saddle point. (b) $h = 0.202499$, the point $\overline{E}$ is stable from right side of the separatrix and unstable from left side of the separatrix. (c) and (d) are the saddle-node bifurcation diagrams.

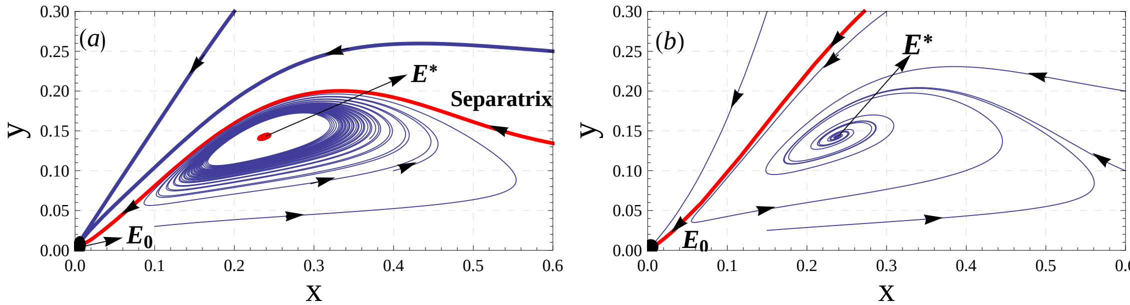


**Figure 6.** The phase portrait diagram of the system (7) for $\alpha = 0.7, \quad \beta = 0.6, \quad c = 0.5, \quad h = 0.36875$, the red trajectory is the separatrix. (a) $\rho = 0.14597$, a stable limit cycle is created around the unique interior equilibrium point. (b) $\rho = 0.2$, the unique interior point is stable.

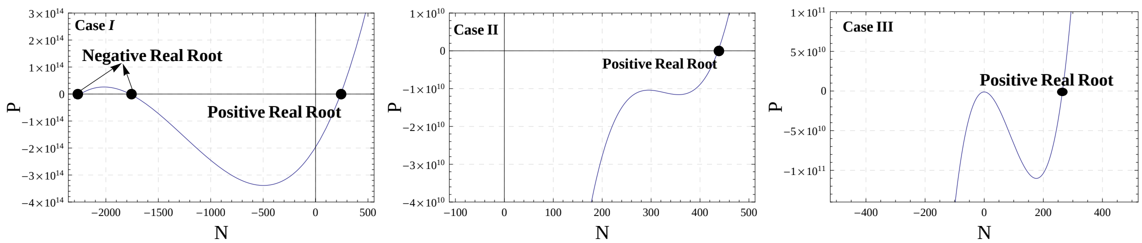


**Figure 7.** These diagrams shows the optimum path in case I, case II, and case III, respectively. The positive real root are the optimum value of prey species. $r = 4, k = 500, A = 1, m = 0.3, p = 5, s = 2, b = 1, m_1 = 0.1, m_2 = 0.2, c = 1, \delta = 0.01$, for case I $q = 2$, for case II $q = 0.1$ and for case III $q = 0.385$.

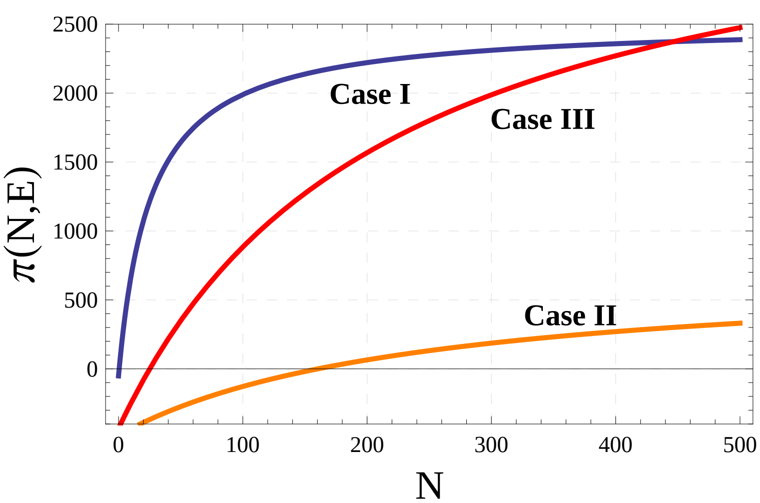


**Figure 8.** This diagram shows the profit for different values of prey species in case I, case II and case III. $\delta = 0.2, r = 4, k = 500, A = 1, m = 0.3, q = 0.1, p = 5, s = 2, b = 1, m_1 = 0.1, m_2 = 0.2, a = 1$.