Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 14, Issue 2 (December 2019), pp. 1279 – 1289

The Weak Hyperedge Tenacity of the Hypercycles

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Received: October 25, 2018; Accepted: April 17, 2019

Abstract

Graphs play an important role in our daily life. For example, the urban transport network can be represented by a graph, as the intersections are the vertices and the streets are the edges of the graph. Suppose that some edges of the graph are removed, the question arises how damaged the graph is. There are some criteria for measuring the vulnerability of graph; the tenacity is the best criteria for measuring it. Since the hypergraph generalize the standard graph by defining any edge between multiple vertices instead of only two vertices, the above question is about the hypergraph. When a hyperedge is omitted from hypergraph, we have two kinds of deletion: strong deletion and weak deletion. Weak hyperedge deletion just deletes the connection between the vertices in the hyperedge and the vertices became in the hypergraph. In this paper, we obtain the tenacity of hypercycles by weak hyperedge deletion.

Keywords: Hypergraphs; Hypercycles; Tenacity; Weak hyperedge deletion

MSC 2010 No.: 05C65

1. Introduction

In some issues of everyday life, items have incidental relationships. Mathematically, we denote this connection with the graph, but sometimes we want to show that there is a certain connection between some of the vertices. In this case, we use hypergraph instead of graph and hyperedges instead of edges. Since 1932, researchers have been working on hypergraphs. The concepts of graphs are defined on hypergraphs, such as path, cycle, connectivity, etc. (Wang et al. (2004), Wu et al. (2004), Cheng (1994)).

The criteria for measuring the vulnerability of the graph, such as connectivity, rupture degree, toughness, tenacity, etc., are also defined for the hypergraph (Bahramian et al. (2015)). We know tenacity is the best criteria for measuring the vulnerability (Bahramian et al. (2015)). The tenacity is studied for different graphs, for example, tenacity of corona product of graphs, tenacity of three classes of Harary Graphs, tenacity of complete graphs, tenacity of trees, etc. (Moazzami (2011), Mamut et al. (2008), Li et al. (2004), Moazzami (1999), Moazzami (2000), Moazzami et al. (2009), Cozzens et al. (1994)). In this paper, we study the relationship between hypercycles and tenacity. We explain some concepts for hypergraph, some lemmas for subsets, main results and conclusion, in these sections, respectively.

2. Some concepts of hypergraph

In this section we explain some definitions of the hypergraph.

Definition 2.1.

A hypergraph H denoted $H = (V, E = (e_i)_{i \in I})$ consists of a set of vertices V and a multiset E of subsets of V, called hyperedges, and their indices are in index set I. When V and I are finite, the hypergraph is called finite. We refer to $|V| = n$ as the order of the hypergraph, and to $|I| = m$ as the size of the hypergraph. If $m = 0$, the hypergraph is called empty. Like parallel edges in the graph, parallel hyperedges are also defined in the hypergraph; if $e_i = e_j$, these hyperedges are called parallel. Two vertices v and w are adjacent when there is a hyperedge, like as e_j, such that $v, w \in e_j$, and we write $[v, w] \subset e_j$. A vertex can repeat in a hyperedge to create a loop; if t is the number of repetition of vertex v, we use the notation $[v; t]$.

A vertex v and a hyperedge e_j are incident if $v \in e_j$ and let $m_{e_j}(v)$ be the number of copies the vertex v in the hyperedge e_j. The degree of a vertex v in the hypergraph H denoted by $d_H(v)$ and defined as $d_H(v) = \Sigma_{i \in I} m_{e_i}(v)$.

The number of vertices in the hyperedge is denoted by $|e_i|$ and it is named the size of the hyperedge e_i. The number of distinguish vertices in the hyperedge e_i is the $supp(e_i)$, i.e. $supp(e_i) = \{v \in e_i | m_{e_i}(v) > 0\}$. The hypergraph H is called r-uniform, if for all i, $i \in I$, we have $|e_i| = r$.

Definition 2.2.

Strong vertex deletion of a vertex v makes a new hypergraph $H' = (V', E' = (e_i)_{i \in I'})$ that $V' = V - \{v\}$ and $I' = \{i \in I | v \notin e_i\}$. It means the vertex v and all hyperedges incident to v are omitted from H. If X is a vertex set, we denote $H \backslash_S X$ to show the hypergraph formed by strongly deleting all vertices of X from H.

Weak vertex deletion of a vertex v creates a new hypergraph $H' = (V', E' = (e'_i)_{i \in I})$ that $V' =$

$V - \{v\}$, $e'_i = e_i$ if $v \notin e_i$ and $e'_i = e_i \backslash [v; m_{e_i}(v)]$ otherwise. It means the vertex v is deleted from V and all hyperedges incident to v. If X is a vertex set, we denote $H \backslash_W X$ to show the hypergraph formed by weakly deleting all vertices of X from H.

Strong hyperedge deletion of a hyperedge e_j creates a new hypergraph $H' = (V', E' = (e'_i)_{i \in I'})$ where $V' = V \backslash \{e_j\}$, $I' = I \backslash \{j\}$ and $e'_i = e_i \backslash [v; m_{e_i}(v) | v \in e_j]$ for $i \neq j$. That is, strong hyperedge deletion of e_j removes e_j from the hypergraph and weakly vertex deletes all the vertices incident with e_j. For any subset F of E, we use $H \backslash_S F$ to denote the hypergraph formed by strongly deleting all the hyperedges of F from H.

Weak hyperedge deletion of a hyperedge e_j makes the hypergraph $H' = (V, E' = (e_i)_{i \in I'})$ where $I' = I \backslash \{j\}$. That is, weak hyperedge deletion of e_j just removes e_j without affecting the rest of the hypergraph. For any subset F of E, we use $H \backslash_W F$ to denote the hypergraph formed by weakly deleting all the hyperedges of F from H.

Definition 2.3.

Consider $m \geq 4$ and fix, and

$$n = \begin{cases} r\lfloor \frac{m}{2} \rfloor + r - 1\,, & m \text{ is odd}, \\ r\frac{m}{2} \,, & m \text{ is even}, \end{cases}$$

then, C^r_m is an r-uniform hypercycle with vertices $v_1, \cdots, v_n$ and hyperedges $e_1, \cdots, e_m$, such that
1. Each hyperedge has r consecutively-labeled vertices modulu m and in particular $e_1 = \{v_1, \cdots, v_r\}$.
2. e_i and e_j intersect if and only if i and j are consecutive modulo m.
3. If i is odd and $1 < i < m$, then $|e_i \cap e_{i-1}| = r - 1$ and $|e_i \cap e_{i+1}| = 1$.
4. If m is odd, then $|e_1 \cap e_m| = 1$. If m is even, then $|e_1 \cap e_m| = r - 1$.
We say C^r_m is odd if m is odd and even otherwise.

Definition 2.4.

Consider a graph $G = (V, E)$, the edge tenacity of a graph is defined as follow

$$T_e(G) = min\left\{\frac{|S| + \tau(G - S)}{\omega(G - S)}\right\},$$

where the minimization is over all subsets S of $E(G)$, $|S|$ is the number of elements of S, $\tau(G - S)$ is the number of edges in the largest components of $G - S$ and $\omega(G - S)$ is the number of components of $G - S$.

Now we take this definition to hypercycles.

3. Some lemmas

With due attention to the definition hypercycles, the below propositions are confirmed.

Proposition 3.1.

Let m be odd. Since $|e_1 \cap e_m| = 1$, then the vertices $v_2, \cdots, v_{r-1}$ are only in hyperedge e_1, and thus, $d(v_2) = \cdots = d(v_{r-1}) = 1$, but other vertices are in two hyperedges, so $d(v_i) = 2$ for $i \neq 2, 3, \cdots, r-1$.

Proposition 3.2.

Let m be even, so all vertices are in two hyperedges, and thus, $d(v) = 2, \forall v \in V$.

We want to apply hyperedge tenacity on hypercycles. Since there are two types of deletions for hyperedges, we have two kinds of hyperedge tenacity. In this paper, we research for hyperedge tenacity by weak deletion hyperedge, and name this tenacity weak hyperedge tenacity. As we know some vertices are just in one hyperedge. By removing this hyperedge, we have some isolated components and a component with one or more hyperedge. Let $\omega_I(H-S)$ be the number of isolated components and $\omega_C(H-S)$ be the number of component with one or more hyperedges, so $\omega(H-S) = \omega_I(H-S) + \omega_C(H-S)$. Briefly we use ω_I, ω_C, ω and τ instead of $\omega_I(H-S)$, $\omega_C(H-S)$, $\omega(H-S)$ and $\tau(H-S)$, respectively.

For each subset S we calculate the amount of $\frac{|S|+\tau(H-S)}{\omega(H-S)}$ and call the *"YIELD AMOUNT OF FRACTION (YAF)"*, made by the set S, and it will be presented by $YAF(S)$.

Proposition 3.3.

For every set S, we have

$$\omega_I = \begin{cases} r - 2 + \Sigma_{e_i, e_j \in S} |e_i \cap e_j|, & e_1 \in S \text{ and } m \text{ is odd}, \\ \Sigma_{e_i, e_j \in S} |e_i \cap e_j|, & \text{otherwise}. \end{cases}$$

Lemma 3.4.

If m is even and $S = \{e_i\}$, then $YAF(S) = m$.

Proof:

Since m is even, there is no isolation by weak deletion e_i and there is only one connected component, so $\omega_I = 0$ and $\omega_C = 1$. Also, $\tau = m - 1$, so $YAF(S) = m$. ∎

Lemma 3.5.

If m is odd and $S = \{e_i\}, i \neq 1$, then $YAF(S) = m$.

Proof:

Since $S = \{e_i\}, i \neq 1$, the proof is as we did in Lemma 3.4. ∎

Lemma 3.6.

Let m be odd and $S = \{e_1\}$, then $YAF(S) = \frac{m}{r-1}$.

Proof:

Because m is odd, so $d(v_2) = \cdots = d(v_{r-1}) = 1$. Therefore, these vertices are isolated by weak deletion e_1. Thus, $\omega_I = r-2$, $\omega_C = 1$ and we have $YAF(S) = \frac{m}{r-1}$. ∎

Lemma 3.7.

Let $S = \{e_i, e_{i+1}\}$ and $i, i+1 \notin \{1, m\}$. If i is even, then $YAF(S) = \frac{m}{r}$ and if i is odd, then $YAF(S) = \frac{m}{2}$.

Proof:

When two hyperedges e_i and e_{i+1} are deleted weakly, there is a connected component such that contains hyperedges $e_{i+2}, e_{i+3}, \cdots, e_{i-2}, e_{i-1}$; so $\omega_C = 1$. Also, the vertices in common between e_i and e_{i+1} become isolate. Thus, $\omega_I = |e_i \cap e_{i+1}|$, and we have $\tau = m-2$. If i is odd, then $\omega_I = 1$ and $YAF(S) = \frac{m}{2}$, and if i is even, so $\omega_I = r-1$ and $YAF(S) = \frac{m}{r}$. ∎

Corollary 3.8.

Let $S_1 = \{e_i, e_{i+1}\}$ that i be odd and $S_2 = \{e_j, e_{j+1}\}$ that j be even, then $YAF(S_2) \leq YAF(S_1)$.

Proof:

By Lemma 4 and as $r \geq 3$, we have $\frac{m}{r} \leq \frac{m}{2}$ and so $YAF(S_2) \leq YAF(S_1)$. ∎

Lemma 3.9.

Consider $S = \{e_i, e_{i+1}, e_{i+2}\}$ that $1, m \notin \{i, i+1, i+2\}$, then $YAF(S) = \frac{m}{1+r}$, $(m > 3)$.

Proof:

Whether i is odd or even, we have $\omega_C = 1$, $\omega_I = r$ and also $\tau = m - |S| = m-3$, therefore, $YAF(S) = \frac{m}{1+r}$. ∎

Lemma 3.10.

If $S = \{e_i, e_{i+1}, e_{i+2}, e_{i+3}\}$ and $m > 4$, then

$$YAF(S) = \begin{cases} \frac{m}{r+2}, & i \text{ is odd,} \\ \frac{m}{2r}, & i \text{ is even.} \end{cases}$$

Proof:

We know $\tau = m - 4$. If i is odd, then $\omega_I = 1 + r$ and $\omega_C = 1$, and if i is even, then $\omega_I = 2r - 1$ and $\omega_C = 1$. Thus, the proof is complete. ■

Corollary 3.11.

Let $S_1 = \{e_i, e_{i+1}, e_{i+2}, e_{i+3}\}$ that i be odd and $S_2 = \{e_j, e_{j+1}, e_{j+2}, e_{j+3}\}$ that j be even, then $YAF(S_2) \leq YAF(S_1)$.

Proof:

Because $r \geq 3$, so $2r \geq 2 + r$ and then $\frac{m}{2r} \leq \frac{m}{r+2}$. ■

Lemma 3.12.

If m is even and $S = \{e_1, e_3, e_5, \cdots, e_{m-1}\}$, then $YAF(S) = \frac{m+2}{m}$.

Proof:

Since the hyperedges in S don't have intersection, so $\omega_I = 0$. Also, by deleting S, hyperedges $e_2, e_4, \cdots, e_m$ remained, thus, $\omega_C = |\{e_2, e_4, \cdots, e_m\}| = \frac{m}{2}$, and $|S| = |\{e_1, e_3, \cdots, e_{m-1}\}| = \frac{m}{2}$. Also, every component has one hyperedge, so $\tau = 1$. By fixing them, we have $YAF(S) = \frac{m+2}{m}$. ■

Lemma 3.13.

Let m be even and $S = \{e_2, e_4, \cdots, e_m\}$. Therefore, $YAF(S) = \frac{m+2}{m}$.

Proof:

The proof is as same as Lemma 3.12. ■

Lemma 3.14.

Let m be odd and $S = \{e_1, e_3, \cdots, e_{m-2}\}$. Then $YAF(S) = \frac{m+3}{m+2r-5}$.

Proof:

We know $|S| = \frac{m-1}{2}$. The largest component of $H - S$ is two hyperedges e_{m-1} and e_m, so $\tau = 2$. Moreover, m is odd and by weak deleting e_1, the vertices $v_2, v_4, \cdots, v_{r-1}$ became isolated, so $\omega_I = r - 2 + \Sigma_{e_i, e_j \in S} |e_i \cap e_j| = r - 2$ and $\omega_C = \frac{m-1}{2}$, therefore, $YAF(S) = \frac{\frac{m-1}{2}+2}{\frac{m-1}{2}+r-2} = \frac{m+3}{m+2r-5}$. ■

Lemma 3.15.

Let m be odd and $S = \{e_1, e_3, \cdots, e_m\}$. Then $YAF(S) = \frac{m+3}{m+2r-3}$.

Proof:

m is odd so $\omega_I = r - 2 + \Sigma_{e_i, e_j \in S} |e_i \cap e_j| = r - 2 + |e_1 \cap e_m| = r - 1$ and also $\omega_C =$

$|\{e_2, e_4, \cdots, e_{m-1}\}| = \frac{m-1}{2}$, also, $\tau = 1$ and $|S| = \frac{m+1}{2}$. Then, we have $YAF(S) = \frac{\frac{m+1}{2}+1}{\frac{m-1}{2}+r-1} = \frac{m+3}{m+2r-3}$. ■

Lemma 3.16.

Let m be odd and $S = \{e_2, e_4, \cdots, e_{m-1}\}$. Then $YAF(S) = \frac{m+3}{m-1}$.

Proof:

The largest component of $C_m^r - S$ is two hyperedges e_1 and e_m, so $\tau = 2$. The hyperedges in S don't have any intersection and $e_1 \notin S$, thus, $\omega_I = 0$ and $\omega_C = \frac{m-1}{2}$. So by substituting, we have $YAF(S) = \frac{\frac{m-1}{2}+2}{\frac{m-1}{2}} = \frac{m+3}{m-1}$. ■

4. Main results

In this section we obtain the best upper bound for weak hyperedge tenacity of C_m^r by using lemmas and relationships in the previous section.

Theorem 4.1.

For every $m \geq 5$ and $r \geq 3$, we have $T_w(C_m^r) \leq \frac{m}{2r}$.

Proof:

By definition of weak hyperedge tenacity and relationships in the previous section, we have

$$T_w(C_m^r) \leq \frac{m}{r}, \;\; T_w(C_m^r) \leq \frac{m}{1+r}, \;\; T_w(C_m^r) \leq \frac{m}{2r},$$

and since $r < r+1 < 2r$, we have

$$T_w(C_m^r) \leq \frac{m}{2r}. \tag{1}$$

■

Theorem 4.2.

Let m be even. Then

$$T_w(C_m^r) \leq min\left\{\frac{m+2}{m}, \frac{m}{2r}\right\}. \tag{2}$$

Proof:

By 1 and Lemma 7 we have $T_w(C_m^r) \leq \frac{m+2}{m}$, therefore, the proof is complete. ■

Theorem 4.3.

Consider m odd, therefore,

$$T_w(C_m^r) \leq min\left\{\frac{m+3}{m+2r-3}, \frac{m}{2r}\right\}. \tag{3}$$

Proof:

By using Lemmas 9, 10 and 11 we have

$$T_w(C_m^r) \leq min\left\{\frac{m+3}{m+2r-5}, \frac{m+3}{m+2r-3}, \frac{m+3}{m-1}\right\},$$

since $m+2r-5 < m+2r-3$, so $T_w(C_m^r) \leq min\{\frac{m+3}{m+2r-3}, \frac{m+3}{m-1}\}$. Also, $r \geq 3$, then

$$2r-3 \geq 1 \Leftrightarrow m+2r-3 \geq m+1 > m-1 \Leftrightarrow \frac{m+3}{m+2r-3} < \frac{m+3}{m-1},$$

By using Theorem 4 and the definition of weak hyperedge tenacity we have $T_w(C_m^r) \leq min\{\frac{m+3}{m+2r-3}, \frac{m}{2r}\}$. ■

Because it's hard to check all the subsets of E, we stop here. With respect to the above content, we obtain an upper bound for weak hyperedge tenacity of hypercycles. In the next section we explain some examples.

5. Examples

In this section, we research upper bound for weak hyperedge tenacity of C_4^3, C_5^3 and C_6^3.

Example 5.1.

Upper bound for weak hyperedge tenacity of C_4^3. Solved. With Definition 2.3, n is 6. If we obtain

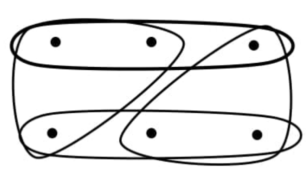

Figure 1. Hypergraph of C_4^3

all subsets of E and play definition of weak hyperedge tenacity, the amount of weak hyperedge tenacity is 1. C_4^3 is in Figure 1.

Example 5.2.

Upper bound for weak hyperedge tenacity of C_6^3.

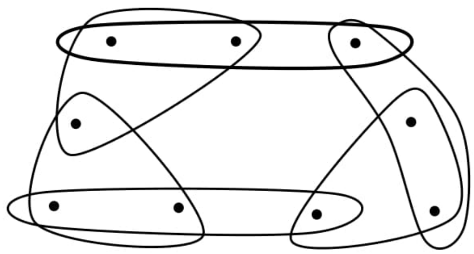

Figure 2. Hypergraph C_6^3

Solved. We know $n = 9$. By Theorem 4.2, we have

$$T_w(C_6^3) \leq min\left\{\frac{m+2}{m}, \frac{m}{2r}\right\} = min\left\{\frac{4}{3}, 1\right\} = 1.$$

But by the definition of weak hyperedge tenacity, $T_w(C_6^3) = \frac{6}{7}$, which the difference is $\frac{1}{7}$.

Example 5.3.

Upper bound for weak hyperedge tenacity of C_5^3. Solved. By Definition 2.3, n is 8 and C_5^3 is in

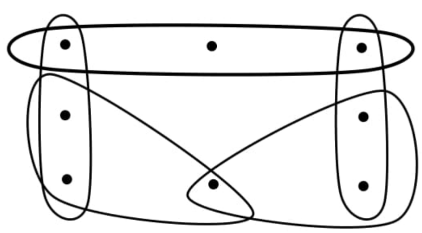

Figure 3. Hypergraph C_5^3

Figure 3. With Theorem 4.3, we have

$$T_w(C_5^3) \leq min\left\{\frac{m+3}{m+2r-3}, \frac{m}{2r}\right\} = min\left\{1, \frac{5}{6}\right\} = \frac{5}{6}.$$

Also, by the definition of weak hyperedge tenacity, $T_w(C_5^3) = \frac{5}{6}$, which is equal to the upper bound.

6. Conclusions and future work

We defined the weak hyperedge tenacity by using weak hyperedge deletion. In this paper, we obtained the upper bound of weak hyperedge tenacity for hypercycles. In some examples the value of the weak hyperedge tenacity and the upper bound of weak hyperedge tenacity are equal. These classes of hypergraphs can be found. Also, we can define strong hyperedge tenacity by using strong hyperedge deletion, and as we did in this paper, we plan to find an upper bound or exact value for it. Also, we can define strong vertex tenacity and weak vertex tenacity by using strong vertex deletion and weak vertex deletion, respectively. In the end, we think that an upper bound or exact value for strong vertex tenacity and weak vertex tenacity are going to be found, which we leave them as future works.

7. *Acknowledgment:*

The authors would like to thank the editor and the anonymous referees for their in-depth reading, criticism of, and helpful comments on an earlier version of this paper.

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