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Existence and Stability Results of Nonlinear Fractional Differential Equations with Nonlinear Integral Boundary Condition on Time Scales

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Abstract

In this paper, we establish the existence and uniqueness of the solution to a nonlinear fractional differential equation with nonlinear integral boundary conditions on time scales. We used the fixed point theorems due to Banach, Schaefer's, nonlinear alternative of Leray Schauder's type and Krasnoselskii's to establish these results. In addition, we study Ulam-Hyer's (UH) type stability result. At the end, we present two examples to show the effectiveness of the obtained analytical results.

Keywords: Fractional differential equations; Existence; Ulam-Hyer's stability; Fixed Point Theorems; Time scales

MSC 2010 No.: 34A26, 34A12, 35F25, 34N05

1. Introduction

Fractional calculus is a branch of mathematics, in which arbitrary order integral and differential operators are studied. In the last decades of the twentieth century, the theory of fractional calculus and fractional differential equations has attracted the attention of many specialists due to lots of practical applications in many areas such as aerodynamics, control theory, signal and image processing and fitting of experimental data, etc., in which most of them are modeled depending upon their application, the effect of coupling, complicated environment, domain and certainly cannot be described through classical differential equation.

The fractional calculus provides one of the excellent tools for describing the allometric scaling laws, long memory processes and long-range interactions (for detail see Bagley and Torvik (1983), Magin (2006), Rudolf (2000)). That is why, in the last few decays, many authors and researchers have been given a lot of attention to the fractional differential equations and investigated the different types of results on the existence of solution of fractional differential equations (please see Alqudah et al. (2019), Malik (2016), Ravichandran et al. (2019), Valliammal et al. (2019) and cited references therein).

Recently, fractional differential equations with initial and boundary conditions have been applied in many problems of applied nature such as, analytical formulations of systems and processes. Moreover, fractional differential equations with integral boundary conditions appear in lots of natural phenomena arising from many fields such as fluid dynamics, chemical kinetics, electronics, and biological models. Most of these phenomena can not be described through ordinary differential equations. Therefore, many authors studied the fractional differential equations with different initial-boundary conditions by using the nonlinear functional analysis and various types of fixed point theorems such as Schauder, Krasnoselskii's, Dhang, Banach and Schaefer (see articles of Agarwal et al. (2010), Ahmad and Nieto (2009), Benchohra et al. (2008), Shuqin (2006), Wang and Feng (2008), Wang et al. (2018)). In Shuqin (2006), the author studied the existence of solutions for a fractional order boundary value problem by using the fixed point theorem due to nonlinear alternative. In Benchohra et al. (2008), authors used the Leray-Schauder, Schaefer's and Banach types fixed point theorems to establish the existence of solutions for a fractional order differential equation with integral boundary conditions.

Stability analysis is the fundamental property of the mathematical analysis which is very important in many fields of engineering and science. In the existing literature, there are many types of stability like Mittag-Leffler, h-stability, exponential and Lyapunov stability. Fixed point approach and Lyapunov method are the main tools to established these results. In the nineteenth-century, Ulam and Hyer presented an interesting type of stability called Ulam-Hyer's stability and nowadays it has been picked up a great deal of consideration due to a wide range of applications in many fields of science such as optimization and mathematical modeling. More recently, few researchers have been worked on UH type stability for the fractional and ordinary differential equations (see the articles by Ding (2018), Wang et al. (2012), Wang et al. (2013), Wang and Li (2016) and the cited references therein).

In the end of nineteenth century, Hilger (1988) introduced the concept of time scales theory which unifies the discrete and continuous analysis into a single theory. In general, one investigates the discrete and continuous dynamical systems separately and most of the results have to be proved for each case (using discrete analysis or continuous analysis). A time scales is a non-empty closed subset of real numbers $\mathbb{R}$. The results obtained on time scales will be true for the continuous, discrete as well as for any non-uniform time domain which are very useful in the study of complex dynamical systems, hence, the study of differential equations on time scales has picked up a lot of worldwide consideration and many scientists have found the applications of time scales in economics, control systems, population dynamics and heat transfer system. For more study on time scales, please see the books Bohner and Peterson (2002), (Bohner and Peterson (2003) and the papers Agarwal and Bohner (1999), András and Mészáros (2013), Kumar and Malik (2019c), Liu and Xiang (2008), Malik and Kumar (2020), and Shen (2017).

More recently, few authors have been worked on fractional differential equations on time scales (Ahmadkhanlu and Jahanshahi (2012), Bastos et al. (2011), Benkhettou et al. (2016), Kumar and Malik (2019a), Kumar and Malik (2019b), Malik and Kumar (2019), Yan et al. (2016)). Particularly, in Ahmadkhanlu and Jahanshahi (2012), authors investigated the results of the existence of solutions for a fractional initial value problem on time scales. In Yan et al. (2016), authors considered a fractional dynamic equation with boundary conditions on time scales and investigated the existence of solutions. In Kumar and Malik (2019a), authors established the existence and UH stability results for a fractional differential equation with impulsive conditions on time scales. In Kumar and Malik (2019b), authors considered a nonlinear implicit fractional dynamical system with impulses on time scales and established the existence, uniqueness and stability results. As per our knowledge, there is not a single manuscript which examined the existence and stability results for the nonlinear fractional order differential equations with integral boundary condition on time scales. Motivated by the above works, in this manuscript, we consider the fractional order differential equations with nonlinear integral boundary condition on time scales.

The rest of the paper is structured as follows: In Section 2, we give some fundamental definitions preliminaries, important lemmas and problem of statement. Section 3 is devoted to the study of existence and stability results for the considered problem. At last, we provide two examples to show the effectiveness of the obtained analytical outcomes.

2. Preliminaries and Problem Statment

Here, we recall some fundamental definitions, basic notations and useful lemmas. The space of all continuous functions $f : I \to \mathbb{R}$ endowed with the norm $||f||_C = \sup_{\theta\in I} |f(\theta)|$ is denoted by $C(I, \mathbb{R})$ and the space of Lebesgue integrable functions from I into $\mathbb{R}$ is denoted by $L^1(I, \mathbb{R})$.

A time scales $\mathbb{T}$ is an arbitrary non-empty closed subset of real numbers. We set $\mathbb{T}^k = \mathbb{T}\backslash\{\max \mathbb{T}\}$ if $\max \mathbb{T}$ exists, otherwise $\mathbb{T}^k = \mathbb{T}$. A time scales interval is defied as $[a, b]_{\mathbb{T}} = \{\theta \in \mathbb{T} : a \le \theta \le b\}$. In a similar way, we can define the intervals $(a, b)_{\mathbb{T}}, [a, b)_{\mathbb{T}}, (a, b]_{\mathbb{T}}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is given by $\sigma(\theta) := \inf\{s \in \mathbb{T} : s > \theta\}$, with $\inf \emptyset = \sup \mathbb{T}$. A positive function $\mu : \mathbb{T} \to [0, \infty)$ given by $\mu(\theta) := \sigma(\theta) - \theta, \forall\, \theta \in \mathbb{T}$ is called a graininess function.

The delta derivative (or $\Delta-$derivative) of a function $z : \mathbb{T}^k \to \mathbb{R}$ at $\theta \in \mathbb{T}^k$ is a number $z^\Delta(\theta)$ (provided it exists), if there exists a neighborhood U of θ and an $\epsilon > 0$ such that

$$|[z(\sigma(\theta)) - z(s)] - z^\Delta(\theta)[\sigma(\theta) - s]| \le \epsilon|\sigma(\theta) - s|, \quad \forall\, s \in U.$$

Theorem 2.1. (Liu and Xiang (2008))

Let $\mathbb{T}$ be a time scales such that $\theta_1, \theta_2 \in \mathbb{T}$ with $\theta_1 \leq \theta_2$. Also, let $z : \mathbb{R} \to \mathbb{R}$ is a continuous non-decreasing function, then the following inequality holds

$$\int_{\theta_1}^{\theta_2} z(s)\Delta s \leq \int_{\theta_1}^{\theta_2} z(s)ds, \tag{1}$$

Definition 2.2. (Ahmadkhanlu and Jahanshahi (2012))

The $\Delta-$fractional integral of a integrable function $z : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is defined as

$$^{\Delta}I_{a^+}^{\vartheta} z(\theta) = \frac{1}{\Gamma(\vartheta)} \int_a^{\theta} (\theta - s)^{\vartheta-1} z(s)\Delta s.$$

Definition 2.3. (Ahmadkhanlu and Jahanshahi (2012))

Let $z : \mathbb{T} \to \mathbb{R}$ be a given function. Then the Caputo $\Delta-$fractional derivative of z is given by

$$^{c}\Delta_{a^+}^{\vartheta} z(\theta) = \frac{1}{\Gamma(n-\vartheta)} \int_a^{\theta} (\theta - s)^{n-\vartheta-1} z^{\Delta^n}(s)\Delta s,$$

where $n = [\vartheta] + 1$.

Lemma 2.4. (Ahmadkhanlu and Jahanshahi (2012))

Let $z : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is an integrable function, then the relation $^{\Delta}I_{a^+}^{\vartheta_1}\,^{\Delta}I_{a^+}^{\vartheta_2} z =^{\Delta} I_{a^+}^{\vartheta_1+\vartheta_2} z$ holds.

Problem Statement

We consider the following fractional fractional order differential equation with integral boundary value problem

$$\begin{aligned} ^{c}\Delta^{\vartheta} u(\theta) &= \Psi(\theta, u(\alpha(\theta))), \quad \theta \in I = [0, T]_{\mathbb{T}},\ \vartheta \in (0, 1), \\ \beta u(0) + \eta u(T) &= \frac{1}{\Gamma(\vartheta)} \int_0^T (T - s)^{\vartheta-1} g(s, u(s))\Delta s, \end{aligned} \tag{2}$$

where $^{c}\Delta^{\vartheta}$ is the Caputo delta-fractional derivative and $0, T \in \mathbb{T}$. $\beta, \eta \in \mathbb{R}$ such that $\beta + \eta \neq 0$, $\alpha : I \to I$ is a continuous function with $\alpha(\theta) \leq \theta$, Ψ and g are some functions which will be specified later.

Remark 2.5.

In comparison to the existing literature, if we set $\beta = 1, \eta u(T) = -u_0, g(s, u(s)) = 0, \alpha(\theta) = \theta$, then the existence results can be found in (Ahmadkhanlu and Jahanshahi (2012)). Also, if we set, $\beta = 1, \eta = 1, g(s, u(s)) = 0$, our problem is converted to a fractional differential equation with anti-periodic boundary condition and by selecting $\beta = 0, \eta = 1$, then our problem is converted to a terminal value problem.

Definition 2.6.

A function $u(\theta) \in C(I, \mathbb{R})$ is said to be a solution of (2), if u satisfies ${}^{c}\Delta^{\vartheta} u(\theta) = \Psi(\theta, u(\alpha(\theta)))$ and $\beta u(0) + \eta u(T) = \dfrac{1}{\Gamma(\vartheta)} \int_0^T (T-s)^{\vartheta-1} g(s, u(s)) \Delta s$.

Lemma 2.7.

Let $\Psi : I \times \mathbb{R} \to \mathbb{R}$ be a rd-continuous function. A function $u(\theta)$ is said to be a solution of equation (2) if $u(\theta)$ is a solution of the following integral equation

$$u(\theta) = \int_0^T G(\theta, s)\Psi(s, u(\alpha(s)))\Delta s \; + \frac{1}{(\beta+\eta)\Gamma(\vartheta)}\bigg(\int_0^T (T-s)^{\vartheta-1} g(s, u(s))\Delta s\bigg), \tag{3}$$

where

$$G(\theta, s) = \begin{cases} \dfrac{(\theta-s)^{\vartheta-1}}{\Gamma(\vartheta)} - \dfrac{\eta}{(\beta+\eta)\Gamma(\vartheta)}(T-s)^{\vartheta-1}, & s \in (0, \theta]_{\mathbb{T}} \\ -\dfrac{\eta}{(\beta+\eta)\Gamma(\vartheta)}(T-s)^{\vartheta-1}, & s \in [\theta, T]_{\mathbb{T}}. \end{cases} \tag{4}$$

Proof:

From the Definition 2.3, we have:

$$\begin{aligned} {}^{c}\Delta^{\vartheta} u(\theta) = & \; \frac{1}{\Gamma(1-\vartheta)} \int_0^\theta (\theta-s)^{-\vartheta} u^{\Delta}(s)\Delta s \\ = & \; {}^{\Delta}I^{1-\vartheta} u^{\Delta}(s). \end{aligned} \tag{5}$$

Now, using the Lemma 2.4, we find ${}^{\Delta}I^{\vartheta}\, {}^{c}\Delta^{\vartheta} u(\theta) = {}^{\Delta}I^{1} u^{\Delta}(\theta) = u(\theta) - c_1$, where $c_1 \in \mathbb{R}$. Hence,

$$u(\theta) = {}^{\Delta}I^{\vartheta}\Psi(\theta, u(\alpha(\theta))) + c_1 = \frac{1}{\Gamma(\vartheta)} \int_0^\theta (\theta-s)^{\vartheta-1}\Psi(s, u(\alpha(s)))\Delta s + c_1.$$

Also, the boundary conditions of equation (2) gives

$$c_1 = \frac{1}{(\beta+\eta)\Gamma(\vartheta)}\bigg(\int_0^T (T-s)^{\vartheta-1} g(s, u(s))\Delta s \; - \; \eta\int_0^T (T-s)^{\vartheta-1}\Psi(s, u(\alpha(s)))\Delta s\bigg).$$

Hence, we get

$$\begin{aligned} u(\theta) = & \frac{1}{\Gamma(\vartheta)} \int_0^\theta (\theta-s)^{\vartheta-1}\Psi(s, u(\alpha(s)))\Delta s \\ & - \frac{\eta}{(\beta+\eta)\Gamma(\vartheta)} \int_0^T (T-s)^{\vartheta-1}\Psi(s, u(\alpha(s)))\Delta s \\ & + \frac{1}{(\beta+\eta)\Gamma(\vartheta)} \int_0^T (T-s)^{\vartheta-1} g(s, u(s))\Delta s. \end{aligned}$$

Subsequently, we get

$$u(\theta) = \int_0^T G(\theta, s)\Psi(s, u(\alpha(s)))\Delta s \; + \frac{1}{(\beta+\eta)\Gamma(\vartheta)} \int_0^T (T-s)^{\vartheta-1} g(s, u(s))\Delta s. \tag{6}$$

Hence, the result follows. ■

Consider the following inequality

$$|{}^{c}\Delta^{\vartheta}v(\theta)-\Psi(\theta,v(\alpha(\theta)))|\leq\epsilon,\quad \theta\in I, \tag{7}$$

where ϵ is a positive number i.e., $\epsilon>0$.

Definition 2.8. (Wang et al. (2012))

Equation (2) is called UH stable if there exists a constant $H_{(L_{\Psi},L_g,\vartheta)}>0$ such that for each solution v of inequality (7) and for $\epsilon>0$, there exists a unique solution u of equation (2) satisfies the following inequality

$$|v(\theta)-u(\theta)|\leq H_{(L_{\Psi},L_g,\vartheta)}\epsilon,\quad \forall\,\theta\in I.$$

Definition 2.9. (Wang et al. (2012))

Equation (2) is called generalized UH stable if there exists $\mathcal{H}_{(L_{\Psi},L_g,\vartheta)}\in C(\mathbb{R}^+,\mathbb{R}^+)$, $\mathcal{H}_{(L_{\Psi},L_g,\vartheta)}(0)=0$, such that for each solution v of inequality (7), there exists a solution u of equation (2) satisfies the following inequality

$$|v(\theta)-u(\theta)|\leq\mathcal{H}_{(L_{\Psi},L_g,\vartheta)}(\epsilon),\quad \forall\,\theta\in I.$$

Remark 2.10.

Definition 2.8 $\implies$ Definition 2.9.

Remark 2.11.

A function $v\in C(I,\mathbb{R})$ is a solution of inequality (7) if and only if there is $\mathcal{G}\in C(I,\mathbb{R})$ such that

(a) $|\mathcal{G}(\theta)|\leq\epsilon,\forall\,\theta\in I.$
(b) ${}^{c}\Delta^{\vartheta}v(\theta)=\Psi(\theta,v(\alpha(\theta)))+\mathcal{G}(\theta),\ \theta\in I.$

From the above remark, we get

$${}^{c}\Delta^{\vartheta}v(\theta)=\Psi(\theta,v(\alpha(\theta)))+\mathcal{G}(\theta),\ \theta\in I. \tag{8}$$

Now, from the Lemma 2.7, we can easily show that the solution v with

$$\beta v(0)+\eta v(T)=\frac{1}{\Gamma(\vartheta)}\int_0^T(T-s)^{\vartheta-1}g(s,v(s))\Delta s,$$

of the equation (8) is given by

$$\begin{aligned} v(\theta) = &\frac{1}{\Gamma(\vartheta)}\int_0^\theta (\theta - s)^{\vartheta-1}\Psi(s, v(\alpha(s)))\Delta s + \frac{1}{\Gamma(\vartheta)}\int_0^\theta (\theta - s)^{\vartheta-1}\mathcal{G}(s)\Delta s \\ &- \frac{\eta}{(\beta+\eta)\Gamma(\vartheta)}\int_0^T (T - s)^{\vartheta-1}\mathcal{G}(s)\Delta s \\ &+ \frac{1}{(\beta+\eta)\Gamma(\vartheta)}\int_0^T (T - s)^{\vartheta-1} g(s, v(s))\Delta s \\ &- \frac{\eta}{(\beta+\eta)\Gamma(\vartheta)}\int_0^T (T - s)^{\vartheta-1}\Psi(s, v(\alpha(s)))\Delta s. \end{aligned}$$

Consequentially, we get

$$\begin{aligned} &\Bigg| v(\theta) - \frac{1}{\Gamma(\vartheta)}\int_0^\theta (\theta - s)^{\vartheta-1}\Psi(s, v(\alpha(s)))\Delta s \\ &\quad - \frac{1}{(\beta+\eta)\Gamma(\vartheta)}\int_0^T (T - s)^{\vartheta-1} g(s, v(s))\Delta s \\ &\quad + \frac{\eta}{(\beta+\eta)\Gamma(\vartheta)}\int_0^T \frac{(T - s)^{\vartheta-1}}{\Gamma(\vartheta)}\Psi(s, v(\alpha(s)))\Delta s \Bigg| \\ &\leq M\epsilon, \end{aligned}$$

where,

$$M = \frac{T^\vartheta}{\Gamma(\vartheta+1)}\left(1 + \frac{|\eta|}{|\beta+\eta|}\right).$$

In order to prove the main results, we need the following assumptions:

(P1): Function $\Psi : I \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following:

(P1a): There exists a constant $M_\Psi > 0$ such that

$$|\Psi(\theta, u)| \leq M_\Psi(1 + |u|), \quad \forall\, \theta \in I,\ u \in \mathbb{R}.$$

(P1b): There exists a constant $L_\Psi > 0$ such that

$$|\Psi(\theta, u) - \Psi(\theta, v)| \leq L_\Psi|u - v|, \quad \forall\, \theta \in I,\ u, v \in \mathbb{R}.$$

(P2): Function $g : I \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following:

(P2a): There exists a constant $M_g > 0$ such that

$$|g(\theta, u)| \leq M_g(1 + |u|), \quad \forall\, \theta \in I,\ u \in \mathbb{R}.$$

(P2b): There exists a positive constant L_g such that

$$|g(\theta, u) - g(\theta, v)| \leq L_g|u - v|, \quad \forall\, \theta \in I,\ u, v \in \mathbb{R}.$$

(P3): $K_1 < 1$, where $K_1 = \dfrac{T^\vartheta}{\Gamma(\vartheta+1)}\left(M_\Psi + \dfrac{M_\Psi|\eta|}{|\beta+\eta|} + \dfrac{M_g}{|\beta+\eta|}\right)$.

(P4): There exists $p(\theta) \in L^1(I, \mathbb{R}^+)$ and a non decreasing continuous function $\phi : [0, \infty) \to (0, \infty)$ such that

$$|\Psi(\theta, u)| \leq p(\theta)\phi(|u|), \quad \text{for each } \theta \in I \text{ and } u \in \mathbb{R}.$$

3. Main Results

Theorem 3.1.

Let the assumptions **(P1)-(P3)** and

$$ML_\Psi + \frac{L_g T^\vartheta}{|\beta + \eta|\Gamma(\vartheta + 1)} < 1, \tag{9}$$

are hold. Then, the equation (2) has a unique solution.

Proof:

For $\delta = \dfrac{K_1}{1 - K_1}$, we consider

$$\mathcal{B} = \{u \in C(I, \mathbb{R}) : \|u\|_C \leq \delta\} \subseteq C(I, \mathbb{R}).$$

Defined an operator $\Upsilon : \mathcal{B} \to \mathcal{B}$ as

$$\begin{aligned}(\Upsilon u)\theta = {} & \frac{1}{\Gamma(\vartheta)} \int_0^\theta (\theta - s)^{\vartheta - 1} \Psi(s, u(\alpha(s)))\Delta s \\ & - \frac{\eta}{(\beta + \eta)\Gamma(\vartheta)} \int_0^T (T - s)^{\vartheta - 1} \Psi(s, u(\alpha(s)))\Delta s \\ & + \frac{1}{(\beta + \eta)\Gamma(\vartheta)} \int_0^T (T - s)^{\vartheta - 1} g(s, u(s))\Delta s.\end{aligned} \tag{10}$$

Firstly, we show that the operator Υ is well defined from $\mathcal{B}$ into $\mathcal{B}$. For any $\theta \in I$ and $u \in \mathcal{B}$, we have:

$$\begin{aligned}|(\Upsilon u)(\theta)| \leq {} & \left| \frac{1}{\Gamma(\vartheta)} \int_0^\theta (\theta - s)^{\vartheta - 1} \Psi(s, u(\alpha(s)))\Delta s \right| \\ & + \left| \frac{1}{(\beta + \eta)\Gamma(\vartheta)} \int_0^T (T - s)^{\vartheta - 1} g(s, u(s))\Delta s \right| \\ & + \left| \frac{1}{(\beta + \eta)\Gamma(\vartheta)} \int_0^T (T - s)^{\vartheta - 1} g(s, u(s))\Delta s \right| \\ \leq {} & \frac{M_\Psi(1 + \delta)}{\Gamma(\vartheta)} \int_0^\theta (\theta - s)^{\vartheta - 1} \Delta s \\ & + \left(\frac{M_\Psi(1 + \delta)|\eta|}{|(\beta + \eta)|\Gamma(\vartheta)} + \frac{M_g(1 + \delta)}{|\beta + \eta|\Gamma(\vartheta)} \right) \int_0^T (T - s)^{\vartheta - 1} \Delta s.\end{aligned}$$

Using the Theorem 2.1, we have

$$\begin{aligned}|(\Upsilon u)(\theta)| &\leq \frac{M_\Psi(1+\delta)}{\Gamma(\vartheta)}\int_0^\theta(\theta-s)^{\vartheta-1}ds\\&\quad+\left(\frac{M_\Psi(1+\delta)|\eta|}{|(\beta+\eta)|\Gamma(\vartheta)}+\frac{M_g(1+\delta)}{|\beta+\eta|\Gamma(\vartheta)}\right)\int_0^T(T-s)^{\vartheta-1}ds\\&\leq\frac{M_\Psi(1+\delta)T^\vartheta}{\Gamma(\vartheta+1)}+\left(\frac{M_\Psi(1+\delta)|\eta|}{|(\beta+\eta)|\Gamma(\vartheta+1)}+\frac{M_g(1+\delta)}{|\beta+\eta|\Gamma(\vartheta+1)}\right)T^\vartheta.\end{aligned}\tag{11}$$

Hence,

$$\|\Upsilon u\|_C\leq\delta.$$

Therefore, Υ is a well defined operator from $\mathcal{B}$ into $\mathcal{B}$. Now, we show that operator Υ is a contractive from $\mathcal{B}$ into $\mathcal{B}$. For any $u, v\in\mathcal{B}$ and $\theta\in I$, we have

$$\begin{aligned}|(\Upsilon u)(\theta)-(\Upsilon v)(\theta)| &\leq\frac{1}{\Gamma(\vartheta)}\int_0^\theta(\theta-s)^{\vartheta-1}|\Psi(s,u(\alpha(s)))-\Psi(s,v(\alpha(s)))|\Delta s\\&\quad+\frac{|\eta|}{|(\beta+\eta)|\Gamma(\vartheta)}\int_0^T(T-s)^{\vartheta-1}|\Psi(s,u(\alpha(s)))-\Psi(s,v(\alpha(s)))|\Delta s\\&\quad+\frac{1}{|(\beta+\eta)|\Gamma(\vartheta)}\int_0^T(T-s)^{\vartheta-1}|g(s,u(s))-g(s,v(s))|\Delta s\\&\leq\frac{L_\Psi}{\Gamma(\vartheta)}\left(\frac{T^\vartheta}{\vartheta}+\frac{T^\vartheta|\eta|}{\vartheta|\beta+\eta|}\right)\|u-v\|_C\;+\;\frac{T^\vartheta L_g}{|\beta+\eta|\Gamma(\vartheta+1)}\|u-v\|_C.\end{aligned}$$

Hence,

$$\|(\Upsilon u)-(\Upsilon v)\|_C\leq L_F\|u-v\|_C,$$

where,

$$L_F=\frac{L_\Psi T^\vartheta}{\Gamma(\vartheta+1)}\left(1+\frac{|\eta|}{|\beta+\eta|}\right)+\frac{L_gT^\vartheta}{|\beta+\eta|\Gamma(\vartheta+1)}.$$

Therefore, Υ is a strict contraction mapping. Application of Banach contraction theorem, Υ has a unique fixed point which is the solution of Equation (2). ∎

Theorem 3.2.

Let the assumptions **(P1a)** and **(P2a)** are fulfilled, then the equation (2) has at least one solution.

Proof:

For the convenience, we divide the proof into four steps as follows:

Step 1: The operator Υ defined in Theorem 3.1 is continuous. Let u_n be a sequence such that

$u_n \to u$ in $C(I,\mathbb{R})$, then for any $\theta \in I$, we have

$$\begin{aligned}|(\Upsilon u_n)(\theta)-(\Upsilon u)(\theta)| \leq & \left|\frac{1}{\Gamma(\vartheta)}\int_0^\theta (\theta-s)^{\vartheta-1}(\Psi(s,u_n(\alpha(s)))-\Psi(s,u(\alpha(s))))\Delta s\right| \\ &+\left|\frac{\eta}{(\beta+\eta)\Gamma(\vartheta)}\int_0^T (T-s)^{\vartheta-1}(\Psi(s,u_n(\alpha(s)))-\Psi(s,u(\alpha(s))))\Delta s\right| \\ &+\left|\frac{1}{(\beta+\eta)\Gamma(\vartheta)}\int_0^T (T-s)^{\vartheta-1}g(s,u_n(s))-g(s,u(s))\Delta s\right|.\end{aligned}$$

Since, the functions $\Psi(\theta,u(\theta))$ and $g(\theta,u(\theta))$ are continuous w.r.t u. Therefore, using the Lebesgue dominated convergence theorem, we have

$$\|\Upsilon u_n-\Upsilon u\|_C \to 0 \text{ as } n\to\infty.$$

Therefore, the operator Υ is continuous.

Step 2: The operator Υ is bounded. From equation (11) we have:

$$\|\Upsilon u\|_C \leq \left(M_\Psi+\frac{M_\Psi|\eta|}{|(\beta+\eta)|}+\frac{M_g}{|\beta+\eta|}\right)\frac{T^\vartheta(1+\sup_{\theta\in I}(|u(\theta)|))}{\Gamma(\vartheta+1)}.$$

Step 3: Let $\eta_1,\eta_2 \in I$ such that $\eta_1<\eta_2$, then we have

$$\begin{aligned}|(\Upsilon u)(\eta_2)-(\Upsilon u)(\eta_1)| \leq & \left|\frac{1}{\Gamma(\vartheta)}\int_0^{\eta_1}\left((\eta_2-s)^{\vartheta-1}-(\eta_1-s)^{\vartheta-1}\right)\Psi(s,u(\alpha(s)))\Delta s\right| \\ &+\left|\frac{1}{\Gamma(\vartheta)}\int_{\eta_1}^{\eta_2}(\eta_2-s)^{\vartheta-1}\Psi(s,u(\alpha(s)))\Delta s\right| \\ \leq & \frac{M_\Psi}{\Gamma(\vartheta)}\int_0^{\eta_1}\left((\eta_2-s)^{\vartheta-1}-(\eta_1-s)^{\vartheta-1}\right)\Delta s \\ &+\frac{M_\Psi}{\Gamma(\vartheta)}\int_{\eta_1}^{\eta_2}(\eta_2-s)^{\vartheta-1}\Delta s.\end{aligned}$$

Since $(\theta-s)^{\vartheta-1}$ is continuous, hence $|(\Upsilon u)(\theta_2)-(\Upsilon u)(\theta_1)|$ tends to zero when $\eta_1\to\eta_2$. Therefore, from the three steps along with the Arzela-Ascoli theorem, Υ is completely continuous.

Step 4: Finally, we need to show that the set

$$\mathfrak{B}=\{u\in C(I,\mathbb{R}): u=\lambda\Upsilon(u),\lambda\in(0,1)\},$$

is bounded. For this let $u\in\mathfrak{B}, 0<\lambda<1$, Thus for each $\theta\in I$, we have:

$$\begin{aligned}u=\lambda\Upsilon(u)=\frac{\lambda}{\Gamma(\vartheta)}\Bigg(&\int_0^\theta(\theta-s)^{\vartheta-1}\Psi(s,u(\alpha(s)))\Delta s \\ &-\frac{\eta}{(\beta+\eta)}\int_0^T(T-s)^{\vartheta-1}\Psi(s,u(\alpha(s)))\Delta s \\ &+\frac{1}{(\beta+\eta)}\int_0^T(T-s)^{\vartheta-1}g(s,u(s))\Delta s\Bigg).\end{aligned}$$

Clearly, the set $\mathfrak{B}$ is bounded (using Equation (11)). Hence, from Step 1 to 4, one can find that all the conditions of Schaefer's fixed point theorem are hold. Thus, Υ has a fixed point which is the solution of Equation (2). ■

Theorem 3.3.

Let there exists a constant $\mathcal{K} > 0$ such that

$$\mathcal{K} > p^* \phi(\mathcal{K}) M + \frac{M_g T^{\vartheta}}{|\beta + \eta| \Gamma(\vartheta + 1)}, \quad \text{where } p^* = \sup_{\theta \in I} p(\theta) \tag{12}$$

and the assumptions **(P2a)** and **(P4)** are satisfied. Then Equation (2) has at least one solution.

Proof:

Let us consider the operator Υ as given in Theorem 3.1. From Theorem 3.2, we can easily shown that Υ is continuous and completely continuous. Also, for $\lambda \in [0, 1],\ \theta \in I$, let there exists a $u(\theta)$ such that $u(\theta) = \lambda(\Upsilon u)\theta$, then we have:

$$\begin{aligned} |u(\theta)| = |\lambda \Upsilon(u)\theta| = \Bigg| \lambda \Bigg(& \frac{1}{\Gamma(\vartheta)} \int_0^{\theta} (\theta - s)^{\vartheta - 1} \Psi(s, u(\alpha(s))) \Delta s \\ & - \frac{\eta}{(\beta + \eta)\Gamma(\vartheta)} \int_0^{T} (T - s)^{\vartheta - 1} \Psi(s, u(\alpha(s))) \Delta s \\ & + \frac{1}{(\beta + \eta)\Gamma(\vartheta)} \int_0^{T} (T - s)^{\vartheta - 1} g(s, u(s)) \Delta s \Bigg) \Bigg|. \end{aligned} \tag{13}$$

Hence,

$$\|u\|_C \leq P^* \phi(\|u\|_C) M + \frac{M_g T^{\vartheta}}{|\beta + \eta| \Gamma(\vartheta + 1)}.$$

Now, from the condition (12) we get number $\mathcal{K} > 0$ such that $\|u\|_C \neq \mathcal{K}$. Let $\mathfrak{N} = \{u \in C(I, \mathbb{R}) : \|u\|_C < \mathcal{K}\}$, then the operator $\Upsilon : \bar{\mathfrak{N}} \to C(I, \mathbb{R})$ is continuous and hence completely continuous. Therefore, from the choice of $\mathfrak{N}$, there is no $u \in \partial(\mathfrak{N})$ such that $u = \lambda \Upsilon u, \lambda \in [0, 1]$. Hence, nonlinear alternative of Leray Schauder's type fixed point theorem immediately gives a fixed point of Υ, which is the solution of the system (2). ■

Theorem 3.4.

Let the assumptions **(P1)-(P2)** be satisfied with $ML_{\Psi} < 1$, then Equation (2) has at least one solution.

Proof:

To prove this result, let us define two maps Υ_1 and Υ_2 such that

$$\begin{aligned} (\Upsilon_1 u)\theta = & \frac{1}{\Gamma(\vartheta)} \int_0^{\theta} (\theta - s)^{\vartheta - 1} \Psi(s, u(\alpha(s))) \Delta s \\ & - \frac{\eta}{(\beta + \eta)\Gamma(\vartheta)} \int_0^{\theta} (T - s)^{\vartheta - 1} \Psi(s, u(\alpha(s))) \Delta s, \end{aligned} \tag{14}$$

$$(\Upsilon_2 u)\theta = \frac{1}{(\beta+\eta)\Gamma(\vartheta)} \int_0^T (T-s)^{\vartheta-1} g(s,u(s))\Delta s \\ - \frac{\eta}{(\beta+\eta)\Gamma(\vartheta)} \int_\theta^T (T-s)^{\vartheta-1} \Psi(s,u(\alpha(s)))\Delta s. \tag{15}$$

Clearly, $\Upsilon = \Upsilon_1 + \Upsilon_2$ and the following steps can be easily prove using the same method as we discussed in the Theorem 3.2.

Step 1: Υ_1 is a contraction map, since,

$$\|(\Upsilon_1 u)\theta - (\Upsilon_1 v)\theta\|_C \leq \frac{L_\Psi T^\vartheta}{\Gamma(\vartheta+1)} \left(1 + \frac{|\eta|}{|\beta+\eta|}\right) \|(u-v)\|_C.$$

Step 2: For each $u \in \mathcal{B}$, we have: $\Upsilon_1 u + \Upsilon_2 u \in \mathcal{B}$.

Step 3: For each $u \in \mathcal{B}$, $\|\Upsilon_2 u\|_C \leq \delta$.

Step 4: Υ_2 is continuous.

Step 5: Υ_2 is equi-continuous. Thus, from the above steps and with the help of Arzela-Ascoli theorem, we conclude that $\Upsilon_2(\mathcal{B})$ is compact. Therefore, collecting the step1-step5, we can conclude that the conditions of Krasnoselskii's fixed point theorem are hold and hence the equation (2) has at least one solution in $\mathcal{B}$. ■

Now, we give the result of UH stability for the equation (2).

Theorem 3.5.

Let the assumptions **(P1)-(P2)** and inequality (9) are fulfilled. Then, Equation (2) is UH stable.

Proof:

Let u be a unique solution of Equation (2) and v be the solution of inequality (7). Therefore, by Lemma 2.7, we have:

$$u(\theta) = \frac{1}{\Gamma(\vartheta)} \int_0^\theta (\theta-s)^{\vartheta-1} \Psi(s,u(\alpha(s)))\Delta s \\ - \frac{\eta}{(\beta+\eta)\Gamma(\vartheta)} \int_0^T (T-s)^{\vartheta-1} \Psi(s,u(\alpha(s)))\Delta s \\ + \frac{1}{(\beta+\eta)\Gamma(\vartheta)} \int_0^T (T-s)^{\vartheta-1} g(s,u(s))\Delta s.$$

Subsequently,

$$\begin{aligned}|v(\theta)-u(\theta)| \leq & \bigg| v(\theta) - \frac{1}{\Gamma(\vartheta)}\int_0^\theta (\theta-s)^{\vartheta-1}\Psi(s,u(\alpha(s)))\Delta s \\ & - \frac{1}{(\beta+\eta)\Gamma(\vartheta)}\int_0^T (T-s)^{\vartheta-1} g(s,u(s))\Delta s \\ & + \frac{\eta}{(\beta+\eta)\Gamma(\vartheta)}\int_0^T (T-s)^{\vartheta-1}\Psi(s,u(\alpha(s)))\Delta s \bigg| \\ \leq & \, \epsilon M + \left| \frac{1}{\Gamma(\vartheta)}\int_0^\theta (\theta-s)^{\vartheta-1}(\Psi(s,v(\alpha(s)))-\Psi(s,u(\alpha(s))))\Delta s \right| \\ & + \left| \frac{1}{(\beta+\eta)\Gamma(\vartheta)}\int_0^T (T-s)^{\vartheta-1}(g(s,v(s))-g(s,u(s)))\Delta s \right| \\ & + \left| \frac{\eta}{(\beta+\eta)\Gamma(\vartheta)}\int_0^T (T-s)^{\vartheta-1}(\Psi(s,v(\alpha(s)))-\Psi(s,u(\alpha(s))))\Delta s \right|.\end{aligned}$$

Hence,

$$\|v-u\|_C \leq \epsilon M + \frac{T^\vartheta}{\Gamma(\vartheta+1)}\left[L_\Psi + \frac{L_\Psi|\eta|}{|\beta+\eta|} + \frac{L_g}{|\beta+\eta|}\right]\|v-u\|_C,$$

which immediately gives

$$\|v-u\|_C \leq \frac{\epsilon M}{1-L_F}.$$

Thus,

$$\|v-u\|_C \leq H_{(L_\Psi,L_g,\vartheta)}\epsilon,$$

where $H_{(L_\Psi,L_g,\vartheta)} = \dfrac{M}{1-L_F}$. Hence, Equation (2) is UH stable. Further, if we set

$$\mathcal{H}_{(L_\Psi,L_g,\vartheta)}(\epsilon) = H_{(L_\Psi,L_g,\vartheta)}\epsilon, \mathcal{H}_{(L_\Psi,L_g,\vartheta)}(0) = 0,$$

then Equation (2) is generalized UH stable. ■

4. Examples

Example 4.1.

We consider the following

$$\begin{aligned} {}^c\Delta^\vartheta u(\theta) &= \frac{|u(\theta)|\sin\theta}{(1+|u(\theta)|)\,(\theta+5)^2}, \quad \theta \in I = [0,1]_{\mathbb{T}^k}, u \in \mathbb{R}, \\ u(0)+u(1) &= \frac{1}{\Gamma(\vartheta)}\int_0^1 \frac{(1-s)^{\vartheta-1}(1+s\cos u(s))}{(s+2)^2}\Delta s, \end{aligned} \tag{16}$$

where $\mathbb{T}$ be any time scales which contains 0 and 1. Here $T=1,\ \beta=1,\ \eta=1$.

Set

$$\Psi(\theta, u(\alpha(\theta))) = \frac{|u(\theta)| \sin\theta}{(1 + |u(\theta)|)\,(\theta+5)^2}, \quad g(\theta, u(\theta)) = \frac{1 + \theta \cos u(\theta)}{(\theta+2)^2}.$$

Since $|\Psi(\theta, u(\alpha(\theta)))| \leq \frac{1}{25}$ and $g(\theta, u(\theta)) \leq \frac{1}{2}, \forall\, \theta \in I,\ u \in \mathbb{R}$. Also, Ψ and g satisfies

$$|\Psi(\theta, u) - \Psi(\theta, v)| \leq \frac{1}{25}|u - v| \text{ and } |g(\theta, u) - g(\theta, v)| \leq \frac{1}{4}|u - v|,\ \forall \theta\ \in I, u, v \in \mathbb{R},$$

with $L_\Psi = \frac{1}{25},\ L_g = \frac{1}{4}$. Hence, the conditions (P1)-(P3) are satisfied with

$$K_1 = \frac{0.56}{\Gamma(\vartheta+1)} < 1,\ \forall\, \vartheta \in (0,1)$$

and

$$\frac{L_\Psi T^\vartheta}{\Gamma(\vartheta+1)}\left(1 + \frac{|\eta|}{|\beta+\eta|}\right) + \frac{L_g T^\vartheta}{|\beta+\eta|\Gamma(\vartheta+1)} = \frac{0.1850}{\Gamma(\vartheta+1)} < 1,\ \ \forall\, \vartheta \in (0,1).$$

Therefore, by Theorem 3.1 and 3.5, Equation (16) has a unique solution which is generalized UH stable.

Example 4.2.

Consider the following problem

$$\begin{aligned} {}^{c}\Delta^{\vartheta} u(\theta) &= 1 + \theta \cos u(\theta), \quad \theta \in I = [0,1]_{\mathbb{T}^k}, u \in \mathbb{R}, \\ u(0)+u(T) &= \frac{1}{\Gamma(\vartheta)} \int_0^1 \frac{5(1-s)^{\vartheta-1} e^{-2s}|u|}{2 + e^s(1+|u|)} \Delta s, \end{aligned} \tag{17}$$

where $\mathbb{T}$ be any time scales which contains 0 and 1. The functions

$$\Psi(\theta, u(\alpha(\theta))) = 1 + \theta \cos u(\theta) \text{ and } g(\theta, u(\theta)) = \frac{5e^{-2t}|u|}{2 + e^{\theta}(1+|u|)},$$

satisfy (P1a)-(P2a) with $|\Psi(\theta, u(\theta))| \leq 2$ and $|g(\theta, u)| \leq \frac{5}{3}, \forall\ \theta \in I,\ u \in \mathbb{R}$. Therefore, by Theorem 3.2, Equation (17) has at least one solution.

5. Conclusion

In this manuscript, we have successfully established the sufficient conditions for the existence of solutions to a fractional dynamical equation with nonlinear integral boundary conditions on time scales. For existence of at least one solution, we used Schaefer, Schauder and Krasnoselskii's fixed point theorem. Further, we used the Banach fixed point theorem for existence of a unique solution. Moreover, we studied the UH stability results for the considered system. In the end, two examples are given to demonstrate the effectiveness of the obtained analytical results.

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