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Jones Polynomial for Graphs of Twist Knots

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Abstract

We frequently encounter knots in the flow of our daily life. Either we knot a tie or we tie a knot on our shoes. We can even see a fisherman knotting the rope of his boat. Of course, the knot as a mathematical model is not that simple. These are the reflections of knots embedded in three-dimensional space in our daily lives. In fact, the studies on knots are meant to create a complete classification of them. This has been achieved for a large number of knots today. But we cannot say that it has been terminated yet. There are various effective instruments while carrying out all these studies. One of these effective tools is graphs. Graphs are have made a great contribution to the development of algebraic topology. Along with this support, knot theory has taken an important place in low dimensional manifold topology. In 1984, Jones introduced a new polynomial for knots. The discovery of that polynomial opened a new era in knot theory. In a short time, this polynomial was defined by algebraic arguments and its combinatorial definition was made. The Jones polynomials of knot graphs and their applications were introduced by Murasugi. T. Uğur and A. Kopuzlu found an algorithm for the Jones polynomials of torus knots $K(2,q)$ in 2006. In this paper, first of all, it has been obtained signed graphs of the twist knots which are a special family of knots. We subsequently compute the Jones polynomials for graphs of twist knots. We will consider signed graphs associated with each twist knot diagrams.

Keywords: Knot; Twist knots; Graph; Knot graph; Signed graph; Spannig subgraph; Jones polynomial

MSC 2010 No.: 57M15, 57M25, 57M27

1. Introduction

A. Kawauchi mentioned that in Kawauchi (1996) "Knot theory is, in a sense, the study of how to determine whether or not two given knots are the same. In order to distinguish two knots, we find and compare a number (or more generally an algebraic system) which is invariant under auto-homeomorphisms of $\mathbb{R}^3$ (or S^3). Such a number or algebraic system is called a knot invariant." In 1984, Jones introduced a new polynomial for knots as a result of these studies. In a short time, this polynomial was defined by algebraic arguments and its combinatorial definition was made. The Jones polynomial has found areas of application in many different disciplines. For instance, American mathematical physicist Edward Witten has devised a relation between knot theory and quantum field theory using the Jones polynomial (Witten (2014)).

The Jones polynomials of knot graphs and their applications were introduced by Murasugi (Murasugi (1991)). T. Uğur and A. Kopuzlu found an algorithm for the Jones polynomials of torus knots $K(2,q)$ in 2006 (Uğur and Kopuzlu (2006)). In 2015, Dong and Jin found real and non-real zeros of Jones polynomial of graphs (Dong and Jin (2015)). In this paper, we compute the Jones polynomials for graphs of twist knots. We will consider signed graphs associated with each twist knot diagrams.

Let us take a piece of rope to our hand. After tying a knot in this rope, we will combine its ends. The result is a rope which has no free ends, beginning or ending points and it is entirely knotted. A knot is just such a knotted rope, except that we consider the rope as having no thickness, its cross-section being a single point. The knot is then a closed curve in space, which does intersect itself nowhere (Adams (1994)). More mathematically, a knot is the embedding of the circle S^1 in $\mathbb{R}^3$ (or S^3). See Adams (1994), Dye (2016), Kawauchi (1996) and Manturov (2018) for details on knots.

A two-dimensional drawing of a knot is called a diagram of the knot. In a diagram, the term crossing is used to describe a location where one portion of the knot passes over another portion of the knot (Johnson and Henrich (2017)). Knot projections were useful in the past for knot tabulation. It provides a bridge between knot theory and graph theory, as it finds use in both theories (Adams (1994)). A graph is denoted by $G=(V,E)$. It consists of a set V, of points called vertices and a set E of edges that connect them. We consider signed planar graphs that lie in the plane. A loop is an edge that (v,v) between the same vertex and a bridge is an edge whose removal disconnects two or more vertices. See Bollobas (1998) for more information about graphs.

Murasugi stated that a graph represents the geometric realization of a finite 1-dim CW-complex in $\mathbb{R}^3$ and a vertex represents a 0-simplex and an edge represents a 1-simplex in Murasugi (1991). Indicate $\beta_i(G)$ the i^{th} Betti number of G. When Poincare proved that Betti numbers were invariants, it has been accepted that they are topological objects. Formally, i^{th} Betti number is the rank of the i^{th} homology group of a topological space. See Henle (1994) for more information. In topological graph theory, the zeroth Betti number $\beta_0(G)$ of a graph G equals the number of connected components of G and the first Betti number $\beta_1(G)$ of a graph G equals $m-n+k$ such that m is the number of the edges of G, n is the number of the vertices of G and k is the number of connected components of G.

Suppose that H is a subgraph of G. $p(H)$ is the number of positive edges in H. $n(H)$ is the number of negative edges in H. Provided that H contains all vertices of G, a subgraph H is called a spanning subgraph of G. Let $S_G(r, s)$ be the set of all spanning subgraphs H of G such that $\beta_0(H) = r + 1$ and $\beta_1(H) = s$ (Murasugi (1991)). We describe

$$J_G(x, y, z) = \sum_{r,s} \Big\{ \sum_{H \in S_G(r,s)} x^{p(H)-n(H)} \Big\} y^r z^s$$

where, the second summation works on all spanning subgraphs H in $S_G(r, s)$. $J_G(x, y, z)$ is called the Jones polynomial of a graph G (Murasugi (1991)).

2. Twist Knots and Their Graphs

Definition 2.1.

A twist knot, which is denoted T_n, is gotten by twisting two parallel strands n times and subsequently hooking the ends together to be alternating knot, as seen in Figure 1 (Johnson and Henrich (2017)).

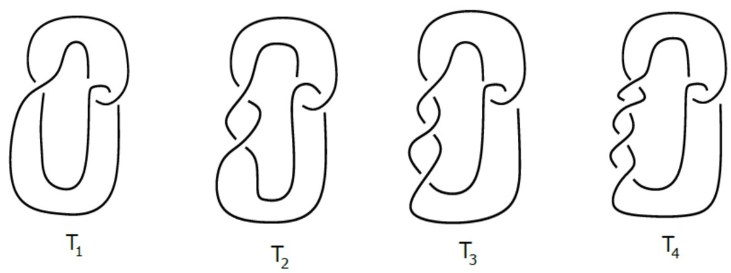

Figure 1. Some twist knots

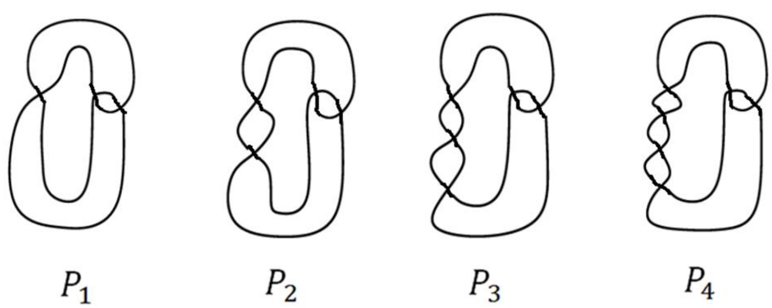

Figure 2. Some projections of twist knots

Now, we proceed in the following manner. At first, we will obtain regular projections of twist knots from their regular diagrams (see Figure 2). Then, we will shadow these projections in a checkered pattern such that the sides of an edge get different colors (see Figure 3). And then, we will get a point in the centers of each dark region. We obtain the graphs of twist knots by combining these points with the paths passing through the crossing points of the dark regions (see Figure 4 and Figure 5).

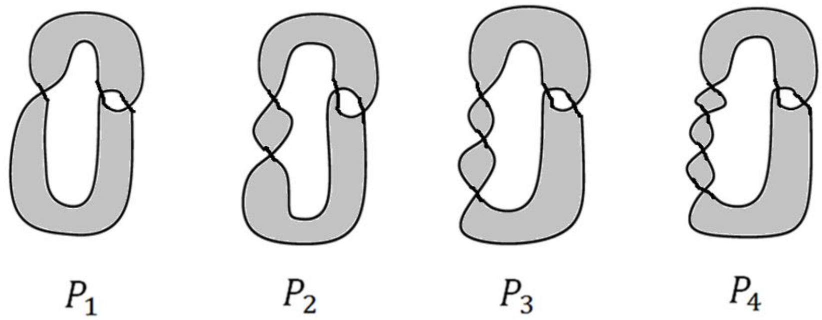

Figure 3. Some projections in a checkered pattern of twist knots

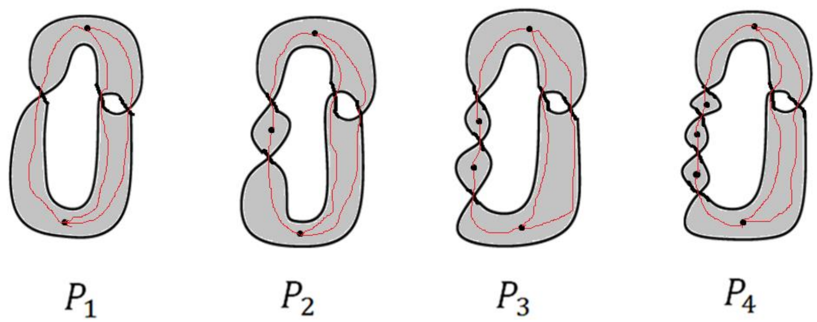

Figure 4. Obtaining some graphs of twist knots

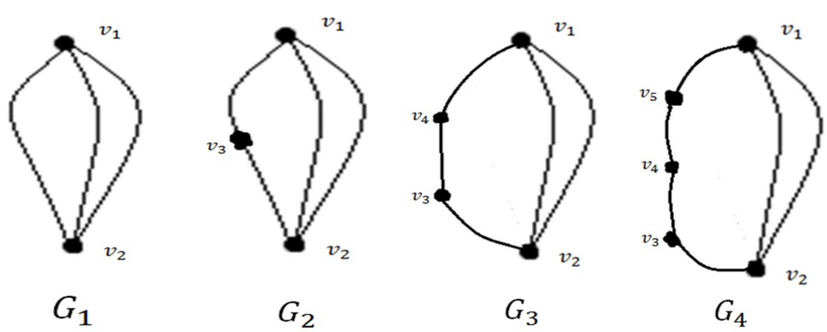

Figure 5. Some graphs of twist knots

Similarly, we can obtain graphs of twist knots T_n for all $n \in \mathbb{N}^+$. Each path into P_n corresponds to a crossing of T_n. In G_n, every paths of P_n are signed with $(+)$ or $(-)$ according to the rule shown in Figure 6 (Murasugi (1991)). We will determine the signed graphs of twist knots accordingly the rule as seen in Figure 7.

3. The Jones Polynomials for Graphs of Twist Knots

Let us compute the Jones polynomials for some graphs of twist knots by using its definition, as expressed in Section 1.

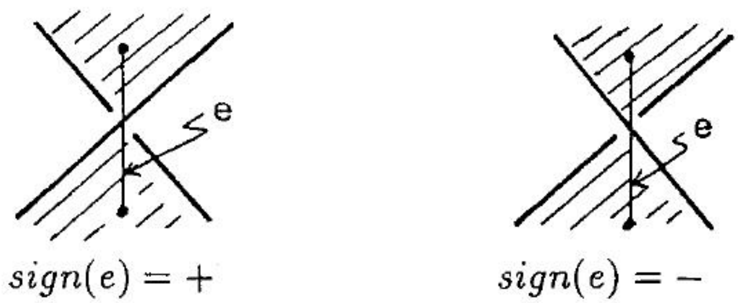

Figure 6. The rule of how to get signed graphs

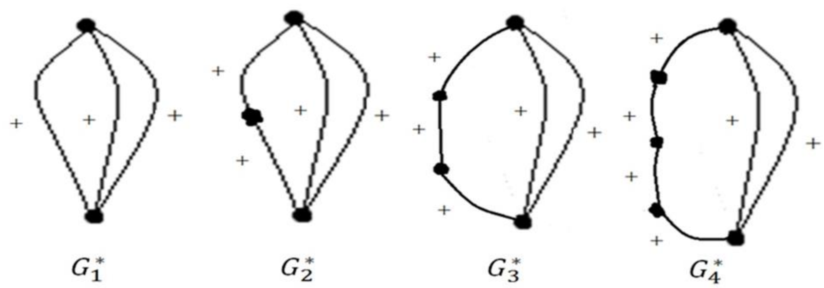

Figure 7. Some signed graphs of twist knots

The all spanning subgraphs of G_1^*:

(i) Spanning subgraphs with no edge:

Figure 8. Spanning subgraphs with no edge of G_1^*

$$r + 1 = 2 \Rightarrow r = 1 \text{ and } s = 0 - 2 + 2 \Rightarrow s = 0$$
$$\Rightarrow J(x, y, z) = y.$$

(ii) Spanning subgraphs with one edge:

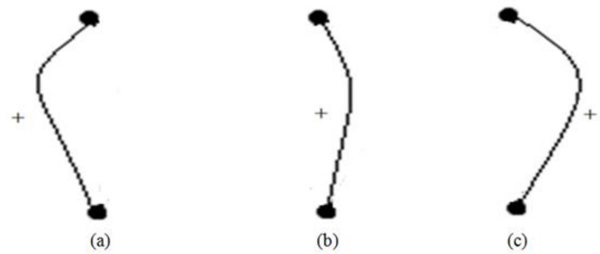

Figure 9. Spanning subgraphs with one edge of G_1^*

$$r+1=1 \Rightarrow r=0 \text{ and } s=1-2+1 \Rightarrow s=0$$
$$\Rightarrow J(x,y,z)=3x.$$

(iii) Spanning subgraphs with two edges:

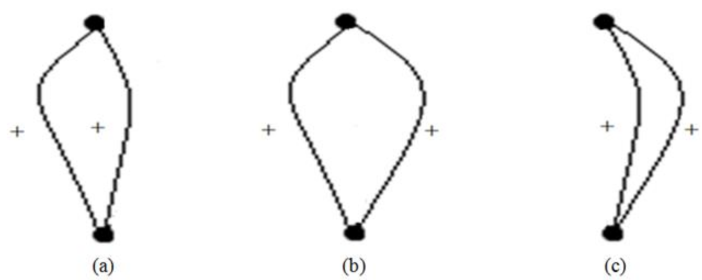

Figure 10. Spanning subgraphs with two edges of G_1^*

$$r+1=1 \Rightarrow r=0 \text{ and } s=2-2+1 \Rightarrow s=1$$
$$\Rightarrow J(x,y,z)=3x^2z.$$

(iv) Spanning subgraphs with three edges:

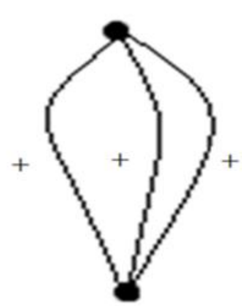

Figure 11. Spanning subgraphs with three edges of G_1^*

$$r+1=1 \Rightarrow r=0 \text{ and } s=3-2+1 \Rightarrow s=2$$
$$\Rightarrow J(x,y,z)=x^3z^2.$$

Then, we have:

$$\Rightarrow J_{G_1^*}(x,y,z)=y+3x+3x^2z+x^3z^2.$$

If we continue similar operations for other graphs G_2^* , G_3^* and G_4^* , we get the following results:

$J_{G_2^*}(x,y,z)=y^2+4xy+5x^2+x^2yz+4x^3z+x^4z^2,$
$J_{G_3^*}(x,y,z)=y^3+5xy^2+9x^2y+x^2y^2z+7x^3+3x^3yz+5x^4z+x^5z^2,$
$J_{G_4^*}(x,y,z)=y^4+6xy^3+14x^2y^2+x^2y^3z+16x^3y+4x^3y^2z+9x^4+6x^4yz+6x^5z+x^6z^2.$

Thus, we can indicate the following theorem for the Jones polynomials for graphs of twist knots.

Theorem 3.1.

For the Jones polynomials of signed graphs G_n^* , such that $n \in \mathbb{N}^+$, of twist knots, we have the following general formula:

$$\begin{aligned} J_{G_n^*}(x,y,z) = (n+2)x^{n+1}z + x^{n+2}z^2 + \Bigg(& \binom{n+2}{0}y^n + \binom{n+2}{1}xy^{n-1} \\ & +x^2\left(\left[\binom{n+2}{2}-\binom{n}{n}\right]y^{n-2}+\binom{n}{n}y^{n-1}z\right) \\ & +x^3\left(\left[\binom{n+2}{3}-\binom{n}{n-1}\right]y^{n-3}+\binom{n}{n-1}y^{n-2}z\right) \\ & +x^4\left(\left[\binom{n+2}{4}-\binom{n}{n-2}\right]y^{n-4}+\binom{n}{n-2}y^{n-3}z\right) \\ & +x^5\left(\left[\binom{n+2}{5}-\binom{n}{n-3}\right]y^{n-5}+\binom{n}{n-3}y^{n-4}z\right)+\cdots\Bigg). \end{aligned}$$

Note that in the third part of the formula, that is, in the parenthesized part, the process must be executed until the term containing x^n is formed for all $n \in \mathbb{N}^+$.

Proof:

We define the Jones polynomial $J_{G_n^*}(x,y,z)$ of graphs of twist knots by $P(n)=J_{G_n^*}(x,y,z)$. We prove the theorem by induction on n. For $n=1$, the truth of the expression $P(n)$ is obvious:

$$P(1)=J_{G_1^*}(x,y,z)=3x^2z+x^3z^2+\left(\binom{3}{0}y^1+\binom{3}{1}xy^0\right)=3x^2z+x^3z^2+y+3x.$$

Suppose that the expression $P(n)$ is true for $n = k$.

$$\begin{aligned} P(k) = J_{G_k^*}(x,y,z) = (k+2)x^{k+1}z + x^{k+2}z^2 + \Bigg(&\binom{k+2}{0} y^k + \binom{k+2}{1} xy^{k-1} \\ &+ x^2\left(\left[\binom{k+2}{2} - \binom{k}{k}\right] y^{k-2} + \binom{k}{k} y^{k-1}z\right) \\ &+ x^3\left(\left[\binom{k+2}{3} - \binom{k}{k-1}\right] y^{k-3} + \binom{k}{k-1} y^{k-2}z\right) \\ &+ \cdots + x^k\left(\left[\binom{k+2}{k} - \binom{k}{k-(k-2)}\right] y^{k-k} + \binom{k}{k-(k-2)} y^{k-(k-1)}z\right)\Bigg). \end{aligned}$$

Now, let us do a data analysis on the table for the results from $n = 1$ to $n = k$.

Table 1. Results of data analysis for $P(n)$.

	$P(1)$	$P(2)$	$P(3)$	$P(4)$	$\ldots$	$P(k)$
C-0	y	y^2	y^3	y^4	$\ldots$	$\binom{k+2}{0}y^k$
C-1	$3x$	$4xy$	$5xy^2$	$6xy^3$	$\ldots$	$\binom{k+2}{1}xy^{k-1}$
C-2	$3x^2z$	$5x^2 + x^2yz$	$9x^2y + x^2y^2z$	$14x^2y^2 + x^2y^3z$	$\ldots$	A
C-3	x^3z^2	$4x^3z$	$7x^3 + 3x^3yz$	$16x^3y + 4x^3y^2z$	$\ldots$	B
C-4		x^4z^2	$5x^4z$	$9x^4 + 6x^4yz$	$\ldots$	C
C-5			x^5z^2	$6x^5z$	$\ldots$	$\vdots$
C-6				x^6z^2	$\ldots$	$\vdots$
$\vdots$					$\ldots$	$\vdots$
C-k					$\ldots$	D
C-(k+1)					$\ldots$	$(k+2)x^{k+1}z$
C-(k+2)					$\ldots$	$x^{k+2}z^2$

In Table 1, the symbol "C-the number" means that "The case of spanning subgraphs with the number edges" and

$$A = x^2\left(\left[\binom{k+2}{2} - \binom{k}{k}\right]y^{k-2} + \binom{k}{k}y^{k-1}z\right),$$

$$B = x^3\left(\left[\binom{k+2}{3} - \binom{k}{k-1}\right]y^{k-3} + \binom{k}{k-1}y^{k-2}z\right),$$

$$C = x^4\left(\left[\binom{k+2}{4} - \binom{k}{k-2}\right]y^{k-4} + \binom{k}{k-2}y^{k-3}z\right),$$

$$D = x^k\left(\left[\binom{k+2}{k} - \binom{k}{k-(k-2)}\right]y^{k-k} + \binom{k}{k-(k-2)}y^{k-(k-1)}z\right).$$

The induction step from $n = k$ to $n = k+1$ performs application of the special case, which considers coefficient increments and increments of the powers of variables in each of spanning subgraphs as indicated in Table 1. It is obvious that the signed graph corresponding the twist knot T_{n+1} has $k+2$ edges that is, the graph G^*_{k+1} has only one excess edge from the graph G^*_k. Thus, the desired result is easily reached as in the case of transition from $n = 1$ to $n = 2$ or in the case of transition from $n = 2$ to $n = 3$ or in the case of transition from $n = 3$ to $n = 4$. This completes the proof. ■

4. Conclusion

Twist knots are a family of knots that have been in-depth studied in previous years. In these studies, topological properties of twist knots were investigated, like Hoste and Shanahan (2001). In addition, various polynomial constants of twist knots were calculated, like Nawata et al. (2012) and Ham and Lee (2016). We have set out with the question of how we can conduct a study on graphs of twist knots that are such an important knot family. Thus, we have firstly obtained graphs of twist knots. Then, we have created signed graphs of twist knots in accordance with a certain rule, provided that knot remained true to the original conditions in three-dimensional space. We have calculated Jones polynomials of these obtained graphs and given a generalized formula for Jones polynomials for graphs of twist knots. In future studies, coloured Jones polynomials and different polynomials of these graphs can be examined. Clique number and chromatic number of these graphs can be examined. In addition, by obtaining adjacency matrices and incident matrices of the graphs of the twist knots their algebraic properties can be examined.

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