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Non-uniform Haar Wavelet Method for Solving Singularly Perturbed Differential Difference Equations of Neuronal Variability

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Abstract

A non-uniform Haar wavelet method is proposed on specially designed non-uniform grid for the numerical treatment of singularly perturbed differential-difference equations arising in neuronal variability. We convert the delay and shift terms using Taylor series upto second order and then the problem with delay and shift is converted into a new problem without the delay and shift terms. Then it is solved by using non-uniform Haar wavelet. Two test examples have been demonstrated to show the accuracy of the non-uniform Haar wavelet method. The performance of the present method yield more accurate results on increasing the resolution level and converges fast in comparison to uniform Haar wavelet.

Keywords: Error analysis; Maximum absolute residual error; Neuronal variability; Non-uniform Haar wavelet; Non-uniform grid; Numerical algorithm; Singular perturbation

MSC 2010 No. : 65M99, 65N35, 65N55, 65L10

1. Introduction

Singularly perturbed differential-difference equations (SPDDE) arises in the mathematical modelling of neuronal variability, mathematical biology, study of human pupil light reflex, control theory, study of bistable devices and various models of physiological process and disease. These type of problems have been studied by Kadalbajoo and Sharma (2005), Khan and Raza (2019), Lange and Miura (1994), Patidar and Sharma (2006), Raza and Khan (2019b), etc.

Basic non-uniform Haar wavelet and non-uniform multi-resolution analysis was introduced by Dubeau et al. (2004). Solution of integral and differential equations using non-uniform Haar wavelet was given by U. Lepik ((2008), (2009)). Solution of singularly perturbed two point boundary value problems using non-uniform Haar wavelet was given by Haq et al. (2011) and Islam et al. (2010). For detailed study of wavelet, we refer to Ahmad and Shah (2013), Chen and Hsiao (1997), Daubechies (1988), O. Oruc (2018), Kumar and Pandit (2015), Mittal and Pandit ((2017), (2018)), Khan and Raza (2019), Jiwari ((2012), (2015)), and Youssri et al. (2015).

In this paper we discuss singularly perturbed differential equations with delay and shift and non-uniform Haar wavelet in Section 2 and method of solution in Section 3. We transform the delay and shift term by using Taylor series up to second order and then the problem with delay and shift is converted into a new problem, which become singularly perturbed differential equations without the delay and shift is solved by non-uniform Haar wavelet. In case of small delay and shift Taylor series is helpful. But in case of large delay and shift or neutral delay, we refer to Raza and Khan (2019a). Numerical algorithm of the scheme is given in Section 4. Error analysis is discussed in Section 5. Numerical examples along with discussion are given in Section 6, and the conclusion in Section 7.

2. Preliminaries

2.1. Singularly perturbed differential equations with delay and shift

We consider singularly perturbed differential equations with delay and shift as follows:

$$\epsilon y''(t) + \alpha(t)y'(t) + a(t)y(t-\delta) + b(t)y(t) + c(t)y(t+\eta) = f(t), \quad t \in [0,1], \tag{1}$$

with the boundary conditions

$$y(t) = \phi(t), \quad on \quad -\delta \le t \le 0, \tag{2}$$

and

$$y(t) = \psi(t), \qquad on \quad 1 \le t \le 1+\eta, \tag{3}$$

where $a(t)$, $b(t)$, $c(t)$,$f(t)$, $\phi(t)$ and $\psi(t)$ are sufficiently smooth functions, ϵ is a perturbation parameter, $0 < \varepsilon << 1$, δ and η are delay and shift parameters which depends on ϵ. In case, if δ

and η are 0 in equation (1), then the SPDDE becomes singularly perturbed differential equations. Singularly perturbed boundary value problems have been discussed by many researchers such as Aziz and Khan (2002), Khan and Khandelwal (2014), Khan et al. (2006), Nafeh (1979), Pandit and Kumar (2014), Shah et al. (2016), El-Ajou et al. (2019), and Kumar et al. (2019).

2.2. Non-uniform Haar wavelet

Definition 2.1.

The non-uniform Haar wavelet family for $t \in [0,1]$ is defined as follows:

$$\mathcal{H}_i(t) = \begin{cases} 1, & \xi_1(i) \leq t < \xi_2(i), \\ -n_i, & \xi_2(i) \leq t < \xi_3(i), \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where i indicates the wavelet number and

$$\xi_1(i) = x(2k\mu), \qquad \xi_2(i) = x((2k+1)\mu),$$
$$\xi_3(i) = x((2k+2)\mu), \qquad \mu = \frac{M}{m},$$

$m = 2^j$, $j = 0, 1, 2..., J$, $M = 2^J$ and integer $k = 0, 1..., m-1$.

The integration of Haar wavelet is given by

$$\mathcal{P}_i(t) = \begin{cases} t - \xi_1(i), & \xi_1(i) \leq t < \xi_2(i), \\ (\xi_3(i) - t)n_i, & \xi_2(i) \leq t < \xi_3(i), \\ 0, & \text{otherwise}. \end{cases} \tag{5}$$

The double integration of Haar wavelet is given as follows:

$$\mathcal{Q}_i(t) = \begin{cases} \frac{1}{2}(t - \xi_1(i))^2, & \xi_1(i) \leq t < \xi_2(i), \\ \mathcal{K} - \frac{1}{2}(\xi_3(i) - t)^2 n_i, & \xi_2(i) \leq t < \xi_3(i), \\ \mathcal{K}, & \xi_3(i) \leq t < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

where $\mathcal{K} = \frac{(\xi_2 - \xi_1)(\xi_3 - \xi_1)}{2}$.

Proceeding in similar manner, the n^{th} integration of Haar wavelet can be written as:

$$I_n\mathcal{H}_i(t) = \begin{cases} 0, & t < \xi_1(i), \\ \frac{1}{n!}[t - \xi_1(i)]^n, & \xi_1(i) \leq t < \xi_2(i), \\ \frac{1}{n!}[(t - \xi_1(i))^n - (1 + n_i)(t - \xi_2(i))^n], & \xi_2(i) \leq t < \xi_3(i), \\ \frac{1}{n!}[(t - \xi_1(i))^n - (1 + n_i)(t - \xi_2(i))^n + n_i(t - \xi_3(i))^n], & \xi_3(i) \leq t, \end{cases} \tag{7}$$

where $n_i = \frac{(\xi_2 - \xi_1)}{(\xi_3 - \xi_1)}$.

Definition 2.2.

The non-uniform grid i.e q-mesh is defined as

$$\hat{t}(j) = \frac{1-q^j}{1-q^N}, \quad j = 0, 1, 2, ..., N, \tag{8}$$

$$t(u) = \frac{\hat{t}(j-1) + \hat{t}(j)}{2}, \quad j = 0, 1, 2, ..., N. \tag{9}$$

The matrix of non-uniform Haar wavelet with respect to the non-uniform grid (9) when $q = 0.99$ is given as follows:

$$\mathcal{H}=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \frac{-939}{902} & \frac{-939}{902} & \frac{-939}{902} & \frac{-939}{902} \\ 1 & 1 & \frac{-201}{197} & \frac{-201}{197} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & \frac{-201}{197} & \frac{-201}{197} \\ 1 & \frac{-100}{99} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{-100}{99} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{-100}{99} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{-100}{99} \end{pmatrix}$$

The matrix of integral and double integral of non-uniform Haar wavelet with respect to the non-uniform grid (9) when $q = 0.99$ are given as:

$$\mathcal{P}=\frac{1}{16}\begin{pmatrix} \frac{235}{3631} & \frac{549}{2837} & \frac{365}{1137} & \frac{301}{673} & \frac{1034}{1807} & \frac{634}{911} & \frac{1564}{1911} & \frac{1916}{2039} \\ \frac{235}{3631} & \frac{549}{2837} & \frac{365}{1137} & \frac{301}{673} & \frac{1959}{4399} & \frac{760}{2401} & \frac{255}{1349} & \frac{408}{6497} \\ \frac{235}{3631} & \frac{549}{2837} & \frac{557}{2888} & \frac{28}{437} & 0 & \frac{988}{5315} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{271}{4359} & \frac{752}{4059} & \frac{376}{6109} \\ \frac{235}{3631} & \frac{235}{3631} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{795}{12533} & \frac{795}{12533} & 0 & \frac{271}{4359} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{271}{4359} & \frac{271}{4359} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{192}{3151} & \frac{192}{3151} \end{pmatrix},$$

$$\mathcal{Q}=\frac{1}{512}\begin{pmatrix} \frac{17}{8117} & \frac{103}{5501} & \frac{464}{9005} & \frac{602}{6019} & \frac{1212}{7403} & \frac{1669}{6892} & \frac{4048}{12087} & \frac{366}{829} \\ \frac{17}{8117} & \frac{103}{5501} & \frac{464}{9005} & \frac{602}{6019} & \frac{197}{1233} & \frac{1475}{7129} & \frac{730}{3069} & \frac{283}{1118} \\ \frac{17}{8117} & \frac{103}{5501} & \frac{101}{2128} & \frac{27}{424} & \frac{211}{3212} & \frac{211}{3212} & \frac{211}{3212} & \frac{211}{3212} \\ 0 & 0 & 0 & 0 & \frac{74}{38291} & \frac{207}{11981} & \frac{6347}{144923} & \frac{325}{5531} \\ \frac{17}{8117} & \frac{137}{9385} & \frac{123}{7378} & \frac{123}{7378} & \frac{123}{7378} & \frac{123}{7378} & \frac{123}{7378} & \frac{123}{7378} \\ 0 & 0 & \frac{53}{26344} & \frac{51}{3637} & \frac{745}{46521} & \frac{745}{46521} & \frac{745}{46521} & \frac{745}{46521} \\ 0 & 0 & 0 & 0 & \frac{74}{38291} & \frac{205}{15219} & \frac{173}{11246} & \frac{173}{11246} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{64}{34475} & \frac{102}{7883} \end{pmatrix}.$$

3. Method for solving singularly perturbed differential difference equations of neuronal variability

We converted the delayed term $y(t-\delta)$ and shift term $y(t+\eta)$ using Taylor series expansion up to second order and then apply non-uniform Haar wavelet method to solve it numerically.

By Taylor series expansion, we obtain the delay and shift term $y(t-\delta)$ and $y(t+\eta)$ as

$$y(t-\delta) \approx y(t) - \delta y'(t) + \frac{\delta^2}{2} y''(t), \tag{10}$$

and

$$y(t+\eta) \approx y(t) + \eta y'(t) + \frac{\eta^2}{2} y''(t), \tag{11}$$

neglecting higher order terms.

Using Equations (10) and (11) in singularly perturbed differential difference equation (1) we get

$$\begin{aligned} \epsilon y''(t) + \alpha(t) y'(t) + a(t)(y(t) - \delta y'(t) + \tfrac{\delta^2}{2} y''(t)) + b(t) y(t) \\ + c(t)(y(t) + \eta y'(t) + \tfrac{\eta^2}{2} y''(t)) = f(t). \end{aligned} \tag{12}$$

On simplifying we get,

$$\begin{aligned} \left(\epsilon + a(t)\frac{\delta^2}{2} + c(t)\frac{\eta^2}{2}\right) y''(t) + (\alpha(t) - a(t)\delta + c(t)\eta) y'(t) \\ + (a(t) + b(t) + c(t)) y(t) = f(t). \end{aligned} \tag{13}$$

In order to solve this problem with boundary conditions (2) and (3) we assume that

$$y''(t) = \sum_{i=1}^{N} a_i \mathcal{H}_i(t). \tag{14}$$

Now integrating (14) from 0 to t, we get

$$y'(t) = \sum_{i=1}^{N} a_i \mathcal{P}_i(t) + y'(0). \tag{15}$$

Further to find $y'(0)$, integrate equation (15) from 0 to 1, we get

$$y'(0) = y(1) - y(0) - \sum_{i=1}^{N} a_i \mathcal{C}_i(t), \tag{16}$$

where $\mathcal{C}_i(t) = \int_0^1 \mathcal{P}_i(t) dt$.

Again integrating Equation (15) from 0 to t, we get

$$y(t) = \sum_{i=1}^{N} a_i \mathcal{Q}_i(t) + t y'(0) + y(0). \tag{17}$$

Now, using Equation (16) in Equations (15) and (17), we get

$$y'(t) = \sum_{i=1}^{N} a_i \mathcal{P}_i(t) + y(1) - y(0) - \sum_{i=1}^{N} a_i \mathcal{C}_i(t), \tag{18}$$

and

$$y(t) = \sum_{i=1}^{N} a_i \mathcal{Q}_i(t) + t(y(1) - y(0) - \sum_{i=1}^{N} a_i \mathcal{C}_i(t)) + y(0). \tag{19}$$

Using equations (14), (18) and (19) in equation (13), we obtain the following system of linear equations:

$$\begin{aligned}&(\epsilon + a(t)\frac{\delta^2}{2} + c(t)\frac{\eta^2}{2}) \sum_{i=1}^{N} a_i \mathcal{H}_i(t) + (\alpha(t) - a(t)\delta + c(t)\eta)(\sum_{i=1}^{N} a_i \mathcal{P}_i(t)\\ &+ y(1) - y(0) - \sum_{i=1}^{N} a_i \mathcal{C}_i(t)) + (a(t) + b(t) + c(t))(\sum_{i=1}^{N} a_i \mathcal{Q}_i(t)\\ &+ t(y(1) - y(0) - \sum_{i=1}^{N} a_i \mathcal{C}_i(t)) + y(0)) = f(t).\end{aligned} \tag{20}$$

$$\begin{aligned}&\sum_{i=1}^{N} a_i[(\epsilon + a(t)\frac{\delta^2}{2} + c(t)\frac{\eta^2}{2})\mathcal{H}_i(t) + (\alpha(t) - a(t)\delta + c(t)\eta)(\mathcal{P}_i(t) - \mathcal{C}_i(t)\\ &+ (a(t) + b(t) + c(t)))(\mathcal{Q}_i(t) - t\mathcal{C}_i(t))] = f(t) + (y(0) - y(1))((\alpha(t)\\ &- a(t)\delta + c(t)\eta) + t(a(t) + b(t) + c(t))) - y(0).\end{aligned} \tag{21}$$

Now, we can easily find the non-uniform Haar wavelet coefficient a_i's by solving system of linear equations (21) with boundary conditions (2) and (3) using any known method. Then put the values of a_i's in equation (19) which is the non-uniform Haar wavelet solution of singularly perturbed differential difference equation.

4. Numerical algorithm of the scheme

Algorithm. Input
Step 1: Compute the matrix of the Haar wavelet $\mathcal{H}_i(t)$ from Equation (4),
Step 2: Compute the matrices of integral of the Haar wavelet $\mathcal{P}_i(t)$ and $\mathcal{Q}_i(t)$ from Equations (5) and (6), respectively,
Step 3: Expand the term which contains delay δ and shift η using Taylor series of Equation (1) up to second order and then convert the problem (1) into new problem (13),
Step 4: Construct the expressions given by Equations (14)-(21),
Step 5: By using Step 4 construct the left hand side matrix of the Equation (21),
Step 6: Compute the unknown vector a_i by solving the system of linear equations (21),
Step 7: Put the vector a_i in Equation (19),
Output: Obtained approximate solution $y(t)$.

5. Error analysis

Lemma 5.1.

Let $y(x)$ be a square integrable function with bounded first derivative on $(0,1)$ and $y(x_j)$ be Haar wavelet approximation of $y(x)$ then the error norm at J^{th} level satisfies the inequality:

$$\|E\| \leq 2D\sqrt{K}\left(\frac{2^{-2(J+1)}}{3}\right)^2, \tag{22}$$

where K is a positive constant and D is given by $y'(t) \leq D$.

Proof:

For the proof see Islam et al. (2010) and Pandit and Kumar (2014). ■

In singularly perturbed differential difference equations with turning point arising in neuronal variability, we do not have the exact solution. Hence the maximum absolute residual error is calculated by the following formula:

$$E = Max.|(\epsilon + a(t_j)\frac{\delta^2}{2} + c(t_j)\frac{\eta^2}{2})y''(t_j) + (\alpha(t_j) - a(t_j)\delta + c(t_j)\eta)y'(t_j) \\ + (a(t_j) + b(t_j) + c(t_j))y(t_j) - f(t_j)|, \tag{23}$$

where $y''(t_j)$, $y'(t_j)$ and $y(t_j)$ are given in equations (14), (18) and (19) respectively and t_j are the discrete points given by Equations (8) and (9), $j = 1, 2, ..., N$.

6. Numerical examples

In this section we demonstrate two linear problems of singularly perturbed differential difference equations arising in neuronal variability to illustrate the non-uniform Haar wavelet method. The results are tabulated for various values and also compared with the exponentially fitted operator finite difference method by Rai and Sharma (2012).

Problem 1. Let us assume the following SPDDE

$$\epsilon y''(t) + 2(t - \tfrac{1}{2})(1 + \tfrac{3.121}{10}(t - \tfrac{1}{2}))y' - (\tfrac{4}{3}(t) + (t - \tfrac{1}{2})\tfrac{2.764}{10})y(t) + \tfrac{2}{10}y(t-\delta) \\ + \tfrac{1}{8}y(t+\eta) = t, \quad t \in [0,1] \tag{24}$$

with boundary conditions

$$y(t) = 0, \ \ -\delta \leq t \leq 0, \quad y(t) = 0, \quad 1 \leq t \leq 1 + \eta. \tag{25}$$

Maximum absolute residual errors (MARE) obtained by non-uniform Haar wavelet method with different resolutions level are given in the Tables 1.1-1.4 for a particular value of perturbation parameter and various values of delay and shift parameters. For the sake of comparison, results of exponentially fitted operator finite difference method by Rai and Sharma (2012) is given in Table

1.5. Also, graph of uniform and non-uniform Haar wavelet (NUHW) solution is given in Figures 1, 2, 3, and 4, respectively.

Table 1.1 MARE on q-mesh for fixed values of $\delta = 0.2$, $\eta = 0.1$ and various values of ϵ using NUHW

$\epsilon \setminus N$	32	64	128	256	512	1024
10^0	4.5103e-17	3.7188e-17	2.4286e-17	1.4122e-17	3.1903e-17	7.0911e-16
10^{-2}	3.4694e-17	1.1883e-16	2.3636e-17	2.2416e-17	3.6782e-17	2.0844e-17
10^{-4}	4.3802e-17	7.9797e-17	2.6455e-17	2.4340e-17	5.1174e-17	1.1081e-16
10^{-6}	5.4210e-17	8.7604e-17	3.1557e-17	2.3066e-17	4.5482e-17	2.0914e-16
10^{-8}	7.5027e-17	1.6220e-16	1.9028e-17	2.2416e-17	6.1366e-17	3.5345e-16
10^{-10}	4.7705e-17	1.5439e-16	1.8377e-17	2.3310e-17	3.5128e-17	2.3386e-16
10^{-12}	4.7705e-17	1.5439e-16	1.8377e-17	2.0708e-17	3.9465e-17	2.3137e-16
10^{-14}	4.7705e-17	1.5439e-16	1.8377e-17	2.0708e-17	3.9465e-17	2.3137e-16
10^{-16}	4.7705e-17	1.5439e-16	1.8377e-17	2.0708e-17	3.9465e-17	2.3137e-16
10^{-18}	4.7705e-17	1.5439e-16	1.8377e-17	2.0708e-17	3.9465e-17	2.3137e-16

Table 1.2 MARE on q-mesh for fixed values of $\delta = 0.2$, $\eta = 0.2$ and various values of ϵ using NUHW

$\epsilon \setminus N$	32	64	128	256	512	1024
10^0	4.4669e-17	3.2960e-17	1.8160e-17	6.2450e-17	5.2204e-17	2.7367e-16
10^{-2}	3.4694e-17	1.9602e-16	2.1684e-17	4.0115e-17	2.6807e-17	2.9118e-17
10^{-4}	4.2934e-17	1.4929e-16	2.5967e-17	3.7080e-17	6.4185e-17	1.3097e-16
10^{-6}	3.1225e-17	1.2577e-16	5.7083e-17	3.2526e-17	5.0741e-17	3.7828e-16
10^{-8}	3.7730e-17	1.4658e-16	2.6021e-17	2.1413e-17	4.9873e-17	3.3979e-16
10^{-10}	3.4694e-17	1.3618e-16	2.9924e-17	2.3419e-17	4.8139e-17	3.8804e-16
10^{-12}	3.4694e-17	1.3618e-16	2.9924e-17	2.3419e-17	5.2909e-17	2.5240e-16
10^{-14}	3.4694e-17	1.3618e-16	2.9924e-17	2.3419e-17	5.2909e-17	2.5240e-16
10^{-16}	3.4694e-17	1.3618e-16	2.9924e-17	2.3419e-17	5.2909e-17	2.5240e-16
10^{-18}	3.4694e-17	1.3618e-16	2.9924e-17	2.3419e-17	5.2909e-17	2.5240e-16

Table 1.3 MARE on q-mesh for fixed values of $\delta = 0.2$, $\eta = 0.4$ and various values of ϵ using NUHW

$\epsilon \setminus N =$	32	64	128	256	512	1024
10^0	3.4694e-17	3.3827e-17	2.0329e-17	4.0766e-17	2.6915e-17	2.3872e-16
10^{-2}	5.4210e-17	5.8113e-17	4.9602e-17	2.3365e-17	3.8489e-17	1.5460e-17
10^{-4}	3.2960e-17	5.6379e-17	2.6455e-17	3.3908e-17	1.6025e-16	1.4472e-15
10^{-6}	3.9465e-17	5.9848e-17	4.4506e-17	2.6895e-17	2.5869e-16	2.2520e-15
10^{-8}	8.7170e-17	9.1073e-17	3.3014e-17	3.2526e-17	1.6155e-16	2.0504e-15
10^{-10}	5.5511e-17	6.9497e-17	4.3531e-17	2.3419e-17	1.4008e-16	2.6264e-15
10^{-12}	5.5511e-17	6.9497e-17	4.3531e-17	2.3419e-17	1.4333e-16	2.1589e-15
10^{-14}	5.5511e-17	6.9497e-17	4.3531e-17	2.3419e-17	1.4333e-16	2.1589e-15
10^{-16}	5.5511e-17	6.9497e-17	4.3531e-17	2.3419e-17	1.4333e-16	2.1589e-15
10^{-18}	5.5511e-17	6.9497e-17	4.3531e-17	2.3419e-17	1.4333e-16	2.1589e-15

Table 1.4 MARE on q-mesh for fixed values of $\delta = 0.4$, $\eta = 0.1$ and various values of ϵ using NUHW

$\epsilon \backslash N$	32	64	128	256	512	1024
10^{-0}	3.0791e-17	3.1225e-17	1.8974e-17	1.9082e-17	4.0766e-17	8.4163e-16
10^{-2}	7.5894e-17	1.9342e-16	6.0444e-17	2.3256e-17	2.7322e-17	2.3080e-17
10^{-4}	3.3827e-17	2.3245e-16	3.1171e-17	5.0659e-17	3.3746e-17	1.0018e-16
10^{-6}	5.0741e-17	2.0817e-16	5.8818e-17	4.7651e-17	3.0358e-17	1.5450e-16
10^{-8}	5.2042e-17	2.1077e-16	4.6566e-17	3.5670e-17	3.4911e-17	2.4199e-16
10^{-10}	4.1633e-17	2.0296e-16	3.5562e-17	3.7513e-17	2.5153e-17	1.9266e-16
10^{-12}	4.1633e-17	2.0296e-16	3.5562e-17	3.2092e-17	2.1034e-17	1.8236e-16
10^{-14}	4.1633e-17	2.0296e-16	3.5562e-17	3.2092e-17	2.1034e-17	1.8236e-16
10^{-16}	4.1633e-17	2.0296e-16	3.5562e-17	3.2092e-17	2.1034e-17	1.8236e-16
10^{-18}	4.1633e-17	2.0296e-16	3.5562e-17	3.2092e-17	2.1034e-17	1.8236e-16

Table 1.5 MARE obtained by Rai and Sharma (2012) for fixed values of $\delta = 0.4$, $\eta = 0.2$ and various values of ϵ

$\epsilon \backslash N$	100	200	400	800
1	4.3770e-05	2.2140e-05	1.1130e-05	5.5810e-06
10^{-1}	6.8910e-04	3.5010e-04	1.7650e-04	8.8610e-05
10^{-2}	1.7000e-03	8.2590e-04	4.0680e-04	2.0180e-04
10^{-4}	5.9670e-03	3.0320e-03	1.3890e-03	6.3170e-04
10^{-6}	6.4170e-03	4.0930e-03	2.5920e-03	1.5930e-03
10^{-8}	6.4170e-03	4.0930e-03	2.5910e-03	1.6350e-03
10^{-10}	6.4170e-03	4.0930e-03	2.591e0-03	1.6350e-03
10^{-12}	6.4170e-03	4.0930e-03	2.5910e-03	1.6350e-03
10^{-14}	6.4170e-03	4.0930e-03	2.5910e-03	1.635e-03
10^{-16}	6.4170e-03	4.0930e-03	2.591e-03	1.6350e-03
10^{-18}	6.4170e-03	4.0930e-03	2.5910e-03	1.6350e-03

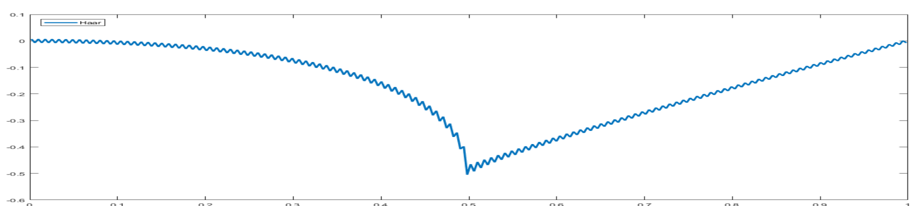

Figure 1. Graph of solution for problem 1 for $\epsilon = 10^{-5}$, $\eta = 0.1$ and $\delta = 0.2$ with $J = 7$ using uniform Haar wavelet

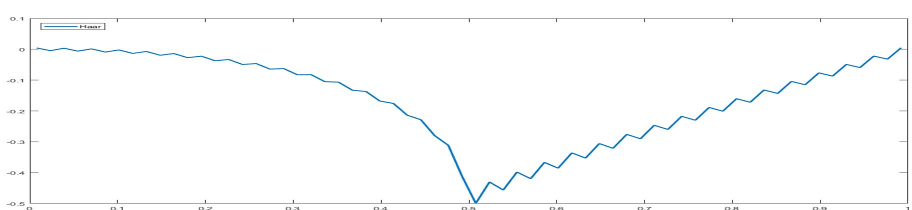

Figure 2. Graph of solution of problem 1 for $\epsilon = 10^{-5}$, $\eta = 0.1$ and $\delta = 0.2$ with $J = 5$ using uniform Haar wavelet

Problem 2. Let us assume the following SPDDE

$$\begin{aligned} &\epsilon y''(t) + (t - \frac{1}{2})(3 + 4(t - \frac{1}{2}))y' - 2y(t) \\ &\quad + 4(t - \frac{1}{2})^2 y(t - \delta) + y(t + \eta) = 1, \quad t \in [0, 1], \end{aligned} \tag{26}$$

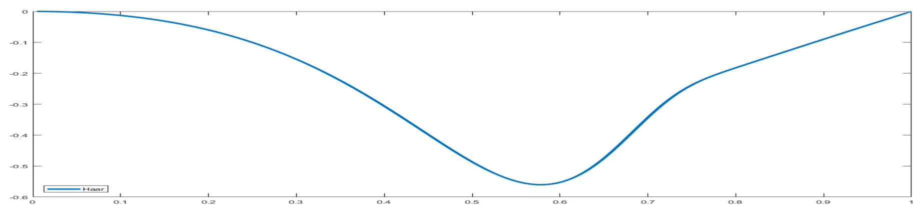

Figure 3. Graph of solution of problem 1 for $\epsilon = 10^{-18}$, $\eta = 0.1$ and $\delta = 0.2$ with $J = 10$ using NUHW

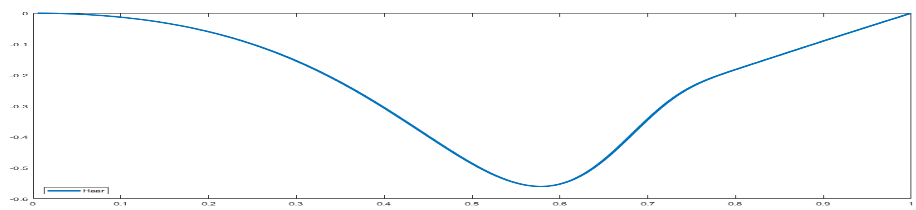

Figure 4. Graph of solution of problem 1 for $\epsilon = 10^{-13}$, $\eta = 0.1$ and $\delta = 0.2$ with $J = 7$ using NUHW

with boundary conditions

$$y(t) = 0, \quad -\delta \leq t \leq 0, \quad y(t) = 1, \quad 1 \leq t \leq 1 + \eta. \tag{27}$$

Maximum absolute residual errors (MARE) using non-uniform Haar wavelet method with different resolutions level are given in the Tables 2.1 – 2.4 for a particular value of perturbation parameter and various values of delay and shift parameters. For the sake of comparison, results of exponentially fitted operator finite difference method by Rai and Sharma (2012) is given in Table 2.5. Also, the graph of uniform and non-uniform Haar wavelet (NUHW) solution is given in Figures 5 and 6-8, respectively.

Table 2.1 MARE on q-mesh for fixed values of $\delta = 0.2$, $\eta = 0.1$ and various values of ϵ using NUHW

$\epsilon \setminus N$	64	128	256	512	1024
10^{0}	7.2858e-17	4.1633e-17	3.3827e-17	4.0766e-17	7.5135e-16
10^{-2}	2.8610e-05	2.6546e-06	8.3612e-07	1.2690e-07	7.1002e-08
10^{-4}	2.4928e-05	1.8036e-06	7.7174e-07	4.3316e-08	3.1363e-08
10^{-6}	2.4928e-05	1.8036e-06	7.7173e-07	4.3307e-08	3.1358e-08
10^{-8}	2.4928e-05	1.8036e-06	7.7173e-07	4.3307e-08	3.1358e-08
10^{-10}	2.4928e-05	1.8036e-06	7.7173e-07	4.3307e-08	3.1358e-08
10^{-12}	2.4928e-05	1.8036e-06	7.7173e-07	4.3307e-08	3.1358e-08
10^{-14}	2.4928e-05	1.8036e-06	7.7173e-07	4.3307e-08	3.1358e-08
10^{-16}	2.4928e-05	1.8036e-06	7.7173e-07	4.3307e-08	3.1358e-08
10^{-18}	2.4928e-05	1.8036e-06	7.7173e-07	4.3307e-08	3.1358e-08

Table 2.2 MARE on q-mesh for fixed values of $\delta = 0.2$, $\eta = 0.2$ and various values of ϵ using NUHW

$\epsilon \backslash N$	64	128	256	512	1024
10^{0}	7.9797e-17	4.8572e-17	5.8547e-17	4.9873e-17	1.1362e-15
10^{-2}	5.4491e-05	7.3087e-06	9.2588e-07	2.4045e-07	1.5236e-07
10^{-4}	5.2413e-05	7.3161e-06	7.3080e-07	3.3181e-07	1.8416e-07
10^{-6}	5.2413e-05	7.3161e-06	7.3078e-07	3.3182e-07	1.8416e-07
10^{-8}	5.2413e-05	7.3161e-06	7.3078e-07	3.3182e-07	1.8416e-07
10^{-10}	5.2413e-05	7.3161e-06	7.3078e-07	3.3182e-07	1.8416e-07
10^{-12}	5.2413e-05	7.3161e-06	7.3078e-07	3.3182e-07	1.8416e-07
10^{-14}	5.2413e-05	7.3161e-06	7.3078e-07	3.3182e-07	1.8416e-07
10^{-16}	5.2413e-05	7.3161e-06	7.3078e-07	3.3182e-07	1.8416e-07
10^{-18}	5.2413e-05	7.3161e-06	7.3078e-07	3.3182e-07	1.8416e-07

Table 2.3 MARE on q-mesh for fixed values of $\delta = 0.2$, $\eta = 0.4$ and various values of ϵ using NUHW

$\epsilon \backslash N$	64	128	256	512	1024
10^{-1}	6.5919e-17	5.7246e-17	6.9823e-17	5.4210e-17	9.5366e-16
10^{-2}	4.1559e-06	2.5940e-06	1.1127e-06	3.8681e-07	3.5636e-07
10^{-4}	9.1955e-06	1.4259e-06	1.4917e-06	5.4775e-07	3.4771e-07
10^{-6}	9.1960e-06	1.4258e-06	1.4918e-06	5.4775e-07	3.4771e-07
10^{-8}	9.1960e-06	1.4258e-06	1.4918e-06	5.4775e-07	3.4771e-07
10^{-10}	9.1960e-06	1.4258e-06	1.4918e-06	5.4775e-07	3.4771e-07
10^{-12}	9.1960e-06	1.4258e-06	1.4918e-06	5.4775e-07	3.4771e-07
10^{-14}	9.1960e-06	1.4258e-06	1.4918e-06	5.4775e-07	3.4771e-07
10^{-16}	9.1960e-06	1.4258e-06	1.4918e-06	5.4775e-07	3.4771e-07
10^{-18}	9.1960e-06	1.4258e-06	1.4918e-06	5.4775e-07	3.4771e-07

Table 2.4 MARE on q-mesh for fixed values of $\delta = 0.4$, $\eta = 0.1$ and various values of ϵ using NUHW

$\epsilon \backslash N$	64	128	256	512	1024
10^{0}	1.0408e-16	4.3368e-17	3.5562e-17	3.2960e-17	1.5654e-15
10^{-2}	4.1981e-05	2.8916e-06	4.5140e-07	1.2783e-07	6.4988e-08
10^{-4}	4.1678e-05	3.3690e-06	4.0005e-07	9.9482e-08	5.5107e-08
10^{-6}	4.1678e-05	3.3690e-06	4.0004e-07	9.9478e-08	5.5105e-08
10^{-8}	4.1678e-05	3.3690e-06	4.0004e-07	9.9478e-08	5.5105e-08
10^{-10}	4.1678e-05	3.3690e-06	4.0004e-07	9.9478e-08	5.5105e-08
10^{-12}	4.1678e-05	3.3690e-06	4.0004e-07	9.9478e-08	5.5105e-08
10^{-14}	4.1678e-05	3.3690e-06	4.0004e-07	9.9478e-08	5.5105e-08
10^{-16}	4.1678e-05	3.3690e-06	4.0004e-07	9.9478e-08	5.5105e-08
10^{-18}	4.1678e-05	3.3690e-06	4.0004e-07	9.9478e-08	5.5105e-08

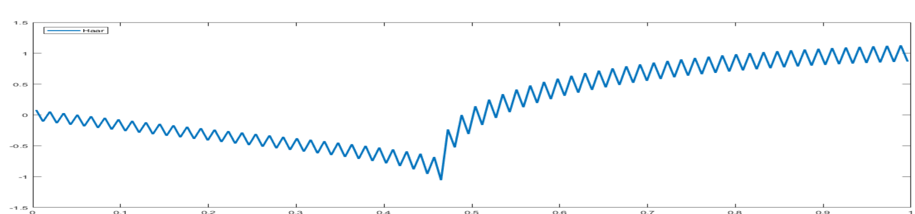

Figure 5. Graph of solution of problem 2 for $\epsilon = 0.1$ and $\eta = 0.9\epsilon$ with $J = 5$ using uniform Haar wavelet

Table 2.5 MARE obtained by Rai and Sharma (2012) for fixed values of $\delta = 0.4$, $\eta = 0.4$ and various values of ϵ

$\epsilon \backslash N$	100	200	400	800
1	5.4660e-05	2.8500e-05	1.4540e-05	7.3470e-06
10^{-1}	1.4950e-03	7.4740e-04	3.7370e-04	1.8680e-04
10^{-2}	4.1570e-03	1.9570e-03	9.4570e-04	4.6440e-04
10^{-4}	1.2110-02	6.3360e-03	2.9810e-03	1.3620e-03
10^{-6}	1.2430e-02	7.7480e-03	4.7970e-03	2.9510e-03
10^{-8}	1.2430e-02	7.7480e-03	4.7970e-03	2.9700e-03
10^{-10}	1.2430e-02	7.7480e-03	4.7970e-03	2.9700e-03
10^{-12}	1.2430e-02	7.7480e-03	4.7970e-03	2.9700e-03
10^{-14}	1.2430e-02	7.7480e-03	4.7970e-03	2.9700e-03
10^{-16}	1.2430e-02	7.7480e-03	2.591e-03	2.9700e-03
10^{-18}	1.2430e-02	7.7480e-03	4.7970e-03	2.9700e-03

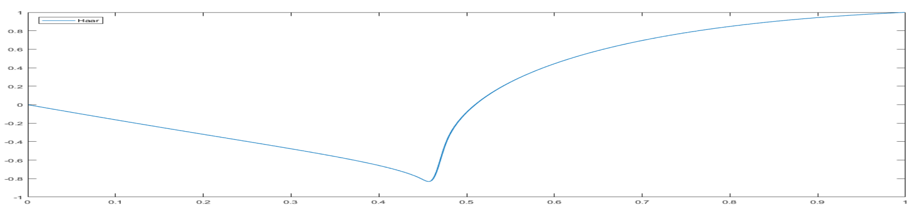

Figure 6. Graph of solution of problem 2 for $\epsilon = 10^{-2}$, $\eta = 0.2$ and $\delta = 0.1$ with $J = 8$ using NUHW

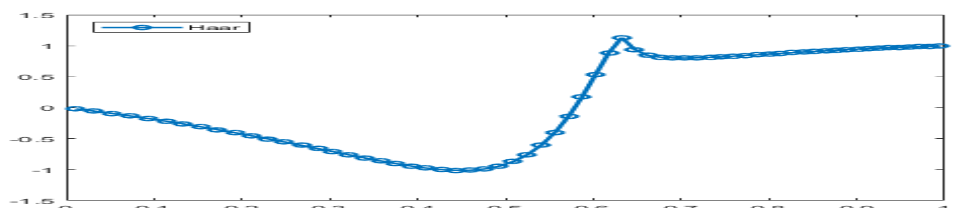

Figure 7. Graph of solution of problem 2 for $\epsilon = 10^{-10}$ and $\eta = 0.9\epsilon$ with $J = 5$ using NUHW

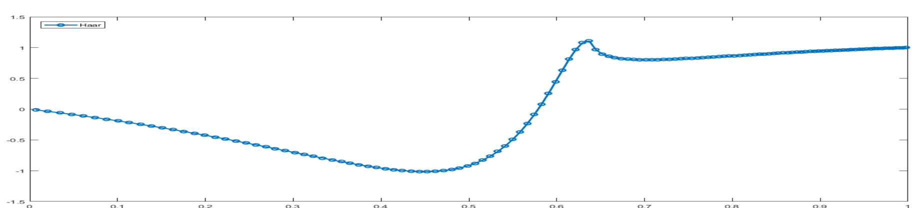

Figure 8. Graph of solution of problem 2 for $\epsilon = 10^{-18}$ and $\eta = 0.9\epsilon$ with $J = 6$ using NUHW

7. Conclusion

We have solved singularly perturbed differential-difference equations arising in neuronal variability by using non-uniform Haar wavelet method. We obtain maximum absolute residual errors and tabulated in the Tables $1.1 - 1.4$ and $2.1 - 2.4$. Further, we compared our maximum absolute residual errors with the exponentially fitted operator finite difference method by Rai and Sharma given in Tables 1.5 and 2.5. Our results are far better than the results given by Rai and Sharma.

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