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Generalized Hermite-based Apostol-Euler Polynomials and Their Properties

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Abstract

The aim of this paper is to study certain properties of generalized Apostol-Hermite-Euler polynomials with three parameters. We have shown that there is an intimate connection between these polynomials and established their elementary properties. We also established some identities by applying the generating functions and deduce their special cases and applications.

Keywords: Euler numbers; Euler polynomials; Apostol-Euler number; Apostol-Euler polynomials; Apostol-Hermite-Euler polynomials and generating functions

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1. Introduction

In mathematics and physics, Hermite polynomials form a well-known class of orthogonal polynomials. In quantum mechanics they appear as eigenfunctions of the harmonic oscillator and in numerical analysis they play a role in Gauss-Hermite quadrature. The functions are named after the French mathematician Charles Hermite.

Importance and potential for applications of Apostol-Hermite polynomials in certain problems in number theory, combinatorics, classical and numerical analysis and other fields of applied mathematics, several kinds of special numbers and polynomials have recently been studied by many authors.

Generalized Apostol-Hermite-Euler's polynomials and generalized Apostol-Euler's numbers have various practical applications in many branches of higher level mathematics such as calculus, differential equations, discrete mathematics, trigonometry, complex analysis, statistics, mathematical physics, etc. They show unique properties that simplify many functional equations and patterns.

A generating function is a way of encoding an infinite sequence of numbers (a_n) by treating them as the coefficients of a power series. These functions are defined by linear polynomials and differential equations, such as functional equations. We derive various functional equations using these generating functions for generalized Apostol-Hermite-Euler polynomials.

2. Properties of Euler Polynomials

In this section, we have obtained certain useful properties of the generalized Apostol-Hermite-Euler polynomials

$$ {}_HE_n^{[m-1,\alpha]}(x,y;a,c,\lambda) $$

defined by Pathan et al. (2015) and the relationship between these polynomials.

Pathan et al. (2015) defined new classes of order $\alpha \in C$ known as generalized Hermite-based Apostol-Euler polynomials $E_n^{[m-1,\alpha]}(x;\lambda)$ for $m \in N$.

The generalized Apostol-Hermite-Euler polynomials ${}_HE_n^{[m-1,\alpha]}(x,y;a,c,\lambda)$ for arbitrary real or complex parameter α and for $a,c \in R^+$, $m \in N$, $\lambda \in C$ are defined in a suitable neighborhood of $t=0$ with $|t\log(a)| < |\log(-\lambda)|$, by means of the following generating functions:

$$ 2^{m\alpha}[B(\lambda,a;t)]^{\alpha}c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HE_n^{[m-1,\alpha]}(x,y;a,c,\lambda)\frac{t^n}{n!}, \tag{1} $$

where $B\ (\lambda, a; t)$ is given below:

$$B(\lambda, a; t) = \left(\lambda a^t + \sum_{h=0}^{m-1} \frac{(t \log a)^h}{h!}\right)^{-1} . \tag{2}$$

.

Theorem 2.1.

For any integer $n \geq 0$, we have

$$_{H}E_n^{[m-1,\alpha]}(x+1, y; a, c, \lambda) = (\ln c)^n \sum_{k=0}^{n} \binom{n}{k} {}_{H}E_k^{[m-1,\alpha]}(x, y; a, c, \lambda). \tag{3}$$

Proof:

Multiplying both sides of Equation (1) by c^t , we get

$$2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{(x+1)t+yt^2} = \sum_{n=0}^{\infty} {}_{H}E_n^{[m-1,\alpha]}(x, y; a, c, \lambda) \frac{(t)^n}{n!} c^t .$$

We can write the above equation as

$$2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{(x+1)t+yt^2} = \sum_{n=0}^{\infty} {}_{H}E_n^{[m-1,\alpha]}(x, y; a, c, \lambda) \sum_{n=0}^{\infty} \frac{(t \ln c)^n}{n!} .$$

Applying Cauchy-Product formula, we obtain

$$2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{(x+1)t+yt^2} = (\ln c)^n \sum_{n=0}^{\infty} \sum_{k=0}^{n} {}_{H}E_n^{[m-1,\alpha]}(x, y; a, c, \lambda) \times \sum_{n=0}^{\infty} \frac{t^k}{k!} \frac{t^{n-k}}{n-k!} .$$

$$2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{(x+1)t+yt^2} = (\ln c)^n \sum_{n=0}^{\infty} \sum_{k=0}^{n} {}_{H}E_n^{[m-1,\alpha]}(x, y; a, c, \lambda) \times \sum_{n=0}^{\infty} \binom{n}{k} \frac{t^n}{n!} . \tag{4}$$

Using (1) in above equation, we can write

$$2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{(x+1)t+yt^2} = \sum_{n=0}^{\infty} {}_{H}E_n^{[m-1,\alpha]}(x+1, y; a, c, \lambda) \frac{t^n}{n!} . \tag{5}$$

Equating the R.H.S. of equations (4) and (5), we get the desired result of Theorem 2.1. ∎

The Complementary Argument Theorem

Theorem 2.2.

If the argument x and $(\alpha - x)$ are complementary, then

$$ {}_H E_n^{[m-1,\alpha]} \left(\alpha - x, y; a, c, \lambda\right) = (-1)^n \, {}_H E_n^{[m-1,\alpha]} \left(x, y; a, c, \lambda\right). \tag{6} $$

Proof:

Using (1) we can write

$$ \sum_{n=0}^{\infty} {}_H E_n^{[m-1,\alpha]}(\alpha - x, y; a, c, \lambda) = 2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{(\alpha-x)t+yt^2} $$

$$ = \frac{2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{(\alpha-x)t+yt^2} \cdot c^{-\alpha t}}{c^{-\alpha t}} $$

$$ = 2^{m\alpha}[B'(\lambda, a; t)]^{\alpha} c^{(-x)t+y(-t)^2} $$

$$ = (-1)^n (2)^{m\alpha}[B'(\lambda, a; t)]^{\alpha} c^{(-x)t+y(-t)^2} $$

$$ = (-1)^n \sum_{n=0}^{\infty} {}_H E_n^{[m-1,\alpha]}(x, y; a, c, \lambda) \frac{(-t)^n}{n!}\,. \tag{7} $$

Equating the coefficient of t^n on both sides of Equation (7), we get the desired result of Theorem 2.2. ■

Theorem 2.3.

For generalized Apostol-Hermite-Euler polynomials,

$$ \frac{d}{dx}[{}_H E_n^{[m-1,\alpha]}(x, y; a, c, \lambda)] = n(\ln c)\, {}_H E_{n-1}^{[m-1,\alpha]}(x, y; a, c, \lambda). \tag{8} $$

Proof:

Differentiating (1) with respect to x on both sides, we get

$$ \sum_{n=0}^{\infty} \frac{d}{dx} {}_H E_n^{[m-1,\alpha]}(\alpha, y; a, c, \lambda) = t \cdot 2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{xt+yt^2} \cdot \ln c $$

$$ = t(\ln c) 2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{xt+yt^2}. \tag{9} $$

Using (1), we can write (9) as

$$\sum_{n=0}^{\infty} \frac{d}{dx} {}_HE_n^{[m-1,\alpha]}(\alpha, y; a, c, \lambda) = (n+1)\ln c \times \sum_{n=0}^{\infty} {}_HE_n^{[m-1,\alpha]}(\alpha, y; a, c, \lambda) \frac{t^{n+1}}{n+1!}. \quad (10)$$

Comparing the coefficient of t^n on both sides of (10), we get the required result of Theorem 2.3. ■

Remark 2.4.

For generalized Apostol-Hermite-Euler polynomials. If $c = e$, then

$$\frac{d}{dx}\left[{}_HE_n^{[m-1,\alpha]}(x, y; a, e, \lambda)\right] = n \times {}_HE_{n-1}^{[m-1,\alpha]}(x, y; a, e, \lambda). \quad (11)$$

Theorem 2.5.

For generalized Apostol-Hermite-Euler polynomials

$$(\ln c) \int_p^q {}_HE_n^{[m-1,\alpha]}(x, y; a, c, \lambda)\, dx = \frac{1}{n+1}[{}_HE_n^{[m-1,\alpha]}(q, y; a, c, \lambda) - {}_HE_n^{[m-1,\alpha]}(p, y; a, c, \lambda)]. \quad (12)$$

Proof:

Integrating (1) with respect to x from p to q on both sides, we get

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_p^q {}_HE_n^{[m-1,\alpha]}(x, y; a, c, \lambda)\, dx &= \int_p^q 2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{xt+yt^2} dx \\ &= 2^{m\alpha}[B(\lambda, a; t)]^{\alpha} \frac{1}{t(\ln c)}\left[c^{qt+yt^2} - c^{pt+yt^2}\right] \\ &= \frac{1}{t(\ln c)} \cdot 2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{qt+yt^2} \\ &\quad - \frac{1}{t(\ln c)} \cdot 2^{m\alpha}[B(\lambda, a; t)]^{\alpha} c^{pt+yt^2}. \end{aligned} \quad (13)$$

Using (1) in RHS of (13), we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_p^q {}_HE_n^{[m-1,\alpha]}(x, y; a, c, \lambda)\, dx = \frac{1}{(\ln c)} \sum_{n=0}^{\infty} \frac{t^{n-1}}{n!} {}_HE_n^{[m-1,\alpha]}(q, y; a, c, \lambda) - \frac{1}{(\ln c)} \sum_{n=0}^{\infty} \frac{t^{n-1}}{n!} {}_HE_n^{[m-1,\alpha]}(p, y; a, c, \lambda). \quad (14)$$

Replacing n by $n+1$ in R.H.S. of (14), we have

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_p^q {}_HE_n^{[m-1,\alpha]}(x,y;a,c,\lambda)\,dx = \frac{1}{(\ln c)}\sum_{n=0}^{\infty}\frac{t^n}{n+1!}{}_HE_{n+1}^{[m-1,\alpha]}(q,y;a,c,\lambda) - \frac{1}{(\ln c)}\sum_{n=0}^{\infty}\frac{t^n}{n+1!}{}_HE_{n+1}^{[m-1,\alpha]}(p,y;a,c,\lambda)$$

$$= \frac{1}{(n+1)(\ln c)}\left[\sum_{n=0}^{\infty}\frac{t^n}{n!}{}_HE_{n+1}^{[m-1,\alpha]}(q,y;a,c,\lambda) - \sum_{n=0}^{\infty}\frac{t^n}{n!}{}_HE_{n+1}^{[m-1,\alpha]}(p,y;a,c,\lambda)\right]. \tag{15}$$

Comparing the coefficient of t^n on both sides of (15), we get the required result of Theorem 2.5. ∎

3. Conclusion

We found a number of interesting properties of Apostol Hermite-Euler polynomials. These formulas of the Apostol-Hermite Euler numbers and polynomials of higher order are further developed and supplement the contents of the recent cognate results developed by us, concerning the generalization of Apostol-Hermite Euler polynomials and numbers of higher order with more variables and more parameters.

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REFERENCES

Apostol, T.M. (1951). On the Lerch Zeta function, Pacific Journal of Mathematics, Vol. 1, pp. 161-167.

Luo, Q.-M. and Srivastava, H. M. (2005). Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials, Journal of Mathematical Analysis and Applications, Vol. 308, No. 1, pp. 290-302.

Luo, Q.-M. and Srivastava, H. M. (2006). Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials, Computers & Mathematics with Applications, An International Journal, Vol. 51, No. 3-4, pp. 631-642.

Pathan, M. A. and Khan, W. A. (2015). Some new classes of generalized Hermite-based Apostol-Euler and Apostol-Genocchi polynomials, Polytechnica Posnaniensis, Institutum Mathematicum, Fasciculi Mathematici, No. 55, pp. 153-170. doi:10.1515/fascmath-2015-0020

Sandor, J. and Crisci. (2000). *Handbook of Number Theory, Vol. II*, Kluwer Academic Publishers, Dordrecht, Boston and London.

Srivastava, H. M. (2000). Some formulas for the Bernoulli and Euler polynomials at rational arguments, Mathematical Proceedings of the Cambridge Philosophical Society, Vol. 129, No. 1, pp. 77-84.

Srivastava, H. M. and Choi, J. (2001). *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston and London.

Srivastava, H.M. and Pinter, A. (2004). Remarks on some relationships between the Bernoulli and Euler polynomials, Applied Mathematics Letters, An International Journal of Rapid Publication, Vol. 17, No. 4, pp. 375-380.

Tremblay, R., Gaboury, S. and Fugere, B. J. (2012a). A further generalization of Apostol-Bernoulli polynomials and related polynomials, Honam Mathematical Journal, Vol. 34, No. 3, pp. 311-326.

Tremblay, R., Gaboury, S. and Fugere, B. J. (2012b). Some new classes of generalized Apostol-Euler and Apostol-Genocchi polynomials, International Journal of Mathematics and Mathematical Sciences, Art. ID 182785, 14. doi:10.1155/2012/182785