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Some Quadratic Transformations and Reduction Formulas associated with Hypergeometric Functions

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Abstract

In this paper, we construct four summation formulas for terminating Gauss' hypergeometric series having argument "two" and with the help of our summation formulas. We establish two quadratic transformations for Gauss' hypergeometric function in terms of finite summation of combination of two Clausen hypergeometric functions. Further, we have generalized our quadratic transformations in terms of general double series identities as well as in terms of reduction formulas for Kampé de Fériet's double hypergeometric function. Some results of Rathie-Nagar, Kim et al. and Choi-Rathie are also obtained as special cases of our findings.

Keywords: Generalized hypergeometric function; Hypergeometric summation theorem; Bounded sequence; Quadratic transformation; Reduction formulas

MSC 2010 No.: 33C05, 33C20

1. Introduction

Functions which are important enough to be given their own name are known as “special function”. These include the well known logarithmic, exponential, and trigonometric functions and extend to cover the gamma, beta, and zeta functions and the class of orthogonal polynomials, among many others. Special functions have broad applications in pure mathematics as well as in applied areas such as quantum mechanics, solutions of wave equations, moments of inertia, fluid dynamics, and heat conduction.

In 1656, John Wallis was the first mathematician who used the term “hypergeometric series" in his treatise “Arithmetica Infinitorum” to explicate the infinite series of the form $1 + a + a(a + 1) + a(a + 1)(a + 2) + \cdots$. In 1730, Euler established nearly all of the significant properties of the Gamma functions. The series ${}_1F_1$ was introduced by Kümmer (1836) and the series ${}_3F_2$ was given by Clausen (1828). In the theory of hypergeometric series a major development was given (although Euler and Pfaff had been found certain important results) by Gauss (1866) on ${}_2F_1$ series. The Hypergeometric functions of two variables was introduced by Appell (1880) and Lauricella (1893) generalized them to several variables.

It is interesting to mention here that the results are very important for the application point of view, whenever hypergeometric functions reduce to gamma functions. Thus, the classical theorems such as Gauss, Kümmer and Bailey for the series ${}_2F_1$ and of Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$ and others play an important role in the theory of hypergeometric function, generalized hypergeometric function and in other applicable sciences. In a series of papers Lavoie et al. (1992, 1994, 1996) studied generalizations of some of the above classical summation theorems. In the theory of special functions, summations, transformations and reduction formulas have received keen attention during the last few years (see Chen and Srivastava (2005), Chen et al. (2006), Chu (2011), Kim (2009), Kim et al. (2012) and Kim et al. (2010)). Recently, in Miller (2009), Miller and Paris (2011a, 2011b) and Miller and Srivastava (2010) examined the generalized hypergeometric series (together with integer parameter differences) and derived several transformations and summation formulas.

Influenced by the recent work of researchers Choi and Rathie (2014), Meethal et al. (2015), Ebisu (2017), Kim, Gaboury et al. (2018, 2018), Miller and Paris (2013), Qureshi et al. (2016), and Srivastava (2016), some new summation formulas, quadratic transformations, generalizations of quadratic transformations and reduction formulas for Kampé de Fériet’s double hypergeometric function are obtained in this article.

2. Preliminaries

In our present investigation, we shall make use of the following standard notations and results.

$\mathbb{N} = \{1, 2, 3, \cdots\}$, $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$ and $\mathbb{Z}_0^- = \mathbb{Z}^- \bigcup \{0\} = \{0, -1, -2, -3, \cdots\}$.

As usual, the symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers, respectively.

The well-known Pochhammer's symbol $(\alpha)_p$ $(\alpha, p \in \mathbb{C})$ is defined by (see Rainville (1960)), Srivastava and Manocha (1984))

$$(\alpha)_p = \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)} = \begin{cases} 1; & (p=0; \alpha \in \mathbb{C}\backslash\{0\}), \\ \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1); & (p=n \in \mathbb{N}; \alpha \in \mathbb{C}), \\ \frac{(-1)^k n!}{(n-k)!}; & (\alpha=-n; p=k; k,n \in \mathbb{N}_0; 0 \leq k \leq n), \\ 0; & (\alpha=-n; p=k; k,n \in \mathbb{N}_0; k>n), \\ \frac{(-1)^n}{(1-\alpha)_n}; & (p=-n; n \in \mathbb{N}; \alpha \in \mathbb{C}\backslash\mathbb{Z}), \end{cases} \tag{1}$$

it being understood usually that $(0)_0 = 1$ and concluded tacitly that the Gamma quotient exists.

Generalized ordinary hypergeometric function of one variable is defined by (see Srivastava and Manocha (1984))

$${}_AF_B\left[\begin{array}{c} (a_A)\,; \\ \\ (b_B)\,; \end{array} z\right] \equiv {}_AF_B\left[\begin{array}{c} a_1,\, a_2,\, \ldots,\, a_A\,; \\ \\ b_1,\, b_2,\, \ldots,\, b_B\,\,; \end{array} z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!}, \tag{2}$$

where denominator parameters $b_1,\, b_2,\, \ldots,\, b_B$ are neither zero nor negative integers, and $A,\; B$ are non-negative integers and numerator parameters $a_1,\, a_2,\, \ldots,\, a_A$ may be zero or negative integers.

Convergence Conditions of Series (2):

Suppose that numerator parameters $a_1,\, a_2,\, \ldots,\, a_A$ are neither zero nor negative integers (otherwise the question of convergence will not arise).

(i) If $A \leq B$, then series ${}_AF_B$ is always convergent for all finite values of z (real or complex), i.e., $|z| < \infty$.
(ii) If $A = B+1$ and $|z| < 1$, then series ${}_AF_B$ is convergent.
(iii) If $A = B+1$ and $|z| = 1$, then series ${}_AF_B$ is absolutely convergent, when
$$\Re\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} > 0,$$
(iv) If $A = B+1$ and $|z| = 1$, but $z \neq 1$, then series ${}_AF_B$ is conditionally convergent, when
$$-1 < \Re\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} \leq 0,$$
where $\Re$ denotes the real part of complex number throughout this paper.

Kampé de Fériet's Double Hypergeometric Function:

Kampé de Fériet (1921) combined and generalized Appell's four double hypergeometric functions $F_1,\ F_2,\ F_3,\ F_4$ (see Srivastava and Manocha (1984)) and their seven confluent forms $\Phi_1,\ \Phi_2,\ \Phi_3,\ \Psi_1,\ \Psi_2,\ \Xi_1,\ \Xi_2$ given by Humbert (1920-21). We reminisce here the definition of general double hypergeometric function of Kampé de Fériet (see Appell and Kampé de Fériet (1926)), Burchnall and Chaundy (1941)) in the slightly refined notation of Srivastava and Panda (1976), Srivastava and Pathan (1979). Thus, the appropriate generalization of the Kampé de Fériet function is given by

$$F^{A:B;D}_{E:G;H}\left[\begin{array}{l}(a_A):(b_B);(d_D);\\ \\ (e_E):(g_G);(h_H);\end{array} x,\ y\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[(a_A)]_{m+n}[(b_B)]_m[(d_D)]_n\,x^m\,y^n}{[(e_E)]_{m+n}[(g_G)]_m[(h_H)]_n\,m!\,n!},\tag{3}$$

where (a_A) denotes the array of A number of parameters $a_1,\ a_2,\ \ldots,\ a_A$ and

$$[(a_A)]_m=\prod_{j=1}^{A}(a_j)_m,$$

with similar interpretation for others.

Convergence Conditions of Double Series (3) (see Srivastava and Panda (1976)):

(i) $A+B<E+G+1,\ A+D<E+H+1,\ \ |x|<\infty,\ |y|<\infty,$ or
(ii) $A+B=E+G+1,\ A+D=E+H+1,$ and

$$\left\{\begin{array}{ll}|x|^{\frac{1}{(A-E)}}+|y|^{\frac{1}{(A-E)}}<1, \text{if} A>E,\\ \\ \max\{|x|,|y|\}<1, & \text{if} A\le E\end{array}\right\}.$$

For absolutely and conditionally convergence of double series (3), the reader can consult an article by Hài et al. (1992).

The Series Decomposition Identity is given by (see Srivastava and Manocha (1984))

$$\sum_{n=0}^{\infty}\Lambda(n)=\sum_{n=0}^{\infty}\Lambda(2n)+\sum_{n=0}^{\infty}\Lambda(2n+1).\tag{4}$$

The Reversal of the Order of Terms in Finite Summation is given by

$$\sum_{r=0}^{n}\Phi(r)=\sum_{r=0}^{n}\Phi(n-r),\qquad (n\in\mathbb{N}_0).\tag{5}$$

The Series Rearrangement Formulas are given as follows (see Srivastava and Manocha (1984))

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Xi(m,\,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\Xi(m,\,n-m), \tag{6}$$

and

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Psi(m,\,n)=\sum_{m=0}^{\infty}\sum_{n=0}^{[\frac{m}{2}]}\Psi(m-2n,\,n), \tag{7}$$

provided that involved double series are absolutely convergent and $[k]$ denotes the greatest integer for $k \in \mathbb{R}$.

The rest of the article is organized as follows. In Section 3, we obtain four summation formulas for terminating Gauss' hypergeometric series having argument "two" by means of summation formula of Rakha and Rathie (2011), and summation formula recorded by Prudnikov et al. (1990). In Section 4, as an application of our four summation formulas, we established two new quadratic transformations for Gauss' hypergeometric function. In Section 5, we have generalized our quadratic transformations in terms of general double series identities having bounded sequences. In Section 6, we obtain two results for the reducibility of Kampé de Fériet double hypergeometric functions. In Section 7, we discuss several special cases of Sections 4 and 6. Any values of parameters and variables leading to the results in following sections which do not make sense are tacitly excluded.

3. Four Summation Formulas

The following four summation formulas for terminating Gauss series will be derived in this section.

First Summation Formula.

$${}_2F_1\left[\begin{array}{c}-2n,\,a;\\ \\ 2a+j;\end{array}\;2\right]=\frac{\Gamma(1-a)}{2^{2a+j}(a)_j\Gamma(1-2a-j)}\sum_{r=0}^{j}\left\{\binom{j}{r}(-1)^r\frac{\Gamma(\frac{1-2a-j+r}{2})(\frac{1-r+j}{2})_n}{\Gamma(\frac{1+r-j}{2})(\frac{1+2a-r+j}{2})_n}\right\}, \tag{8}$$

$$\left(a,1-a,2a+j,1-2a-j\in\mathbb{C}\backslash\mathbb{Z}_0^-\,;n,j\in\mathbb{N}_0\right).$$

Second Summation Formula.

$${}_2F_1\left[\begin{array}{c}-2n-1,\,a;\\ \\ 2a+j;\end{array}\;2\right]=\frac{-\,\Gamma(1-a)}{2^{2a+j}(a)_j\Gamma(1-2a-j)}\sum_{r=0}^{j}\left\{\binom{j}{r}(-1)^r\frac{\Gamma(\frac{-2a-j+r}{2})(\frac{2-r+j}{2})_n}{\Gamma(\frac{r-j}{2})(\frac{2+2a-r+j}{2})_n}\right\}, \tag{9}$$

$$\left(a, 1-a, 2a+j, 1-2a-j \in \mathbb{C}\backslash\mathbb{Z}_0^- \,; n, j \in \mathbb{N}_0\right).$$

Third Summation Formula.

$$ {}_2F_1\left[\begin{array}{c} -2n,\ a; \\ \\ 2a-j; \end{array} 2\right] = \frac{\Gamma(1-a)}{2^{2a-j}\Gamma(1-2a+j)} \sum_{r=0}^{j} \left\{ \binom{j}{r} \frac{\Gamma(\frac{1-2a+j+r}{2})(\frac{1-r+j}{2})_n}{\Gamma(\frac{1+r-j}{2})(\frac{1+2a-r-j}{2})_n} \right\}, \tag{10}$$

$$\left(a, 1-a, 2a-j, 1-2a+j \in \mathbb{C}\backslash\mathbb{Z}_0^- \,; n, j \in \mathbb{N}_0\right).$$

Fourth Summation Formula.

$$ {}_2F_1\left[\begin{array}{c} -2n-1,\ a; \\ \\ 2a-j; \end{array} 2\right] = \frac{-\Gamma(1-a)}{2^{2a-j}\Gamma(1-2a+j)} \sum_{r=0}^{j} \left\{ \binom{j}{r} \frac{\Gamma(\frac{-2a+j+r}{2})(\frac{2-r+j}{2})_n}{\Gamma(\frac{r-j}{2})(\frac{2+2a-r-j}{2})_n} \right\}, \tag{11}$$

$$\left(a, 1-a, 2a-j, 1-2a+j \in \mathbb{C}\backslash\mathbb{Z}_0^- \,; n, j \in \mathbb{N}_0\right).$$

where the notation $\binom{j}{r}$ is the binomial coefficient.

Remark 3.1.

We have verified above summation formulas for suitable numerical values of n, a and j.

In order to start the derivation of above summation formulas consider the series,

$$ {}_2F_1\left[\begin{array}{c} -n,\ a; \\ \\ 2a+j; \end{array} 2\right] = \sum_{r=0}^{n} \frac{(-n)_r\,(a)_r\,2^r}{(2a+j)_r\,r!} = \sum_{r=0}^{n} \frac{n!\,(-1)^r\,(a)_r\,2^r}{(2a+j)_r\,r!\,(n-r)!},$$

which, upon replacing r by $n-r$ and applying reversal of the order of summation (5) and some algebraic properties of Pochhammer's symbol (1), becomes

$$ {}_2F_1\left[\begin{array}{c} -n,\ a; \\ \\ 2a+j; \end{array} 2\right] = \frac{(-1)^n\,(a)_n\,2^n}{(2a+j)_n}\, {}_2F_1\left[\begin{array}{c} -n,\ 1-2a-j-n; \\ \\ 1-a-n; \end{array} \tfrac{1}{2}\right]. \tag{12}$$

Proof:

In order to derive the first summation formula (8), we begin as follows.

Replacing n by $2n$ on both sides of equation (12), we obtain

$$ {}_2F_1\left[\begin{matrix} -2n,\, a; \\ \\ 2a+j; \end{matrix}\; 2\right] = \frac{(-1)^{2n}\,(a)_{2n}\,2^{2n}}{(2a+j)_{2n}}\,{}_2F_1\left[\begin{matrix} -2n,\, 1-2a-j-2n; \\ \\ 1-a-2n; \end{matrix}\; \tfrac{1}{2}\right]. \tag{13} $$

We now applying the summation formula of Rakha and Rathie (2011) in the right-hand side of equation (13), after straightforward calculation we easily arrive at the right-hand side of equation (8).

Similarly, by using the result (12) in conjugation with the result of Rakha and Rathie (2011), we can prove the second summation formula (9). ■

Proof:

In order to find the third summation formula (10), we continue as follows.

Changing n by $2n$ and j by $-j$ on both sides of equation (12), to yield

$$ {}_2F_1\left[\begin{matrix} -2n,\, a; \\ \\ 2a-j; \end{matrix}\; 2\right] = \frac{(-1)^{2n}\,(a)_{2n}\,2^{2n}}{(2a-j)_{2n}}\,{}_2F_1\left[\begin{matrix} -2n,\, 1-2a+j-2n; \\ \\ 1-a-2n; \end{matrix}\; \tfrac{1}{2}\right]. \tag{14} $$

By employing the summation formula recorded by Prudnikov et al. (1990) in the right-hand side of equation (14), after little algebra we obtain the right-hand side of equation (10).

Similarly, by utilizing the result (12) and summation formula recorded by Prudnikov et al. (1990), we can derive the fourth summation formula (11). ■

4. Two Quadratic Transformations

In this section, we shall establish the following two new quadratic transformations for Gauss series by means of series rearrangement technique and use of hypergeometric summation formulas (8) to (11).

First Transformation. The following transformation holds true:

$$(1+z)^{-2b}\,{}_2F_1\left[\begin{array}{l}2b,\ a;\\ \\ 2a+j;\end{array}\ \frac{2z}{1+z}\right]=\frac{\Gamma(1-a)}{2^{2a+j}(a)_j\Gamma(1-2a-j)}\sum_{r=0}^{j}\binom{j}{r}(-1)^r\left\{\frac{\Gamma(\frac{1-2a-j+r}{2})}{\Gamma(\frac{1+r-j}{2})}\right.$$
$$\times\,{}_3F_2\left[\begin{array}{l}b,\ b+\frac{1}{2},\ \frac{1-r+j}{2};\\ \\ \frac{1+2a+j-r}{2},\ \frac{1}{2};\end{array}\ z^2\right]+2bz\frac{\Gamma(\frac{-2a-j+r}{2})}{\Gamma(\frac{r-j}{2})}$$
$$\left.\times\,{}_3F_2\left[\begin{array}{l}b+\frac{1}{2},\ b+1,\ \frac{2-r+j}{2};\\ \\ \frac{2+2a+j-r}{2},\ \frac{3}{2};\end{array}\ z^2\right]\right\}, \tag{15}$$

$$\left(b,a,1-a,2a+j,1-2a-j\in\mathbb{C}\backslash\mathbb{Z}_0^-\,;j\in\mathbb{N}_0\right).$$

Second Transformation. The following transformation holds true:

$$(1+z)^{-2b}\,{}_2F_1\left[\begin{array}{l}2b,\ a;\\ \\ 2a-j;\end{array}\ \frac{2z}{1+z}\right]=\frac{\Gamma(1-a)}{2^{2a-j}\Gamma(1-2a+j)}\sum_{r=0}^{j}\binom{j}{r}\left\{\frac{\Gamma(\frac{1-2a+j+r}{2})}{\Gamma(\frac{1+r-j}{2})}\right.$$
$$\times\,{}_3F_2\left[\begin{array}{l}b,\ b+\frac{1}{2},\ \frac{1-r+j}{2};\\ \\ \frac{1+2a-j-r}{2},\ \frac{1}{2};\end{array}\ z^2\right]+2bz\frac{\Gamma(\frac{r-2a+j}{2})}{\Gamma(\frac{r-j}{2})}$$
$$\left.\times\,{}_3F_2\left[\begin{array}{l}b+\frac{1}{2},\ b+1,\ \frac{2-r+j}{2};\\ \\ \frac{2+2a-j-r}{2},\ \frac{3}{2};\end{array}\ z^2\right]\right\}, \tag{16}$$

$$\left(b,a,1-a,2a-j,1-2a+j\in\mathbb{C}\backslash\mathbb{Z}_0^-\,;j\in\mathbb{N}_0\right).$$

Proof:

In order to establish first transformation (15), we proceed as follows.

Denoting the series expansion of left-hand side of transformation (15) by ζ and after some simplifications, we have

$$\begin{aligned}
\zeta &= \sum_{m=0}^{\infty} \frac{(2b)_m\,(a)_m\,2^m\,z^m}{(2a+j)_m\,m!}\;{}_1F_0\left[\begin{matrix} 2b+m; \\ \\ \text{——}; \end{matrix}\quad -z\right] \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2b)_{m+n}\,(a)_m\,2^m\,(-1)^n\,z^{n+m}}{(2a+j)_m\,m!\,n!} \\
&= \sum_{n=0}^{\infty} \frac{(2b)_n\,(-1)^n\,z^n}{n!} \sum_{m=0}^{n} \frac{(a)_m\,2^m\,(-n)_m}{(2a+j)_m\,m!} \\
&= \sum_{n=0}^{\infty} \frac{(2b)_n\,(-1)^n\,z^n}{n!}\;{}_2F_1\left[\begin{matrix} -n,\,a; \\ \\ 2a+j; \end{matrix}\quad 2\right].
\end{aligned} \tag{17}$$

Now, use series decomposition identity (4) in the right-hand side of Equation (17), to yield

$$\zeta = \sum_{n=0}^{\infty} \frac{(2b)_{2n}\,(-1)^{2n}\,z^{2n}}{(2n)!}\;{}_2F_1\left[\begin{matrix} -2n,\,a; \\ 2 \\ 2a+j; \end{matrix}\right] + \sum_{n=0}^{\infty} \frac{(2b)_{2n+1}\,(-1)^{2n+1}\,z^{2n+1}}{(2n+1)!}\;{}_2F_1\left[\begin{matrix} -2n-1,\,a; \\ 2 \\ 2a+j; \end{matrix}\right]. \tag{18}$$

Finally, by employing summation formulas (8) and (9) in the right-hand side of equation (18), after some simplifications we get the right-hand side of transformation (15). ■

Proof:

In order to establish second transformation (16), we proceed as follows.

Denoting the series expansion of left-hand side of transformation (16) by ω and after some simplifications, we have

$$\begin{aligned}
\omega &= \sum_{m=0}^{\infty} \frac{(2b)_m\,(a)_m\,2^m\,z^m}{(2a-j)_m\,m!}\;{}_1F_0\left[\begin{matrix} 2b+m; \\ \\ \text{——}; \end{matrix}\quad -z\right] \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2b)_{m+n}\,(a)_m\,2^m\,(-1)^n\,z^{n+m}}{(2a-j)_m\,m!\,n!} \\
&= \sum_{n=0}^{\infty} \frac{(2b)_n\,(-1)^n\,z^n}{n!} \sum_{m=0}^{n} \frac{(a)_m\,2^m\,(-n)_m}{(2a-j)_m\,m!} \\
&= \sum_{n=0}^{\infty} \frac{(2b)_n\,(-1)^n\,z^n}{n!}\;{}_2F_1\left[\begin{matrix} -n,\,a; \\ \\ 2a-j; \end{matrix}\quad 2\right].
\end{aligned} \tag{19}$$

Applying series decomposition identity (4) in Equation (19), we find

$$\omega = \sum_{n=0}^{\infty} \frac{(2b)_{2n}\,(-1)^{2n}\,z^{2n}}{(2n)!}\,{}_2F_1\left[\begin{array}{c}-2n,\ a;\\ \\ 2a-j;\end{array}\ 2\right] + \sum_{n=0}^{\infty} \frac{(2b)_{2n+1}\,(-1)^{2n+1}\,z^{2n+1}}{(2n+1)!}\,{}_2F_1\left[\begin{array}{c}-2n-1,\ a;\\ \\ 2a-j;\end{array}\ 2\right]. \tag{20}$$

By using summation formulas (10) and (11) in the right-hand side of Equation (20), after some simplifications we easily get the right-hand side of transformation (16). ■

5. General Double Series Identities

Here we generalize our quadratic transformations (15) and (16) as in the following theorems.

Theorem 5.1.

Let $\{\Phi(n)\}_{n=1}^{\infty}$ be a bounded sequence of essentially arbitrary complex numbers such that $\Phi(0) \neq 0$. Then, we have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Phi(m+n)\frac{(a)_m\,(2z)^m\,(-z)^n}{(2a+j)_m\,m!\,n!} = \frac{\Gamma(1-a)}{2^{2a+j}(a)_j\Gamma(1-2a-j)}\sum_{r=0}^{j}\binom{j}{r}(-1)^r\left\{\frac{\Gamma(\frac{1-2a-j+r}{2})}{\Gamma(\frac{1+r-j}{2})}\times\sum_{n=0}^{\infty}\Phi(2n)\frac{(\frac{1-r+j}{2})_n\,z^{2n}}{(\frac{1+2a+j-r}{2})_n\,(2n)!} + \frac{\Gamma(\frac{-2a-j+r}{2})}{\Gamma(\frac{r-j}{2})}\times\sum_{n=0}^{\infty}\Phi(2n+1)\frac{(\frac{2-r+j}{2})_n\,z^{2n+1}}{(\frac{2+2a+j-r}{2})_n\,(2n+1)!}\right\}, \tag{21}$$

$$\left(a, 1-a, 2a+j, 1-2a-j \in \mathbb{C}\backslash\mathbb{Z}_0^-\,; j \in \mathbb{N}_0\right),$$

provided that single and double series involved are absolutely convergent.

Theorem 5.2.

Let $\{\Phi(n)\}_{n=1}^{\infty}$ be a bounded sequence of essentially arbitrary complex numbers such that $\Phi(0) \neq 0$. Then, we have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Phi(m+n)\frac{(a)_m\,(2z)^m\,(-z)^n}{(2a-j)_m\,m!\,n!}=\frac{\Gamma(1-a)}{2^{2a-j}\Gamma(1-2a+j)}\sum_{r=0}^{j}\binom{j}{r}\left\{\frac{\Gamma(\frac{1-2a+j+r}{2})}{\Gamma(\frac{1+r-j}{2})}\right.$$
$$\times\sum_{n=0}^{\infty}\Phi(2n)\frac{(\frac{1-r+j}{2})_n\,z^{2n}}{(\frac{1-r+2a-j}{2})_n\,(2n)!}+\frac{\Gamma(\frac{r-2a+j}{2})}{\Gamma(\frac{r-j}{2})}$$
$$\left.\times\sum_{n=0}^{\infty}\Phi(2n+1)\frac{(\frac{2-r+j}{2})_n\,z^{2n+1}}{(\frac{2-r+2a-j}{2})_n\,(2n+1)!}\right\}, \tag{22}$$

$$\left(a,1-a,2a-j,1-2a+j\in\mathbb{C}\backslash\mathbb{Z}_0^-\,;j\in\mathbb{N}_0\right),$$

provided that single and double series involved are absolutely convergent.

Proof:

In order to prove Theorem 5.1, we proceed as follows.

Denote the left-hand side of double-series identity (21) by S and replacing n by $n-m$, after some simplifications, we have

$$S=\sum_{n=0}^{\infty}\Phi(n)\frac{(-1)^n\,(z)^n}{n!}\,{}_2F_1\left[\begin{matrix}-n,\,a;\\ \\ 2a+j;\end{matrix}\;2\right]. \tag{23}$$

Using series decomposition identity (4) in Equation (23), we get

$$S=\sum_{n=0}^{\infty}\Phi(2n)\frac{(-1)^{2n}\,(z)^{2n}}{2n!}\,{}_2F_1\left[\begin{matrix}-2n,\,a;\\ \\ 2a+j;\end{matrix}\;2\right]$$
$$+\sum_{n=0}^{\infty}\Phi(2n+1)\frac{(-1)^{2n+1}\,(z)^{2n+1}}{2n+1!}\,{}_2F_1\left[\begin{matrix}-2n-1,\,a;\\ \\ 2a+j;\end{matrix}\;2\right]. \tag{24}$$

Finally, by applying summation formulas (8) and (9) in the right-hand side of Equation (24), after simplification we easily arrive at the right-hand side of Equation (21). ∎

Proof:

In order to prove Theorem 5.2, we proceed as follows.

Denote the left-hand side of double-series identity (22) by T and replacing n by $n-m$, after some simplifications, we have

$$T=\sum_{n=0}^{\infty}\Phi(n)\frac{(-1)^n\,(z)^n}{n!}\,{}_2F_1\left[\begin{array}{c}-n,\,a;\\ \\ 2a-j;\end{array}\;2\right]. \tag{25}$$

Using series decomposition identity (4) in Equation (25), to yield

$$T=\sum_{n=0}^{\infty}\Phi(2n)\frac{(-1)^{2n}\,(z)^{2n}}{2n!}\,{}_2F_1\left[\begin{array}{c}-2n,\,a;\\ \\ 2a-j;\end{array}\;2\right]$$

$$+\sum_{n=0}^{\infty}\Phi(2n+1)\frac{(-1)^{2n+1}\,(z)^{2n+1}}{2n+1!}\,{}_2F_1\left[\begin{array}{c}-2n-1,\,a;\\ \\ 2a-j;\end{array}\;2\right]. \tag{26}$$

By employing summation formulas (10) and (11) in the right-hand side of Equation (26), after some simplifications we easily obtain the right-hand side of Equation (22). ∎

6. Reducibility of Kampé de Fériet Functions

In the assertions (21) and (22), putting $\Phi(n)=\left(\prod_{j=1}^{D}(d_j)_n\right)\left(\prod_{j=1}^{E}(e_j)_n\right)^{-1}$; $n\in\mathbb{N}_0$ and after straightforward calculation we obtain presumably new hypergeometric reduction formulas for the Kampé de Fériet double hypergeometric functions in two variables. Under the common convergence conditions of Kampé de Fériet double hypergeometric function $F^{D:1;0}_{E:1;0}(2z,-z)$ and generalized hypergeometric function ${}_{2D+1}F_{2E+2}(z^2)$ given below
(i) If $2D\leq 2E+1$, then $|z|<\infty$,
(ii) If $D=E+1$, then $|z|<\frac{1}{3}$, the following hypergeometric reduction formulas hold true.

First Reduction Formula.

$$F^{D:1;0}_{E:1;0}\left[\begin{array}{c}(d_D):\;a;\text{—};\\ \\ (e_E):2a+j;\text{—};\end{array}\;2z,\,-z\right]=\frac{\Gamma(1-a)}{2^{2a+j}(a)_j\Gamma(1-2a-j)}\sum_{r=0}^{j}\binom{j}{r}(-1)^r\left\{\frac{\Gamma(\frac{1-2a-j+r}{2})}{\Gamma(\frac{1+r-j}{2})}\right.$$

$$\times{}_{2D+1}F_{2E+2}\left[\begin{array}{c}\frac{(d_D)}{2},\;\frac{1+(d_D)}{2},\;\frac{1-r+j}{2};\\ \\ \frac{(e_E)}{2},\;\frac{1+(e_E)}{2},\;\frac{1+2a-r+j}{2},\;\frac{1}{2};\end{array}\;4^{(D-E-1)}\,z^2\right]$$

$$\left.+\frac{z\Gamma(\frac{-2a-j+r}{2})\prod_{i=1}^{D}(d_i)}{\Gamma(\frac{r-j}{2})\prod_{i=1}^{E}(e_i)}\,{}_{2D+1}F_{2E+2}\left[\begin{array}{c}\frac{1+(d_D)}{2},\;\frac{2+(d_D)}{2},\;\frac{2-r+j}{2};\\ \\ \frac{1+(e_E)}{2},\;\frac{2+(e_E)}{2},\;\frac{2+2a+j-r}{2},\;\frac{3}{2};\end{array}\;4^{(D-E-1)}\,z^2\right]\right\}, \tag{27}$$

$$\left(a,1-a,2a+j,1-2a-j,d_1,d_2,\ldots,d_D,e_1,e_2,\ldots,e_E\in\mathbb{C}\backslash\mathbb{Z}_0^-;\,j\in\mathbb{N}_0\right).$$

Second Reduction Formula.

$$F^{D:1;0}_{E:1;0}\left[\begin{array}{l}(d_D): a; —;\\ \\ (e_E): 2a-j; —;\end{array} 2z,\ -z\right] = \frac{\Gamma(1-a)}{2^{2a-j}\Gamma(1-2a+j)}\sum_{r=0}^{j}\binom{j}{r}\left\{\frac{\Gamma(\frac{1-2a+j+r}{2})}{\Gamma(\frac{1+r-j}{2})}\right.$$
$$\times {}_{2D+1}F_{2E+2}\left[\begin{array}{l}\frac{(d_D)}{2},\ \frac{1+(d_D)}{2},\ \frac{1-r+j}{2};\\ \\ \frac{(e_E)}{2},\ \frac{1+(e_E)}{2},\ \frac{1+2a-r-j}{2},\ \frac{1}{2};\end{array} 4^{(D-E-1)}\, z^2\right]$$
$$\left.+\frac{z\Gamma(\frac{r-2a+j}{2})\prod_{i=1}^{D}(d_i)}{\Gamma(\frac{r-j}{2})\prod_{i=1}^{E}(e_i)}\,{}_{2D+1}F_{2E+2}\left[\begin{array}{l}\frac{1+(d_D)}{2},\ \frac{2+(d_D)}{2},\ \frac{2-r+j}{2};\\ \\ \frac{1+(e_E)}{2},\ \frac{2+(e_E)}{2},\ \frac{2+2a-j-r}{2},\ \frac{3}{2};\end{array} 4^{(D-E-1)}\, z^2\right]\right\}, \quad (28)$$

$$\left(a, 1-a, 2a-j, 1-2a+j, d_1, d_2, \ldots, d_D, e_1, e_2, \ldots, e_E \in \mathbb{C}\backslash\mathbb{Z}_0^-\,; j \in \mathbb{N}_0\right).$$

7. Special Cases

(i) Set $D = E = 0$, in Equation (27). We get

$$e^{-z}\,{}_1F_1\left[\begin{array}{c}a;\\ \\ 2a+j;\end{array} 2z\right] = \frac{\Gamma(1-a)}{2^{2a+j}(a)_j\Gamma(1-2a-j)}\sum_{r=0}^{j}\binom{j}{r}(-1)^r\left\{\frac{\Gamma(\frac{1-2a-j+r}{2})}{\Gamma(\frac{1+r-j}{2})}\right.$$
$$\left.\times {}_1F_2\left[\begin{array}{l}\frac{1-r+j}{2};\\ \\ \frac{1+2a-r+j}{2},\ \frac{1}{2};\end{array} \frac{z^2}{4}\right] + \frac{z\Gamma(\frac{-2a-j+r}{2})}{\Gamma(\frac{r-j}{2})}\,{}_1F_2\left[\begin{array}{l}\frac{2-r+j}{2};\\ \\ \frac{2+2a+j-r}{2},\ \frac{3}{2};\end{array} \frac{z^2}{4}\right]\right\}, (29)$$

$$\left(a, 1-a, 2a+j, 1-2a-j \in \mathbb{C}\backslash\mathbb{Z}_0^-\,; j \in \mathbb{N}_0\right).$$

(ii) Put $D = E = 0$, in Equation (28). We obtain

$$e^{-z}\,{}_1F_1\left[\begin{array}{c}a;\\ \\ 2a-j;\end{array} 2z\right] = \frac{\Gamma(1-a)}{2^{2a-j}\Gamma(1-2a+j)}\sum_{r=0}^{j}\binom{j}{r}\left\{\frac{\Gamma(\frac{1-2a+j+r}{2})}{\Gamma(\frac{1+r-j}{2})}\right.$$
$$\left.\times {}_1F_2\left[\begin{array}{l}\frac{1-r+j}{2};\\ \\ \frac{1+2a-r-j}{2},\ \frac{1}{2};\end{array} \frac{z^2}{4}\right] + \frac{z\Gamma(\frac{r-2a+j}{2})}{\Gamma(\frac{r-j}{2})}\,{}_1F_2\left[\begin{array}{l}\frac{2-r+j}{2};\\ \\ \frac{2+2a-j-r}{2},\ \frac{3}{2};\end{array} \frac{z^2}{4}\right]\right\}, (30)$$

$$\left(a, 1-a, 2a-j, 1-2a+j \in \mathbb{C}\backslash\mathbb{Z}_0^- ; j \in \mathbb{N}_0\right).$$

(iii) Set $j = 0$ in Equations (15) and (16); we get Kümmer's transformation (see Kümmer (1836)).

(iv) Put $j = 0,$ in Equation (29); we obtain another transformation of Kümmer (see Rainville (1960)).

(v) Take $j = 1,$ in Equations (29) and (30); we have known results of Rathie and Nagar (see Rathie and Nagar (1995)).

(vi) Fix $j = 0, 1,$ in Equations (27) and (28); we recover known results of Kim (see Kim (2009)).

(vii) Substitute $j = 2, 3,$ in Equations (29) and (30); we get known results of Choi and Rathie (see Choi and Rathie (2014)) and see also Kim et al. (2010).

8. Conclusion

In this paper, from Section 3 to Section 7, some interesting results are obtained for single and double hypergeometric functions, which may be potentially useful to non-specialists who are interested in Applied Mathematics or Mathematical Physics. We conclude our present analysis by observing that several interesting summation formulas, quadratic transformations, reduction formulas, corresponding multiple series identities and their hypergeometric representations can be obtained with the help of some newly derived terminating Clausen series of argument "two" in an analogous manner.

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