Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Special Issue No. 6 (April 2020), pp. 87 – 102

Class of Integrals Involving Generalized Hypergeometric Function

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Received: February 29, 2020; Accepted: March 15, 2020

Abstract

In this paper, we establish some definite integrals involving generalized hypergeometric function, product of algebraic functions, Jacobi function, Legendre function and general class of polynomials. Certain special cases of the main results are also pointed out.

Keywords: Generalized hypergeometric functions; Srivastava's polynomials; Definite integrals

MSC 2010 No.: 26A33, 33C45, 33C60

1. Introduction

In past few decades, many researchers are attracted to diverse techniques of special functions and their uses in many other areas of mathematics. These functions as a part of the theory of hypergeometric functions, are important special functions and they are closely related to physics and engineering applications. The great applications in a wide variety of fields (Srivastava et al. (2014)) have played a pivotal role in the advancements of further research in special functions.

Several further properties of each of generalized incomplete hypergeometric functions and some classes of incomplete H-functions associated with them can be found in the subsequent developments presented in recent articles (Bansal et al. (2019); Bansal et al. (2020)). In view of their importance and popularity in recent years, the theory of operators of fractional calculus have been developed widely and extensively by many researchers (Chaurasia and Singh (2015); Suthar, Parmar et al. (2018)) and various generalizations associated with integrals have been studied (Chaurasia and Singh (2014); Kumar et al. (2018); Khan (2017); Nisar et al. (2017); Sarbia and Kalla (2002); Suthar et al. (2017); Suthar, Agarwal et al. (2018)).

2. Preliminaries

For our investigation, we need extended generalized hypergeometric function (Srivastava et al. (2014)) defined as:

$$ {}_rF_s\,[\,x] = {}_rF_s\left[\begin{matrix}(\varsigma_1,p),\varsigma_2,\cdots,\varsigma_r; \\ \vartheta_1,\cdots,\vartheta_s;\end{matrix}\; x\right] = \sum_{n=0}^{\infty}\frac{(\varsigma_1;p)_n\,(\varsigma_2)_n\,,\cdots,(\varsigma_r)_n}{(\vartheta_1)_n\,,\cdots,(\vartheta_s)_n}\frac{(x)^n}{n\,!}, \tag{1} $$

where $(\lambda;p)_n$ denotes the Pochhammer symbol (see (Srivastava et al. (2014)), p. 485 Eq. (1.66)) and is defined as follow:

$$ (\lambda;p)_n = \begin{cases} \frac{\Gamma_p(\lambda+n)}{\Gamma(\lambda)}, & (\Re(p)>0;\,\lambda,\,n\in\mathbb{C}), \\ (\lambda)_p, & (p=0;\,\lambda,\,n\in\mathbb{C}). \end{cases} \tag{2} $$

The generalized gamma function (Chaudhary and Zubair (2001), p. 9, Eq (1.66)) is defined as:

$$ \Gamma_p(z) = \begin{cases} \int_0^\infty x^{z-1}\exp(-x-p/x)dx, & (\Re(p)>0;\,z\in\mathbb{C}), \\ \Gamma(z), & (p=0;\,\Re(z)>0). \end{cases} \tag{3} $$

The analogous extensions of the Gauss hypergeometric and the confluent hypergeometric functions are represented as follows:

$$ {}_2F_1\left[\begin{matrix}(\varsigma,p),\vartheta; \\ \gamma;\end{matrix}\; x\right] = \sum_{n=0}^{\infty}\frac{(\varsigma,\,p)_n\,(\vartheta)_n}{(\gamma)_n}\frac{(x)^n}{n\,!}, \tag{4} $$

and

$$ {}_1F_1\left[\begin{matrix}(\varsigma,p); \\ \gamma;\end{matrix}\; x\right] = \sum_{n=0}^{\infty}\frac{(\varsigma,p)_n}{(\gamma)_n}\frac{(x)^n}{n!}. \tag{5} $$

For the current investigation, we need the following definitions.

Definition 2.1.

The Gamma function (Rainville (1960)) can be defined as:

$$\Gamma(\lambda) = \int_0^\infty e^{-t} t^{\lambda-1} dt, \quad \Re(\lambda > 0). \tag{6}$$

Definition 2.2.

The Beta function is defined (Olver et al. (2010)) as:

$$B(u,v) = \int_0^1 t^{u-1} (1-t)^{v-1} \, dt = \frac{\Gamma(u)\,\Gamma(v)}{\Gamma(u+v)} = B(v,u), \; (u,v \in \Re). \tag{7}$$

Definition 2.3.

The Jacobi polynomial $P_n^{(\tau,v)}(x)$ (Rainville (1960), p. 254) is defined as:

$$P_n^{(\tau,v)}(x) = \frac{(\tau+1)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, 1+\tau+v+n; \\ 1+\tau; \end{matrix} \; \frac{1-x}{2} \right], \tag{8}$$

where ${}_2F_1$ is the classical hypergeometric functions defined as:

$${}_2F_1 \left[\begin{matrix} -n, \, 1+\tau+v+n; \\ 1+\tau; \end{matrix} \; \frac{1-x}{2} \right] = \sum_{l=0}^{\infty} \frac{(-n)_l \, (1+\tau+v+n)_l}{(1+\tau)_l} \frac{(1-x)^l}{2^l \, l!}.$$

When $\tau = v = 0$, then the polynomial (8) becomes the Legendre polynomial(Rainville (1960), p. 157) and also, we have

$$P_n^{(\tau,v)}(1) = \frac{(\tau+1)_n}{n!}. \tag{9}$$

Definition 2.4.

The solution of Legendre equation (Rainville (1960), Sec. 3.1)

$$\left(1-x^2\right) \frac{d^2 f}{dx^2} - 2x \frac{df}{dx} + \left[\varepsilon(\varepsilon+1) - \tau^2(1-x^2)^{-1}\right] f = 0, \tag{10}$$

is presented in the form of the Legendre function. If we put $f = (x^2-1)^{\frac{1}{2\tau}} \varepsilon$, then Equation (10) reduced to

$$\left(1-x^2\right) \frac{d^2 \varepsilon}{dx^2} - 2(\tau+1)x \frac{d\varepsilon}{dx} + \left[\varepsilon(\tau-\varepsilon)(\tau+\varepsilon+1)^{-1}\right] \; = \; 0, \tag{11}$$

and if we set $\delta = \frac{1}{2} - \frac{1}{2}x$ as the independent variable, the above differential equation (10) becomes as follows:

$$\delta\left(1-\delta\right)\frac{d^2\varepsilon}{d\delta^2}+\left(\tau+1\right)\left(1-2\delta\right)\frac{d\varepsilon}{d\delta}+\left[\varepsilon\left(\varepsilon-\tau\right)\left(\tau+\varepsilon+1\right)\right]=0. \tag{12}$$

The solution of Equation (10) in the representation of Gauss hypergeometric equation with $a=\tau-\varepsilon,\ b=\tau+\varepsilon+1$ and $c=\tau+1$ is as follows:

$$f=\ P_{\varepsilon}^{\tau}(x)=\frac{1}{\Gamma\left(1-\tau\right)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2\tau}}F\left[-\varepsilon,\varepsilon+1;\ 1-\tau;\tfrac{1}{2}-\tfrac{1}{2}x\right],\ |1-x|<2, \tag{13}$$

where $P_{\varepsilon}^{\tau}(x)$ is well-known Legendre function of the first kind (see Erdélyi et al. (1953)).

Definition 2.5.

The general class of polynomials $S_{n_1,\cdots,n_r}^{m_1,\cdots,m_r}\left[x\right]$ (Srivastava (1985), p. 185, Eq. (7)) is presented in the following manner:

$$S_{n_1,\cdots,n_r}^{m_1,\cdots,m_r}\left[x\right]=\sum_{l_1=0}^{\left[\frac{n_1}{m_1}\right]}\cdots\sum_{l_r=0}^{\left[\frac{n_r}{m_r}\right]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}x^{l_i}, \tag{14}$$

where $n_1,\cdots,n_r\ =\ 0,1,2,\cdots;m_1,\cdots,m_r$ are arbitrary positive integers and the coefficient $A_{n_i,l_i}(n_i,l_i\geq 0)$ are arbitrary constants, (real or complex). On suitably specializing the coefficients A_{n_i,l_i}, $S_{n_1,\cdots,n_r}^{m_1,\cdots,m_r}\left[x\right]$ yields a number of known polynomials as its special cases. These includes Laguerre polynomials, Bessel polynomials, Jacobi polynomials, Hermite polynomials, Brafman polynomials, Gould-Hopper polynomials and several others (Srivastava and Singh (1983), pp. 158-161).

The objective of this paper is to establish some definite integrals involving generalized hypergeometric function, product of algebraic functions, Jacobi function, Legendre function and general class of polynomials.

During the course of this paper, let $\mathbb{C}$ be sets of complex numbers, $\Re$ - real, $\Re^+$ -positive numbers, $\mathbb{Z}_0^-$ - non-positive and $\mathbb{N}$ -positive integers.

3. Integrals involving generalized hypergoemetric functions with algebraic function

In this section, we evaluate the following integral formulas involving generalized hypergeometric function with some algebraic functions:

$$I_1 = \int_0^1 x^{-\xi}(1-x)^{\xi-\omega-1}{}_rF_s\left[zx\right]dx$$

$$= \int_0^1 x^{-\xi}(1-x)^{\xi-\omega-1}\sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{(zx)^n}{n!}dx$$

$$= \sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}\int_0^1 x^{n-\xi}(1-x)^{\xi-\omega-1}dx$$

$$= \sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}\frac{\Gamma(n-\xi+1)\,\Gamma(\xi-\omega)}{\Gamma(n-\omega+1)}$$

$$= B(1-\xi,\xi-\omega)\,{}_{r+1}F_{s+1}\left[\begin{matrix}(\varsigma_1,p),\varsigma_2,...,\varsigma_r,(1-\xi)\,; \\ \vartheta_1,...,\vartheta_s,(1-\omega)\,;\end{matrix}\, z\right]. \tag{15}$$

$$I_2 = \int_0^1 x^{\xi-1}(1-x)^{\omega-1}{}_rF_s\left[zx\right]dx$$

$$= \int_0^1 x^{\xi-1}(1-x)^{\omega-1}\sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{(zx)^n}{n!}dx$$

$$= \sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}\int_0^1 x^{n+\xi-1}(1-x)^{\omega-1}dx$$

$$= \sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}\frac{\Gamma(\omega)\Gamma(n+\xi)}{\Gamma(\omega+\xi+n)},$$

$$= B(\xi,\omega)_{r+1}F_{s+1}\left[\begin{matrix}(\varsigma_1,p),\varsigma_2,...,\varsigma_r,\xi; \\ \vartheta_1,...,\vartheta_s,\omega+\xi;\end{matrix}\, z\right]. \tag{16}$$

$$I_3 = \int_1^{\infty} x^{-\xi}(x-1)^{\omega-1}{}_rF_s\left[\frac{z}{x}\right]dx$$

$$= \sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n(\xi)_n}{(\vartheta_1)_n...(\vartheta_s)_n(\omega+\xi)_n}\frac{z^n}{n!}\int_1^{\infty} x^{-\xi-n}(x-1)^{\omega-1}dx.$$

By letting $x-1=t$ and taking as $x\to\infty, t\to\infty$ and using the formula

$$\Gamma(\varsigma)\Gamma(\vartheta)=\Gamma(\varsigma+\vartheta)\int_0^{\infty} x^{\varsigma-1}(1+x)^{-(\varsigma+\vartheta)}dx, \tag{17}$$

we get the following result

$$\begin{aligned}I_3&=\sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}\int_0^{\infty} t^{\omega-1}\,(1+t)^{-\xi-n}dt\\
&=\sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}\frac{\Gamma(\omega)\,\Gamma(\xi-\omega+n)}{\Gamma(\xi+n)}\\
&=\frac{\Gamma(\omega)\Gamma(\xi-\omega)}{\Gamma(\xi)}\sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n(\xi-\omega)_n}{(\vartheta_1)_n...(\vartheta_s)_n(\xi)_n}\frac{z^n}{n!}\\
&=B(\omega,\xi-\omega)_{r+1}F_{s+1}\left[\begin{matrix}(\varsigma_1,p),\varsigma_2,...,\varsigma_r,\xi-\omega;\\ \vartheta_1,...,\vartheta_s,\xi;\end{matrix}\,z\right].\end{aligned} \tag{18}$$

$$\begin{aligned}I_4&=\int_{-1}^{1}(1-x)^{\xi}(1+x)^{\omega}{}_rF_s\left[z(1-x)^{\tau}\right]dx\\
&=\sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}\int_{-1}^{1}(1-x)^{\xi+\tau\,n+1-1}(1+x)^{\omega+1-1}dx.\end{aligned}$$

By using the formula (Rainville (1960), p. 26)

$$\int_{-1}^{1}(1-x)^{\xi+n}(1+x)^{\omega+n}dx=2^{2n+\xi+\omega+1}B(\xi+n+1,\omega+n+1), \tag{19}$$

we evaluate the following integral:

$$\begin{aligned}I_4&=\sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}2^{\tau n+\omega+\xi+1}B(1+\xi+\tau\,n,1+\omega)\\
&=2^{\omega+\xi+1}\Gamma(1+\omega)\sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{\Gamma(1+\xi+\tau n)}{\Gamma(\xi+\omega+2+\tau n)}\frac{(2^{\tau}z)^n}{n!}.\end{aligned} \tag{20}$$

In particular, if $\tau = 1$, then Equation (20) becomes

$$I_4 = 2^{\omega+\xi+1}\frac{\Gamma(1+\omega)\Gamma(1+\xi)}{\Gamma(\xi+\omega+2)}\sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n(1+\xi)_n}{(\vartheta_1)_n...(\vartheta_s)_n(\xi+\omega+2)_n}\frac{(2z)^n}{n!}$$

$$= 2^{\omega+\xi+1}\frac{\Gamma(1+\omega)\Gamma(1+\xi)}{\Gamma(\xi+\omega+2)}\,{}_{r+1}F_{s+1}\left[\begin{matrix}(\varsigma_1,p),\varsigma_2,...,\varsigma_r,1+\xi;\\ \vartheta_1,...,\vartheta_s,\xi+\omega+2;\end{matrix}2z\right]. \tag{21}$$

4. Integrals involving generalized hypergoemetric functions with Jacobi polynomials

In this section, we derive the following integral formulas involving generalized hypergoemetric functions multiplied with Jacobi Polynomials:

$$I_5 = \int_{-1}^{1} x^{\lambda}\,(1-x)^{\tau}\,(1+x)^{\mu}\,P_n^{(\tau,v)}(x)\,{}_rF_s\left[z\,(x+1)^{l}\right]dx$$

$$= \sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}\int_{-1}^{1} x^{\lambda}\,(1-x)^{\tau}\,(1+x)^{\mu+nl}\,P_n^{(\tau,v)}(x)dx. \tag{22}$$

Next using the following formula

$$\int_{-1}^{1} x^{\lambda}\,(1-x)^{\varsigma}\,(1+x)^{\mu}\,P_n^{(\varsigma,\vartheta)}(x)dx$$

$$= \frac{(-1)^n\,2^{\varsigma+\mu+1}\Gamma(\mu+1)\,\Gamma(n+\varsigma+1)\,\Gamma(\mu+\vartheta+1)}{n!\Gamma(\mu+\vartheta+n+1)\,\Gamma(\mu+\varsigma+n+2)}$$

$$\times{}_3F_2\left[\begin{matrix}-\lambda,\tau+\vartheta+1,\tau+1;\\ \tau+\vartheta+n+1,\tau+\varsigma+n+2;\end{matrix}1\right], \tag{23}$$

where $\varsigma > -1$ and $\vartheta > -1$. Also ${}_3F_2$ is the special case of generalized hyergeometric series. Then using Equation (22) in (23), we have

$$I_5 = \sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}$$

$$\times\frac{(-1)^n\,2^{\tau+\mu+nl+1}\Gamma(\mu+nl+1)\,\Gamma(n+\tau+1)\,\Gamma(\mu+nl+v+1)}{n!\Gamma(\mu+nl+v+n+1)\,\Gamma(\mu+nl+\tau+n+2)}$$

$$\times{}_3F_2\left[\begin{matrix}-\lambda,\mu+nl+v+1,\mu+nl+1;\\ \mu+nl+v+n+1,\mu+nl+\tau+n+2;\end{matrix}1\right].$$

After some rearrangements, we get

$$I_5 = 2^{\tau+\mu+1}\Gamma(\tau+1)\,{}_{r+1}F_s\left[\begin{matrix}(\varsigma_1,p),\varsigma_2,...,\varsigma_r,\tau+1;\\ \vartheta_1,...\vartheta_s;\end{matrix}-2^l z\right]$$

$$\times\frac{\Gamma(\mu+nl+1)\Gamma(\mu+nl+v+1)}{n!\Gamma(\mu+n(l+1)+v+1)\Gamma(\mu+n(l+1)+\tau+2)}$$

$$\times{}_3F_3\left[\begin{matrix}-\lambda,\mu+nl+v+1,\mu+nl+1;\\ \mu+nl+v+n+1,\mu+nl+\tau+n+2,\ 1;\end{matrix}1\right]. \tag{24}$$

From Equation (24), we can justify the special case.

If $l=1$, then using the basic property $(\varsigma+1)_{2n}=\left(\frac{\varsigma+1}{2}\right)_n\left(\frac{\varsigma+2}{2}\right)_n$, we get

$$I_5 = 2^{\tau+\mu+1}\Gamma(\tau+1)\,{}_{r+1}F_s\left[\begin{matrix}(\varsigma_1,p),\varsigma_2,...,\varsigma_r,\tau+1;\\ \vartheta_1,...\vartheta_s;\end{matrix}-2z\right]$$

$$\times\,{}_3F_3\left[\begin{matrix}-\lambda,\mu+n+v+1,\mu+n+1;\\ \mu+2n+v+1,\mu+n+\tau+n+2,\ 1;\end{matrix}1\right]$$

$$\times\frac{(\mu+1)_n\Gamma(\mu+1)(\mu+v+1)_n\Gamma(\mu+v+1)}{\left(\frac{\mu+v+1}{2}\right)_n\left(\frac{\mu+v+2}{2}\right)_n\Gamma(\mu+v+1)\left(\frac{\mu+\tau+2}{2}\right)_n\left(\frac{\mu+\tau+3}{2}\right)_n\Gamma(\mu+\tau+2)}$$

$$= 2^{\tau+\mu+1}\Gamma(\tau+1)\frac{\Gamma(\mu+1)\Gamma(\mu+v+1)}{\Gamma(\mu+v+1)\Gamma(\mu+\tau+2)}$$

$$\times{}_{r+3}F_{s+4}\left[\begin{matrix}(\varsigma_1,p),\varsigma_2,...,\varsigma_r,\tau+1,\mu+1,\mu+v+1;\\ \vartheta_1,...\vartheta_s,\frac{\mu+v+1}{2},\frac{\mu+v+2}{2},\frac{\mu+\tau+2}{2},\frac{\mu+\tau+3}{2};\end{matrix}-2z\right]$$

$$\times{}_3F_3\left[\begin{matrix}-\lambda,\mu+n+v+1,\mu+n+1;\\ \mu+2n+v+1,\mu+n+\tau+n+2,\ 1;\end{matrix}1\right]. \tag{25}$$

$$I_6 = \int_{-1}^{1}(1-x)^{\delta}(1+x)^{\nu}P_n^{(\tau,\nu)}(x)\,{}_rF_s\left[z(1-x)^l\right]dx$$

$$= {}_rF_s\left[\begin{matrix}(\varsigma_1,p),\varsigma_2,...,\varsigma_r;\\ \vartheta_1,...,\vartheta_s;\end{matrix}z\right]\int_{-1}^{1}(1-x)^{\delta+nl}(1+x)^{\nu}dx$$

$$\times\frac{(1+\tau)_n}{n!}{}_2F_1\left[\begin{matrix}-n,1+\tau+\nu+n;\\ 1+\tau;\end{matrix}\frac{1-x}{2}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\varsigma_1, p)_n (\varsigma_2)_n ... (\varsigma_r)_n}{(\vartheta_1)_n ... (\vartheta_s)_n} \frac{(1+\tau)_n}{n!} \frac{z^n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\tau+\nu+n)_k}{(1+\tau)_k 2^k k!}$$

$$\times \int_{-1}^{1} (1-x)^{\delta+nl+k} (1+x)^{\nu} \, dx. \tag{26}$$

Using (19), we obtain the following result

$$I_6 = \sum_{n=0}^{\infty} \frac{(\varsigma_1, p)_n (\varsigma_2)_n ... (\varsigma_r)_n \ (1+\tau)_n}{(\vartheta_1)_n ... (\vartheta_s)_n} \frac{z^n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\tau+\nu+n)_k (n+1)_k}{(1+\tau)_k 2^k k! (n+k)!}$$

$$\times 2^{\delta+nl+k+\nu+1} \frac{\Gamma(1+\delta+nl+k)\Gamma(1+\nu)}{\Gamma(\delta+nl+k+\nu+2)}$$

$$= 2^{\delta+\nu+1}\Gamma(1+\nu) \, {}_{r+1}F_s \left[\begin{matrix} (\varsigma_1, p), \varsigma_2, ..., \varsigma_r, 1+\tau; \\ \vartheta_1, ..., \vartheta_s; \end{matrix} 2^l z \right]$$

$$\times \sum_{k=0}^{\infty} \frac{(-n)_k (1+\tau+\nu+n)_k (n+1)_k (1+\delta+nl)_k}{(1+\tau)_k k! (n+k)! (\delta+nl+\nu+2)_k} \frac{\Gamma(1+\delta+nl)}{\Gamma(\delta+nl+\nu+2)}. \tag{27}$$

By substituting $(n+k)! = (n+1)_k \, n! = (n+1)_k \, (1)_n$ into Equation (27), we get

$$I_6 = 2^{\delta+\nu+1}\Gamma(1+\nu) \, {}_{r+1}F_s \left[\begin{matrix} (\varsigma_1, p), \varsigma_2, ..., \varsigma_r, 1+\tau; \\ \vartheta_1, ..., \vartheta_s; \end{matrix} 2^l z \right]$$

$$\times \sum_{k=0}^{\infty} \frac{(-n)_k (1+\tau+\nu+n)_k (n+1)_k (1+\delta+nl)_k}{(1+\tau)_k (n+1)_k (1)_n (\delta+nl+\nu+2)_k} \frac{1^k}{k!} \frac{\Gamma(1+\delta+nl)}{\Gamma(\delta+nl+\nu+2)}$$

$$= 2^{\delta+\nu+1}\Gamma(1+\nu) \, {}_{r+1}F_{s+1} \left[\begin{matrix} (\varsigma_1, p), \varsigma_2, ..., \varsigma_r, 1+\tau; \\ \vartheta_1, ..., \vartheta_s, 1; \end{matrix} z2^l \right]$$

$$\times {}_3F_2 \left[\begin{matrix} -n, 1+\tau+\nu+n, 1+\delta+nl; \\ 1+\tau, \delta+nl+\nu+2; \end{matrix} 1 \right] \frac{\Gamma(1+\delta+nl)}{\Gamma(\delta+nl+\nu+2)}. \tag{28}$$

In particular, if $l = 1$, then

$$I_6 = 2^{\delta+\nu+1} B(1+\nu, 1+\delta) {}_{r+2}F_{s+2} \left[\begin{matrix} (\varsigma_1, p), \varsigma_2, ..., \varsigma_r, 1+\tau, 1+\delta; \\ \vartheta_1, ..., \vartheta_s, 1, \delta+\nu+2; \end{matrix} 2z \right]$$

$$\times {}_3F_2 \left[\begin{matrix} -n, 1+\tau+\nu+n, 1+\delta+n; \\ 1+\tau, \delta+n+\nu+2; \end{matrix} 1 \right]. \tag{29}$$

$$I_7 = \int_{-1}^{1} (1-x)^{\xi} (1+x)^{\omega} P_n^{(\tau,\nu)}(x)_r F_s \left[z (1-x)^l (1+x)^h \right] dx$$

$$= \sum_{n=0}^{\infty} \frac{(\varsigma_1, p)_n (\varsigma_2)_n ... (\varsigma_r)_n}{(\vartheta_1)_n ... (\vartheta_s)_n} \frac{z^n}{n!} \int_{-1}^{1} (1-x)^{\xi+nl} (1+x)^{\omega+nh} P_n^{(\tau,\nu)} dx$$

$$= {}_{r+1}F_s \left[\begin{matrix} (\varsigma_1, p), \varsigma_2, ..., \varsigma_r, 1+\tau; \\ \vartheta_1, ..., \vartheta_s; \end{matrix} z \right] \sum_{k=0}^{\infty} \frac{(-n)_k (1+\tau+\nu+n)_k}{(1+\tau)_k k! 2^k n!}$$

$$\times \int_{-1}^{1} (1-x)^{\xi+nl+k} (1+x)^{\omega+nh} dx.$$

On using the result (19), we obtain

$$I_7 = {}_{r+1}F_s \left[\begin{matrix} (\varsigma_1, p), \varsigma_2, ..., \varsigma_r, 1+\tau; \\ \vartheta_1, ..., \vartheta_s; \end{matrix} z \right] \sum_{k=0}^{\infty} \frac{(-n)_k (1+\tau+\nu+n)_k}{(1+\tau)_k k! 2^k n!}$$

$$\times 2^{\xi+nl+k+\omega+nh+1} B (1+\xi+nl+k, 1+\omega+nh)$$

$$= 2^{\xi+\omega+1} {}_{r+1}F_s \left[\begin{matrix} (\varsigma_1, p), \varsigma_2, ..., \varsigma_r, 1+\tau; \\ \vartheta_1, ..., \vartheta_s; \end{matrix} 2^{l+h} z \right] \sum_{k=0}^{\infty} \frac{(-n)_k (1+\tau+\nu+n)_k}{(1+\tau)_k \; n!} \frac{1}{k!}$$

$$\times \frac{\Gamma(1+\xi+nl+k) \Gamma(1+\omega+nh)}{\Gamma(\xi+n(l+h)+k+\omega+2)}$$

$$= 2^{\xi+\omega+1} {}_{r+1}F_{s+1} \left[\begin{matrix} (\varsigma_1, p), \varsigma_2, ..., \varsigma_r, 1+\tau; \\ \vartheta_1, ..., \vartheta_s, 1; \end{matrix} 2^{l+h} z \right]$$

$$\times {}_3F_2 \left[\begin{matrix} -n, 1+\tau+\nu+n, 1+\xi+nl; \\ 1+\tau, \xi+n(l+h)+\omega+2; \end{matrix} 1 \right] \frac{\Gamma(1+\xi+nl) \Gamma(1+\omega+nh)}{\Gamma(\xi+n(l+h)+\omega+2)}. \tag{30}$$

For the particular value of $l = 1$ and $h = 1$, we arrive at the following result

$$I_7 = 2^{\xi+\omega+1} B(1+\xi, 1+\omega)_{r+3} F_{s+3} \left[\begin{matrix} (\varsigma_1, p), \varsigma_2, ..., \varsigma_r, 1+\tau, 1+\xi, 1+\omega; \\ \vartheta_1, ..., \vartheta_s, 1, \frac{\xi+\omega+2}{2}, \frac{\xi+\omega+3}{2}; \end{matrix} 4z \right]$$

$$\times {}_3F_2 \left[\begin{matrix} -n, 1+\tau+\nu+n, 1+\xi+n; \\ 1+\tau, \xi+2n+\omega+2; \end{matrix} 1 \right]. \tag{31}$$

$$I_8 = \int_{-1}^{1} (1-x)^{\xi} (1+x)^{\omega} P_n^{(\tau,v)}(x)_r F_s \left[z (1+x)^{-l} \right] dx$$

$$= \sum_{n=0}^{\infty} \frac{(\varsigma_1, p)_n (\varsigma_2)_n ... (\varsigma_r)_n}{(\vartheta_1)_n ... (\vartheta_s)_n} \frac{z^n}{n!} \int_{-1}^{1} (1-x)^{\xi} (1+x)^{\omega - nl} P_n^{(\tau,\upsilon)}(x) dx,$$

$$= {}_{r+1}F_s \left[\begin{matrix} (\varsigma_1, p), \varsigma_2, ..., \varsigma_r, 1+\tau; \\ \vartheta_1, ..., \vartheta_s; \end{matrix} z \right] \sum_{k=0}^{\infty} \frac{(-n)_k (1+\tau+\upsilon+n)_k}{(1+\tau)_k k! n! 2^k}$$

$$\times \int_{-1}^{1} (1-x)^{\xi+k} (1+x)^{\omega - nl} dx,$$

$$= 2^{\xi+\omega+1} {}_{r+1}F_s \left[\begin{matrix} (\varsigma_1, p), \varsigma_2, ..., \varsigma_r, 1+\tau; \\ \vartheta_1, ..., \vartheta_s; \end{matrix} 2^{-l} z \right]$$

$$\times \sum_{k=0}^{\infty} \frac{(-n)_k (1+\tau+\upsilon+n)_k}{(1+\tau)_k k! n!} \frac{\Gamma(1+\xi+k)\Gamma(1+\omega-nl)}{\Gamma(\xi+\omega+k-nl+2)}$$

$$= 2^{\xi+\omega+1} \Gamma(1+\xi) {}_{r+1}F_{s+1} \left[\begin{matrix} (\varsigma_1, p), \varsigma_2, ..., \varsigma_r, 1+\tau; \\ \vartheta_1, ..., \vartheta_s, 1; \end{matrix} 2^{-l} z \right]$$

$$\times \sum_{k=0}^{\infty} \frac{(-n)_k (1+\tau+\upsilon+n)_k (1+\xi)_k}{(1+\tau)_k k!} \frac{\Gamma(1+\omega-nl)}{\Gamma(\xi+\omega+k-nl+2)}$$

$$= 2^{\xi+\omega+1} \Gamma(1+\xi) {}_{r+1}F_{s+1} \left[\begin{matrix} (\varsigma_1, p), \varsigma_2, ..., \varsigma_r, 1+\tau; \\ \vartheta_1, ..., \vartheta_s, 1; \end{matrix} 2^{-l} z \right]$$

$$\times {}_3F_2 \left[\begin{matrix} -n, 1+\tau+\upsilon+n, 1+\xi; \\ 1+\tau, \xi+\omega-nl+2; \end{matrix} 1 \right] \frac{\Gamma(1+\omega-nl)}{\Gamma(\xi+\omega-nl+2)}. \tag{32}$$

However, if $l = -1$, then (32) becomes

$$I_8 = 2^{\xi+\omega+1} B(1+\xi, 1+\omega) {}_{r+2}F_{s+2} \left[\begin{matrix} (\varsigma_1, p), \varsigma_2, ..., \varsigma_r, 1+\tau, 1+\omega; \\ \vartheta_1, ..., \vartheta_s, 1, \xi+\omega+2; \end{matrix} 2z \right]$$

$$\times {}_3F_2 \left[\begin{matrix} -n, 1+\tau+\upsilon+n, 1+\xi; \\ 1+\tau, \xi+\omega+n+2; \end{matrix} 1 \right]. \tag{33}$$

$$I_9 = \int_{-1}^{1} (1-x)^{\xi} (1+x)^{\omega} P_n^{(\tau,\nu)}(x) {}_rF_s \left[z (1-x)^{l} (1+x)^{-h} \right] dx$$

$$= \sum_{n=0}^{\infty} \frac{(\varsigma_1, p)_n (\varsigma_2)_n ... (\varsigma_r)_n}{(\vartheta_1)_n ... (\vartheta_s)_n} \frac{z^n}{n!} \int_{-1}^{1} (1-x)^{\xi+nl} (1+x)^{\omega-nh} P_n^{(\tau,\nu)}(x) dx$$

$$= \sum_{n=0}^{\infty} \frac{(\varsigma_1, p)_n (\varsigma_2)_n ... (\varsigma_r)_n}{(\vartheta_1)_n ... (\vartheta_s)_n} \frac{z^n}{n!} \frac{(1+\tau)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\tau+\nu+n)_k}{(1+\tau)_k k! 2^k}$$

$$\times \int_{-1}^{1} (1-x)^{\xi+nl+k}\,(1+x)^{\omega-nh}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n(1+\tau)_n}{(\vartheta_1)_n...(\vartheta_s)_n(1)_n}\frac{z^n}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k\,(1+\tau+\nu+n)_k}{(1+\tau)_k\,k!2^k}$$

$$\times 2^{\xi+\omega+1+n(l-h)+k}\,\frac{\Gamma(\xi+nl+k+1)\Gamma(\omega-nh+1)}{\Gamma(\xi+\omega+nl-nh+k+2)}$$

$$= 2^{\xi+\omega+1}{}_{r+1}F_{s+1}\left[\begin{matrix}(\varsigma_1,p),\varsigma_2,...,\varsigma_r,1+\tau;\\ \vartheta_1,...,\vartheta_s,1;\end{matrix}\,2^{l-h}z\right]$$

$$\times {}_3F_2\left[\begin{matrix}-n,1+\tau+\nu+n,\xi+nl+1;\\ 1+\tau,\xi+\omega+nl-nh+2;\end{matrix}\,1\right]\frac{\Gamma(\xi+nl+1)\Gamma(\omega-nh+1)}{\Gamma(\xi+\omega+nl-nh+2)}.$$

If $l=1$ and $h=-1$, then

$$I_9 = 2^{\xi+\omega+1}B(\omega+1,\xi+1)_{r+3}F_{s+3}\left[\begin{matrix}(\varsigma_1,p),\varsigma_2,...,\varsigma_r,1+\tau,\xi+1,\omega+1;\\ \vartheta_1,...,\vartheta_s,1,((\xi+\omega)/2)+1,((\xi+\omega)/2)+1;\end{matrix}\,4z\right]$$

$$\times {}_3F_2\left[\begin{matrix}-n,1+\tau+\nu+n,\xi+n+1;\\ 1+\tau,\xi+\omega+2n++2;\end{matrix}\,1\right]. \tag{34}$$

5. Integrals involving generalized hypergoemetric functions with Legendre function

In this part, we derive integral formulas involving generalized hypergoemetric functions multiplied with Legendre function.

$$I_{10} = \int_0^1 x^{\omega-1}\left(1-x^2\right)^{\frac{\tau}{2}} P_\nu^\tau(x)\,{}_rF_s\left[zx^\xi\right]dx$$

$$= \int_0^1 x^{\omega-1}\left(1-x^2\right)^{\frac{\tau}{2}} P_\nu^\tau(x)\sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}x^{n\xi}dx$$

$$= \sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}\int_0^1 x^{\omega+n\xi-1}\left(1-x^2\right)^{\frac{\tau}{2}} P_\nu^\tau(x)dx. \tag{35}$$

Using the formula (Erdélyi et al. (1953), Sec 3.12) for $\Re(\omega)>0,\ \tau\in\mathbb{N}$

$$\int_0^1 x^{\omega-1}\left(1-x^2\right)^{\frac{\tau}{2}} P_\nu^\tau(x)dx = \frac{(-1)^\tau\,2^{-\omega-\tau}\pi^{\frac{1}{2}}\Gamma(\omega)\Gamma(1+\tau+\nu)}{\Gamma(1-\tau+\nu)\Gamma\left(\frac{1}{2}+\frac{\omega}{2}+\frac{\tau-\nu}{2}\right)\Gamma\left(1+\frac{\omega}{2}+\frac{\tau+\nu}{2}\right)}, \tag{36}$$

then the integral (35) becomes

$$I_{10}=\sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}\frac{(-1)^{\tau}\,2^{-\omega-\tau-n\xi}\pi^{\frac{1}{2}}\Gamma(\omega+n\xi)\Gamma(1+\tau+\nu)}{\Gamma(1-\tau+\nu)\Gamma\left(\frac{1+\omega+n\xi+\tau-\nu}{2}\right)\Gamma\left(1+\frac{\omega+n\xi+\tau+\nu}{2}\right)}$$
$$=\frac{(-1)^{\tau}\,2^{-\omega-\tau}\pi^{\frac{1}{2}}\Gamma(1+\tau+\nu)}{\Gamma(1-\tau+\nu)}\sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{(2^{-\xi}z)^n}{n!}$$
$$\times\frac{\Gamma(\omega+n\xi)}{\Gamma\left(\frac{1+\omega+n\xi+\tau-\nu}{2}\right)\Gamma\left(1+\frac{\omega+n\xi+\tau+\nu}{2}\right)}. \tag{37}$$

In particular, if $\xi=2$ in (37), then

$$I_{10}=\frac{(-1)^{\tau}\,2^{-\omega-\tau}\pi^{\frac{1}{2}}\Gamma(1+\tau+\nu)\Gamma(\omega)}{\Gamma(1-\tau+\nu)\Gamma\left(\frac{1+\omega+\tau-\nu}{2}\right)\Gamma\left(1+\frac{\omega+\tau+\nu}{2}\right)}$$
$$\times{}_{r+2}F_{s+2}\left[\begin{matrix}(\varsigma_1,p),\varsigma_2,...,\varsigma_r,\frac{\omega}{2},\frac{\omega+1}{2}; & z\\ \vartheta_1,...,\vartheta_s,\frac{1+\omega+\tau-\nu}{2},1+\frac{\omega+\tau+\nu}{2}; & \overline{4}\end{matrix}\right]. \tag{38}$$

$$I_{11}=\int_0^1 x^{\omega-1}\left(1-x^2\right)^{\frac{-\tau}{2}}P_\nu^\tau(x)\,{}_rF_s\left[zx^{\xi}\right]dx$$
$$=\sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}\int_0^1 x^{\omega+n\xi-1}\left(1-x^2\right)^{\frac{-\tau}{2}}P_\nu^\tau(x)dx.$$

Using the formula (Erdélyi et al. (1953), Sec 3.12) for $\Re(\omega)>0,\ \tau\in\mathbb{N}$,

$$\int_0^1 x^{\omega-1}\left(1-x^2\right)^{\frac{-\tau}{2}}P_\nu^\tau(x)dx=\frac{2^{\tau-\omega}\pi^{\frac{1}{2}}\Gamma(\omega)}{\Gamma\left(\frac{1+\omega-\tau-\nu}{2}\right)\Gamma\left(1+\frac{\omega-\tau-\nu}{2}\right)}, \tag{39}$$

we get

$$I_{11}=\sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}\frac{2^{\tau-\omega-n\xi}\pi^{\frac{1}{2}}\Gamma(\omega+n\xi)}{\Gamma\left(\frac{1+\omega+n\xi-\tau-\nu}{2}\right)\Gamma\left(1+\frac{\omega+n\xi-\tau-\nu}{2}\right)}$$
$$=2^{\tau-\omega}\pi^{\frac{1}{2}}{}_rF_s\left[\begin{matrix}(\varsigma_1,p),\varsigma_2,...,\varsigma_r; & 2^{-\xi}z\\ \vartheta_1,...,\vartheta_s; & \end{matrix}\right]\frac{\Gamma(\omega+n\xi)}{\Gamma\left(\frac{1+\omega+n\xi-\tau-\nu}{2}\right)\Gamma\left(1+\frac{\omega+n\xi-\tau-\nu}{2}\right)}.$$

Particularly, if $\xi=2$, then

$$I_{11}=2^{\tau-\omega}\pi^{\frac{1}{2}}\frac{\Gamma(\omega)}{\Gamma\left(\frac{1+\omega-\tau-\nu}{2}\right)\Gamma\left(1+\frac{\omega-\tau-\nu}{2}\right)}$$

$$\times {}_{r+2}F_{s+2}\left[\begin{matrix}(\varsigma_1,p),\varsigma_2,...,\varsigma_r,\frac{\omega}{2},\frac{\omega+1}{2};\\ \vartheta_1,...,\vartheta_s,\frac{1+\omega-\tau-\nu}{2},1+\frac{\omega-\tau-\nu}{2};\end{matrix}\frac{z}{4}\right]. \tag{40}$$

6. Integrals involving generalized hypergeometric functions with general class of polynomials

From the basic definitions, we obtained the following results.

$$\begin{aligned}
I_{12} &= \int_{-1}^{1}(1-x)^{\xi-1}(1+x)^{\omega-1}S_{n_1,\dots,n_r}^{m_1,\dots,m_r}\left[z(1-x)^{\tau}(1+x)^{\nu}\right]\\
&\quad\times {}_rF_s\left[z(1-x)^{h}(1+x)^{k}\right]dx\\
&= \int_{-1}^{1}(1-x)^{\xi-1}(1+x)^{\omega-1}\sum_{l_1=0}^{\left[\frac{n_1}{m_1}\right]}...\sum_{l_r=0}^{\left[\frac{n_r}{m_r}\right]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}z^{l_i}(1-x)^{\tau l_i}\\
&\quad\times(1+x)^{\nu l_i}\sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}(1-x)^{nh}(1+x)^{nk}dx\\
&= \sum_{l_1=0}^{\left[\frac{n_1}{m_1}\right]}...\sum_{l_r=0}^{\left[\frac{n_r}{m_r}\right]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}z^{l_i}\sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}\\
&\quad\times\int_{-1}^{1}(1-x)^{\xi+\tau l_i+nh-1}(1+x)^{\omega+\nu l_i+nk-1}dx
\end{aligned}\tag{41}$$

$$\begin{aligned}
&= \sum_{l_1=0}^{\left[\frac{n_1}{m_1}\right]}...\sum_{l_r=0}^{\left[\frac{n_r}{m_r}\right]}\prod_{i=1}^{r}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}z^{l_i}\sum_{n=0}^{\infty}\frac{(\varsigma_1,p)_n(\varsigma_2)_n...(\varsigma_r)_n}{(\vartheta_1)_n...(\vartheta_s)_n}\frac{z^n}{n!}\\
&\quad\times 2^{\omega+\xi+(\tau+\nu)l_i+n(h+k)-1}\frac{\Gamma(\xi+\tau l_i+nh)\,\Gamma(\omega+\nu l_i+nk)}{\Gamma(\omega+\xi+(\tau+\nu)l_i+n(h+k))}\\
&= S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[z]\,{}_rF_s\left[\begin{matrix}(\varsigma_1,p),\varsigma_2...,\varsigma_r;\\ \vartheta_1,...,\vartheta_s;\end{matrix}z\right]2^{\omega+\xi+(\tau+\nu)l_i+n(h+k)-1}\\
&\quad\times\frac{\Gamma(\xi+\tau l_i+nh)\,\Gamma(\omega+\nu l_i+nk)}{\Gamma(\omega+\xi+(\tau+\nu)l_i+n(h+k))}.
\end{aligned}\tag{42}$$

In particular, if $h=k=1$, then

$$\begin{aligned}
I_{12} &= S_{n_1,\dots,n_r}^{m_1,\dots,m_r}[z]\,{}_{r+2}F_{s+2}\left[\begin{matrix}(\varsigma_1,p),\varsigma_2...,\varsigma_r,\xi+\tau l_i,\omega+\nu l_i;\\ \vartheta_1,...,\vartheta_s,\frac{\omega+\xi+(\tau+\nu)l_i}{2},\frac{\omega+\xi+(\tau+\nu)l_i+1}{2};\end{matrix}4z\right]\\
&\quad\times 2^{\omega+\xi+(\tau+\nu)l_i-1}B(\xi+\tau l_i,\omega+\nu l_i).
\end{aligned}\tag{43}$$

7. Conclusion

We conclude here by remarking that the numerous further consequences of our results can easily be derived by using some known and new relationships between generalized hypergeometric-functions, which is an elegant unification of various special functions such as: Gauss's hypergeometric and confluent hypergeometric functions, after some suitable parametric replacements. The results obtained here are basic in nature and are likely to find useful applications in the study of simple and multiple variable hypergeometric series which in turn are useful in statistical mechanics, electrical networks, and probability theory.

Acknowledgment:

The authors are thankful to the referees for their valuable remarks and comments for the improvement of the paper.

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