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Boundedness and Square Integrability in Neutral Differential Systems of Fourth Order

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Abstract

The aim of this paper is to study the asymptotic behavior of solutions to a class of fourth-order neutral differential equations. We discuss the stability, boundedness and square integrability of solutions for the considered system. The technique of proofs involves defining an appropriate Lyapunov functional. Our results obtained in this work improve and extend some existing well-known related results in the relevant literature which were obtained for nonlinear differential equations of fourth order with a constant delay. The obtained results here are new even when our equation is specialized to the forms previously studied and include many recent results in the literature. Finally, an example is given to show the feasibility of our results.

Keywords: Lyapunov functional; Neutral differential equations of fourth order; Uniform asymptotic stability; Square integrability

MSC 2010 No.: 34C11, 34D20, 34D23

1. Introduction

The study of qualitative properties of differential equations has a long history, and qualitative theories have been developed for various kinds of ordinary differential equations. It is well known that the qualitative analysis of differential equations is related to both pure and applied mathematics. Its applications to various fields such as science, engineering, and ecology have been extensively developed.

Applications of neutral differential equations include electrodynamics, control systems, neutron transportation, mixing liquids and population models and many other fields in real life. Asymptotic behavior and stability of solutions of such systems play an important role when one studies qualitative properties of those systems.

In literature we find some results concerning second order differential equations of neutral type (Guiling et al. (2014)), but in the case of third and fourth order of neutral type there are very few results. While for the delay differential equations, the literature of third and fourth order is full of results on qualitative properties (boundedness, stability, square integrability) (see Abou-El-Ela et al. (2009), Bereketoglu (1998), Remili and Beldjerd (2017), Remili and Oudjedi (2016), Greaf et al. (2015), Kang et al. (2010), Rahmane and Remili (2015), Remili and Beldjerd (2016), Sadek (2004), Sinha (1973), Tejumola and Tchegnani(2000), Tunç (2010)). Some of the previous results inspire us to study.

2. Assumptions and main results

In this article, we develop the conditions under which all the solutions of the following equation are stable, bounded and square integrable:

$$\begin{aligned}&\left(q(t)\left(x'''\left(t\right)+\rho x'''\left(t-r\right)\right)\right)'+a\left(t\right)x'''(t)+b\left(t\right)x''(t)+c\left(t\right)x'(t)\\&\quad+d\left(t\right)h\left(x\left(t\right)\right)=p(t,x(t),x'(t),x''(t),x'''(t)),\end{aligned} \tag{1}$$

where ρ and r are positive constants to be determined later and $a(.), b(.), c(.), d(.), q(.)$ and $h(x)$ are continuous functions depending only on the arguments shown and $h'(x)$ exists and is continuous.

For the sake of convenience we introduce the following notation,

$$X(t) = x(t) + \rho x(t-r).$$

By a solution of (1) we mean a continuous function $x : [t_x, \infty) \to \mathbb{R}$ such that $X(t) \in C^3([t_x,\infty),\mathbb{R})$ and which satisfies Equation (1) on $[t_x,\infty)$. Without further mention, we will assume throughout that every solution $x(t)$ of (1) under consideration here is continuable to the right and nontrivial, i.e, $x(t)$ is defined on some ray $[t_x,\infty)$. Moreover, we assume that (1) possesses such solutions.

Suppose that there exist positive constants $a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, h_0, q_0, q_1, \delta$ and δ_0 such that the following conditions hold,

i) $0 < a_0 \leq a(t) \leq a_1, \quad 0 < b_0 \leq b(t) \leq b_1, \quad 0 < c_0 \leq c(t) \leq c_1, \quad 0 < d_0 \leq d(t) \leq d_1,$
$0 < q_0 \leq q(t) \leq q_1 < 1$ and $d'(t) \leq 0$ for $t \geq 0$.

ii) $h(0) = 0\,, \ \dfrac{h(x)}{x} \geq \delta > 0$ for $x \neq 0$.

iii) $h_0 - \dfrac{a_0\delta_0}{d_1} \leq h'(x) \leq \dfrac{h_0}{2}$ for $x \in \mathbb{R}$.

iv) $b_0 > \dfrac{c_1}{a_0} + \dfrac{a_1 h_0 d_1}{c_0} + \dfrac{\delta_0}{a_0} = \kappa.$

The following lemma will be useful in the proof of the next theorem.

Lemma 2.1. (Hara (1974))

Let $h(0) = 0,\ xh(x) > 0\ (x \neq 0)$ and $\delta(t) - h'(x) \geq 0\ (\delta(t) > 0)$. Then,

$$2\delta(t)H(x) \geq h^2(x)\,, \qquad \text{where} \quad H(x) = \int_0^x h(s)ds.$$

The main objective of this paper is to prove the following theorem.

Theorem 2.2.

Further to assumptions (i)-(iv), assume that there are positive constants η_1 and η_2 such that the following conditions are satisfied

H1) $\int_0^{+\infty} \left(|a'(t)| + |b'(t)| + |c'(t)| + |q'(t)| - d'(t)\right) dt < \eta_1.$

H2) $|p(t, x, x', x'', x''')| \leq |e(t)|$ and $\displaystyle\int_0^{+\infty} |e(t)|\, dt < \eta_2.$

Then, there exists a finite positive constant K_0 such that every solution $x(.)$ of (1) and their derivatives $x'(.), x''(.), x'''(.)$ and $X'''(.)$ satisfy

1. $|x(t)| \leq K_0, |\, x'(t)| \leq K_0,\ |x''(t)| \leq K_0,\ |X'''(t)| \leq K_0,$ for all $t \geq 0$,

2. $\displaystyle\int_0^{\infty} \left(x^2(s) + x'^2(s) + x''^2(s) + x'''^2(s)\right) ds < \infty,$

provided that

$$\rho < \min\Big\{1,\ \frac{2\varepsilon}{\alpha h_0},\ \frac{2\varepsilon c_0}{\alpha c_1 + \alpha d_1 \lambda_0},\ 2\frac{b_0 - \kappa - \varepsilon\,(a_1 + c_1)}{\alpha b_1 + \beta + \alpha d_1 \lambda_0 + \alpha d_1},\ \frac{2\varepsilon a_0}{\alpha(2a_1 + b_1 + c_1 + d_1) + 5 + \beta}\Big\},$$

where

$$\alpha = \frac{1}{a_0} + \varepsilon\ ,\ \ \beta = \frac{d_1 h_0}{c_0} + \varepsilon \quad \text{and} \quad \varepsilon < \min\left\{\frac{1}{a_0}\ ,\ \frac{d_1 h_0}{c_0}\ ,\ \frac{b_0 - \kappa}{a_1 + c_1}\right\}. \tag{2}$$

Proof:

We can write Equation (1) in the differential system form as

$$\begin{aligned} &x' = y,\\ &y' = z,\\ &z' = w,\\ &W' = \frac{1}{q(t)}\big[-a\,(t)\,w + -b\,(t)\,z - c\,(t)\,y - d\,(t)\,h\,(x) + p(t,x,y,z,w) - q'(t)W\big]. \end{aligned} \tag{3}$$

System (3) is obtained from Equation (1) by setting

$$X'(t) = x'(t) + \rho x'(t-r) = y(t) + \rho y(t-r) = Y(t),$$

$$X''(t) = x''(t) + \rho x''(t-r) = z(t) + \rho z(t-r) = Z(t),$$

$$X'''(t) = x'''(t) + \rho x'''(t-r) = w(t) + \rho w(t-r) = W(t).$$

We define a functional $U = U(t,x,y,z,w)$ given by

$$U = e^{-\frac{1}{\eta}\int_0^t \gamma\,(s)\,ds} V, \tag{4}$$

where

$$\gamma\,(t) = |a'\,(t)| + |b'\,(t)| + |c'\,(t)| + |q'\,(t)| - d'\,(t)\,,$$

the function $V = V(t,x,y,z,w)$ defined by

$$\begin{aligned} 2V &= a\,(t)\,z^2 + 2\beta a\,(t)\,yz + 2\beta q(t)yW + 2q(t)zW + 2\alpha c\,(t)\,yz + c\,(t)\,y^2\\ &\quad + 2d\,(t)\,h\,(x)\,y + 2\alpha d\,(t)\,h\,(x)\,Z + \big[\beta b\,(t) - \alpha h_0 d\,(t)\big]y^2 - \beta q(t)z^2 + \alpha q(t)W^2\\ &\quad + \alpha\rho d\,(t)\,(z\,(t-r))^2 + 2\beta d\,(t)\,H\,(x) + \alpha b\,(t)\,z^2\\ &\quad + \mu_1 \int_{t-r}^t z^2\,(s)\,ds + \mu_2 \int_{t-r}^t w^2\,(s)\,ds, \end{aligned}$$

and η is a positive constant, which will be determined later in the proof. By adding and subtracting some terms we can rewrite $2V$ as

$$2V = V_1 + V_2 + V_3 + V_4 + a(t)\left[\frac{q(t)W}{a(t)} + z + \beta y\right]^2 + c(t)\left[\frac{d(t)h(x)}{c(t)} + y + \alpha z\right]^2$$
$$+ \frac{d^2(t)h^2(x)}{c(t)} + \mu_1\int_{t-r}^{t} z^2(s)\,ds + \mu_2\int_{t-r}^{t} w^2(s)\,ds,$$

where

$$V_1 = 2d(t)\int_0^x h(s)\left[\frac{d_1h_0}{c_0} - 2\frac{d(t)}{c(t)}h'(s)\right]ds,$$
$$V_2 = \left[\alpha b(t) - \beta q(t) - \alpha^2 c(t)\right]z^2,$$
$$V_3 = \left[\beta b(t) - \alpha h_0 d(t) - \beta^2 a(t)\right]y^2 + \left[\frac{\alpha}{q(t)} - \frac{1}{a(t)}\right]q^2(t)W^2,$$
$$V_4 = 2\varepsilon d(t)H(x) + 2\alpha\rho d(t)h(x)z(t-r) + \alpha\rho d(t)\left(z(t-r)\right)^2.$$

To prove that V is positive definite it suffices to show that V_1, V_2, V_3 and V_4 are positives. Remark that the estimate (2) implies

$$\frac{1}{a_0} < \alpha < 2\frac{1}{a_0} \text{ and } \frac{d_1h_0}{c_0} < \beta < 2\frac{d_1h_0}{c_0}. \tag{5}$$

Then, using conditions (i) through (iv), and inequalities (2) and (5) we obtain the following,

$$V_1 \geq 2d(t)\int_0^x h(s)\frac{d_1}{c_0}\left[h_0 - 2h'(s)\right]ds$$
$$\geq 4\frac{d_0d_1}{c_0}\int_0^x h(s)\left[\frac{h_0}{2} - h'(s)\right]ds \geq 0.$$

Rearranging V_2 we obtain the estimate

$$V_2 = \alpha\Big[b(t) - \beta a(t) - \alpha c(t)\Big]z^2 + \beta\Big[\alpha a(t) - q(t)\Big]z^2$$
$$\geq \alpha\Big[b(t) - \big(\frac{d_1h_0}{c_0} + \varepsilon\big)a(t) - \big(\frac{1}{a_0} + \varepsilon\big)c(t)\Big]z^2 + \beta\left[\frac{a(t)}{a_0} - 1\right]z^2$$
$$\geq \alpha\Big[b_0 - \frac{a_1d_1h_0}{c_0} - \frac{c_1}{a_0} - \varepsilon\big(a_1 + c_1\big)\Big]z^2$$
$$\geq \alpha\Big[b_0 - \kappa - \varepsilon\left(a_1 + c_1\right)\Big]z^2 \geq 0.$$

We also have,

$$V_3 \geq \beta\left(b_0 - \frac{\alpha}{\beta}h_0d_1 - \beta a_1\right)y^2 + \left(\alpha - \frac{1}{a_0}\right)q_0^2W^2$$
$$\geq \beta\left(b_0 - \frac{c_0}{a_0} - a_1\frac{d_1h_0}{c_0} - \varepsilon(c_0 + a_1)\right)y^2 + \varepsilon q_0^2W^2$$
$$\geq \beta\big(b_0 - \kappa - \varepsilon(c_1 + a_1)\big)y^2 + \varepsilon q_0^2W^2 \geq 0,$$

and by the estimate of ρ, we have

$$
\begin{aligned}
V_4 &= 2\varepsilon d(t)\int_0^x h(\xi)d\xi + \alpha\rho d(t)\left[(z(t-r)+h(x))^2 - h^2(x)\right] \\
&\geq 2\varepsilon d(t)\int_0^x h(\xi)d\xi - 2\alpha\rho d(t)\int_0^x h'(\xi)h(\xi)d\xi \\
&\geq 2d(t)\int_0^x \left(\varepsilon - \frac{\alpha\rho h_0}{2}\right)h(\xi)d\xi \\
&\geq 2d_0\left(\varepsilon - \frac{\alpha\rho h_0}{2}\right)H(x).
\end{aligned}
$$

Thus, there exists a positive number D_0 such that

$$2V \geq D_0\left(y^2+z^2+W^2+H(x)\right).$$

By Lemma 2.1 and condition iii) we conclude that there exists a positive number D_1 such that

$$2V \geq D_1\left(x^2+y^2+z^2+W^2\right). \tag{6}$$

Thus, V is positive definite. Then, we can find positive definite functions $U_1(\|\xi\|)$ and $U_2(\|\xi\|)$ such that $U_1(\|\xi\|) \leq V \leq U_2(\|\xi\|)$. By (4) and inequality (6), we get

$$U \geq D_2(x^2+y^2+z^2+W^2), \tag{7}$$

where $D_2 = \frac{D_1}{2}e^{-\frac{\eta_1}{\eta}}$. Therefore, by conditions H1 and H2 we can find positive definite functions $W_1(\|\xi\|)$ and $W_2(\|\xi\|)$ such that $W_1(\|\xi\|) \leq U \leq W_2(\|\xi\|)$.

Now we prove that $\dot{U}$ is a negative definite function using the following derivative,

$$\frac{d}{dt}\big(\alpha q(t)W^2(t)\big) = -\alpha q'^2 + 2\alpha W(t)\frac{d}{dt}\big(q(t)W(t)\big).$$

Calculating the time derivative of the function V, along any solution $(x(t), y(t), z(t), w(t))$ of system (3), we have

$$2\dot{V}_{(3)} = V_5+V_6+V_7+V_8+V_9+2(\beta y+z+\alpha W)p(t,x,y,z,w),$$

where

$$
\begin{aligned}
V_5 &= -2\left(\frac{d_1h_0}{c_0}c(t) - d(t)h'(x)\right)y^2 - 2\alpha d(t)\big(h_0 - h'(x)\big)yz, \\
V_6 &= -2\big(b(t) - \alpha c(t) - \beta a(t)\big)z^2, \\
V_7 &= -2\big(\alpha a(t) - q(t)\big)w^2, \\
V_8 &= -2\varepsilon c(t)y^2 - 2\alpha\rho a(t)w_tw - 2\alpha\rho b(t)zw_t - 2\alpha\rho c(t)yw_t + 2\alpha\rho d(t)h'(x)yz_t \\
&\quad + \mu_1z^2 + \mu_2w^2 - \mu_1z_t^2 - \mu_2w_t^2 + 2\alpha\rho d(t)z_tw_t + 2\rho q(t)ww_t + 2\beta\rho q(t)zw_t,
\end{aligned}
$$

and

$$
\begin{aligned}
V_9 = {} & d'(t)\left[2\beta H(x) - \alpha h_0y^2 + 2h(x)y + 2\alpha h(x)z\right] + c'(t)\left[y^2+2\alpha yz\right] \\
&+ b'(t)\left[\alpha z^2+\beta y^2\right] + a'(t)\left[z^2+2\beta yz\right] - \alpha q'(t)W^2 - \beta q'(t)z^2 \\
&+ \alpha\rho d'(t)\left[z(t-r)+h(x)\right]^2 - \alpha\rho d'(t)h^2(x).
\end{aligned}
$$

Again using conditions i), iii), iv), and inequalities (2) and (5) we get

$$\begin{aligned} V_5 &\leq -2\left[d\left(t\right)h_0 - d\left(t\right)h'\left(x\right)\right]y^2 - 2\alpha d\left(t\right)\left[h_0 - h'\left(x\right)\right]yz \\ &\leq -2d\left(t\right)\left[h_0 - h'\left(x\right)\right]y^2 - 2\alpha d\left(t\right)\left[h_0 - h'\left(x\right)\right]yz \\ &\leq -2d\left(t\right)\left[h_0 - h'\left(x\right)\right]\left[\left(y + \frac{\alpha}{2}z\right)^2 - \left(\frac{\alpha}{2}z\right)^2\right] \\ &\leq \frac{\alpha^2}{2}d\left(t\right)\left[h_0 - h'\left(x\right)\right]z^2. \end{aligned}$$

Therefore,

$$\begin{aligned} V_5 + V_6 &\leq -2\left[b\left(t\right) - \alpha c\left(t\right) - \beta a\left(t\right) - \frac{\alpha^2}{4}d\left(t\right)\left[h_0 - h'\left(x\right)\right]\right]z^2 \\ &\leq -2\left[b_0 - \left(\frac{1}{a_0} + \varepsilon\right)c_1 - \left(\frac{d_1 h_0}{c_0} + \varepsilon\right)a_1 - \frac{\alpha^2}{4}\left(a_0\delta_0\right)\right]z^2 \\ &\leq -2\left[b_0 - \frac{c_1}{a_0} - \frac{d_1 h_0 a_1}{c_0} - \frac{\delta_0}{a_0} - \varepsilon\left(a_1 + c_1\right)\right]z^2 \\ &\leq -2\left[b_0 - \kappa - \varepsilon\left(a_1 + c_1\right)\right]z^2 \leq 0, \end{aligned}$$

$$V_7 \leq -2\left[\alpha a_0 - 1\right]w^2 = -2\varepsilon a_0 w^2 \leq 0,$$

and

$$\begin{aligned} V_8 &\leq -2\varepsilon c\left(t\right)y^2 + \alpha\rho a_1 w_t^2 + \alpha\rho a_1 w^2 + \alpha\rho b_1 z^2 + \alpha\rho b_1 w_t^2 + \alpha\rho c_1 y^2 \\ &\quad + \alpha\rho c_1 w_t^2 + \alpha\rho d_1\lambda_0 y^2 + \alpha\rho d_1\lambda_0 z_t^2 + \mu_1 z^2 + \mu_2 w^2 - \mu_1 z_t^2 - \mu_2 w_t^2 \\ &\quad + \alpha\rho d_1 z_t^2 + \alpha\rho d_1 w_t^2 + 2\rho w^2 + \beta\rho z^2 + 2\rho w_t^2 + \beta\rho w_t^2 - 2\rho|ww_t| + (\rho - \rho^2)w_t^2 \\ &\leq -\left(2\varepsilon c_0 - \alpha\rho c_1 - \alpha\rho d_1\lambda_0\right)y^2 + \left(\alpha\rho b_1 + \beta\rho + \mu_1\right)z^2 + \left(\alpha\rho a_1 + 2\rho + \mu_2\right)w^2 \\ &\quad + \left(\alpha\rho d_1\lambda_0 + \alpha\rho d_1 - \mu_1\right)z_t^2 + \left(\alpha\rho a_1 + \alpha\rho b_1 + \alpha\rho c_1 + \alpha\rho d_1 + \beta\rho + 3\rho - \mu_2\right)w_t^2 \\ &\quad - \rho^2 w_t^2 - 2\rho|ww_t|, \end{aligned}$$

where

$$\lambda_0 = \max\left\{\frac{h_0}{2},\ \left|h_0 - \frac{a_0\delta_0}{d_1}\right|\right\}.$$

By taking

$$\begin{cases} \mu_1 = \alpha\rho d_1\lambda_0 + \alpha\rho d_1, \\ \mu_2 = \alpha\rho a_1 + \alpha\rho b_1 + \alpha\rho c_1 + \alpha\rho d_1 + \beta\rho + 3\rho, \end{cases}$$

we obtain

$$\begin{aligned} V_8 &\leq -\left(2\varepsilon c_0 - \alpha\rho c_1 - \alpha\rho d_1\lambda_0\right)y^2 + \left(\alpha\rho b_1 + \beta\rho + \mu_1\right)z^2 + \left(\alpha\rho a_1 + 2\rho + \mu_2\right)w^2 \\ &\quad - \rho^2 w_t^2 - 2\rho|ww_t|. \end{aligned}$$

Then, we have

$$
\begin{aligned}
V_5+V_6+V_7+V_8 \leq & -\rho^2 w_t^2-2\rho|ww_t|-\left(2\varepsilon c_0-\alpha\rho c_1-\alpha\rho d_1\lambda_0\right) y^2 \\
& -2\left[b_0-\kappa-\varepsilon\left(a_1+c_1\right)-\frac{1}{2}\rho\left(\alpha b_1+\beta+\alpha d_1\lambda_0+\alpha d_1\right)\right] z^2 \\
& -\Bigg(2\varepsilon a_0-\rho\left(2\alpha a_1+5+\alpha b_1+\alpha c_1+\alpha d_1+\beta\right)\Bigg) w^2 .
\end{aligned}
$$

Hence, there exists a positive constant D_3 such that,

$$
\begin{aligned}
V_5+V_6+V_7+V_8 & \leq -2D_3\left(y^2+z^2+w^2+\rho^2 w_t^2+2\rho|ww_t|\right) \\
& \leq -2D_3\left(y^2+z^2+W^2\right),
\end{aligned}
$$

provided that

$$
\rho<\min\left\{1,\ \frac{2\varepsilon}{\alpha h_0},\ \frac{2\varepsilon c_0}{\alpha c_1+\alpha d_1\lambda_0},\ 2\frac{b_0-\kappa-\varepsilon\left(a_1+c_1\right)}{\alpha b_1+\beta+\alpha d_1\lambda_0+\alpha d_1}, \frac{2\varepsilon a_0}{\alpha(2a_1+b_1+c_1+d_1)+5+\beta}\right\}.
$$

Using condition iii) and Lemma 2.1, we obtain

$$
h^2(x) \leq h_0 H(x),
$$

and consequently

$$
\begin{aligned}
|V_9| \leq & -d'(t)\left[2\beta H(x)+\alpha h_0 y^2+\left(h^2(x)+y^2\right)+\alpha\left(h^2(x)+z^2\right)+\alpha\rho h^2(x)\right] \\
& +|c'(t)|\left[y^2+\alpha\left(y^2+z^2\right)\right]+|b'(t)|\left[\alpha z^2+\beta y^2\right]-\alpha q'(t)W^2-\beta q'(t)z^2 \\
& +|a'(t)|\left[z^2+\beta\left(y^2+z^2\right)\right] \\
\leq & \lambda_2\left[|a'(t)|+|b'(t)|+|c'(t)|+|q'(t)|-d'(t)\right]\left(y^2+z^2+W^2+H(x)\right) \\
\leq & 2\frac{\lambda_2}{D_0}\left[|a'(t)|+|b'(t)|+|c'(t)|+|q'(t)|-d'(t)\right] V,
\end{aligned}
$$

such that $\lambda_2=\max\left\{2\beta+(\alpha\rho+\alpha+1)h_0, \alpha h_0+\alpha+2\beta+2, 1+2\beta+3\alpha\right\}$.

By taking $\frac{1}{\eta}=\frac{1}{D_0}\lambda_2$, we obtain

$$
\begin{aligned}
\dot{V}_{(3)} \leq & -D_3(y^2+z^2+W^2)+\frac{1}{\eta}\Big(|a'(t)|+|b'(t)|+|c'(t)|+|q'(t)|-d'(t)\Big)V \\
& +\left(\beta y+z+\alpha W\right)p(t,x,y,z,w). \qquad (8)
\end{aligned}
$$

From H2, (7), (8) and the Cauchy-Schwartz inequality, we get

$$\begin{aligned}\dot{U}_{(3)} &= \left(\dot{V}_{(3)} - \frac{1}{\eta}\gamma\,(t)\,V\right)G(t)\\ &\le \Big(-D_3\left(y^2+z^2+W^2\right) + \big(\beta y + z + \alpha W\big)p(t,x,y,z,w)\Big)G(t)\\ &\le \left(\beta|y|+|z|+\alpha|W|\right)|p(t,x,y,z,w)|\\ &\le D_4\left(|y|+|z|+|W|\right)|e(t)|\\ &\le D_4\left(3+y^2+z^2+W^2\right)|e(t)|\\ &\le 3D_4|e(t)| + \frac{D_4}{D_2}U|e(t)|, \end{aligned} \tag{9}$$

where $G(t) = e^{-\frac{1}{\eta}\int_0^t \gamma\,(s)\,ds}$ and $D_4 = \max\{\alpha,\beta,1\}$. Integrating (9) from 0 to t, and using the condition H2 and the Gronwall-Reid-Bellman inequality, we obtain

$$\begin{aligned}U(t,x,y,z,W) &\le U\big(0,x(0),y(0),z(0),W(0)\big) + 3D_4\eta_2\\ &\quad + \frac{D_4}{D_2}\int_0^t U\big(s,x(s),y(s),z(s),W(s)\big)|e(s)|ds\\ &\le \Big(U\big(0,x(0),y(0),z(0),W(0)\big) + 3D_4\eta_2\Big)e^{\frac{D_4}{D_2}\int_0^t |e(s)|ds}\\ &\le \Big(U\big(0,x(0),y(0),z(0),W(0)\big) + 3D_4\eta_2\Big)e^{\frac{D_4}{D_2}\eta_2} = K_1 < \infty. \end{aligned} \tag{10}$$

In view of inequalities (7) and (10), we get

$$(x^2+y^2+z^2+W^2) \le \frac{1}{D_2}U \le K_0^2, \tag{11}$$

where $K_0^2 = \dfrac{K_1}{D_2}$. Clearly (11) implies that

$$|x(t)| \le K_0,\ |y(t)| \le K_0,\ |z(t)| \le K_0,\ |W(t)| \le K_0 \quad \text{for all} \quad t \ge 0.$$

Hence,

$$|x(t)| \le K_0,\ |x'(t)| \le K_0,\ |x''(t)| \le K_0,\ |X'''(t)| \le K_0 \quad \text{for all} \quad t \ge 0. \tag{12}$$

Now, we prove the square integrability of solutions and their derivatives of Equation (1).

First, from (8) we obtain

$$\dot{V}_{(3)} \le -D_3(y^2+z^2+w^2) + \frac{1}{\eta}\gamma\,(t)\,V + 2\big(\beta y + z + \alpha W\big)p(t,x,y,z,w),$$

thus,

$$\begin{aligned}\dot{U}_{(3)} &= \left(\dot{V}_{(3)} - \frac{1}{\eta}\gamma\,(t)\,V\right)G(t)\\ &\le \Big(-D_3\left(y^2+z^2+w^2\right) + \big(\beta y + z + \alpha W\big)p(t,x,y,z,w)\Big)G(t). \end{aligned} \tag{13}$$

Now, we define $F_t = F(t, x(t), y(t), z(t), w(t))$ as

$$F_t = U + \sigma \int_0^t \left(y^2(s) + z^2(s) + w^2(s)\right) ds,$$

where $\sigma > 0$. It is easy to see that F_t is positive definite, since $U = U(t, x, y, z, w)$ is already positive definite. Using the following estimate $e^{-\frac{\eta_1}{\eta}} \leq G(t) \leq 1$ by H1 and (13) imply

$$\begin{aligned}\dot{F}_{t(3)} \leq & -D_3\Big(y^2(t) + z^2(t) + w^2(t)\Big)e^{-\frac{\eta_1}{\eta}} + D_4\Big(|y(t)| + |z(t)| + |W(t)|\Big)|p(t, x, y, z, w)| \\ & +\sigma\Big(y^2(t) + z^2(t) + w^2(t)\Big),\end{aligned}$$

where D_4 is positive constant. By choosing $\sigma = D_3 e^{-\frac{\eta_1}{\eta}}$ we obtain

$$\begin{aligned}\dot{F}_{t(3)} &\leq D_4\Big(3 + y^2(t) + z^2(t) + W^2(t)\Big)|e(t)| \\ &\leq D_4\Big(3 + \frac{1}{D_2}U\Big)|e(t)| \\ &\leq 3D_4|e(t)| + \frac{D_4}{D_2}F_t|e(t)|. \end{aligned} \tag{14}$$

Integrating the last inequality (14) from 0 to t, and using again the Gronwall-Reid-Bellman inequality and the condition H2, we get

$$\begin{aligned}F_t &\leq F_0 + 3D_4\eta_2 + \frac{D_4}{D_2}\int_0^t F_s|e(s)|ds \\ &\leq \Big(F_0 + 3D_4\eta_2\Big)e^{\frac{D_4}{D_2}\int_0^t |e(s)|ds} \\ &\leq \Big(F_0 + 3D_4\eta_2\Big)e^{\frac{D_4}{D_2}\eta_2} = K_2 < \infty.\end{aligned}$$

Therefore,

$$\int_0^\infty y^2(s)ds < K_2 \;\; , \;\; \int_0^\infty z^2(s) < K_2 \text{ and } \int_0^\infty w^2(s)ds < K_2,$$

which implies that

$$\int_0^\infty x'^2(s)ds \leq K_2 \;\; , \;\; \int_0^\infty x''^2(s)ds \leq K_2 \;\; , \;\; \int_0^\infty x'''^2(s)ds \leq K_2. \tag{15}$$

Next, multiply (1) by $x(t)$ and integrate by parts from 0 to t. We obtain

$$\int_0^t d(s)x(s)h(x(s))ds = I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + L_0, \tag{16}$$

where

$$\begin{aligned}I_1(t) &= q(t)x'(t)X''(t) - q(t)x(t)X'''(t) - \int_0^t q'(s)x'''(s)ds - \rho\int_0^t q''(s)x''(s-r)ds\\ &\quad - \int_0^t q(s)x''^2(s)ds - \rho\int_0^t q(s)x''(s)x''(s-r)ds,\\ I_2(t) &= -a(t)x(t)x''(t) + \int_0^t a'''(s)ds + \int_0^t a(s)x'(s)x''(s)ds,\\ I_3(t) &= -b(t)x(t)x'(t) + \int_0^t b''(s)ds + \int_0^t b(s)x'^2(s)ds,\\ I_4(t) &= -\frac{1}{2}c(t)x^2(t) + \frac{1}{2}\int_0^t c'^2(s)ds,\\ I_5(t) &= \int_0^t x(s)p(t,x(s),x'(s),x''(s),x'''(s))ds,\end{aligned}$$

and

$$\begin{aligned}L_0 &= q(0)x(0)X'''(0) - q(0)x'(0)X''(0) + a(0)x(0)x''(0)\\ &\quad + b(0)x(0)x'(0) + \frac{1}{2}c(0)x^2(0).\end{aligned}$$

From (12), (15) and the conditions (i) and (H1), we have

$$\begin{aligned}I_1(t) &\le (2+\rho)q_1K_0^2 + (1+\rho)K_0^2\int_0^t |q'(s)|ds + \frac{1}{2}\rho q_1\int_0^t x''^2(s)ds\\ &\quad + \frac{1}{2}\rho q_1\int_0^t x''^2(s-r)ds,\\ &\le (2+\rho)q_1K_0^2 + (1+\rho)K_0^2\int_0^t |q'(s)|ds + \frac{1}{2}\rho q_1\int_0^t x''^2(s)ds\\ &\quad + \frac{1}{2}\rho q_1K_0^2 r + \frac{1}{2}\rho q_1\int_0^{t-r} x''^2(s)ds,\\ I_2(t) &\le a_1K_0^2 + K_0^2\int_0^t |a'(s)|ds + a_1\int_0^t x'(s)x''(s)ds,\\ &\le a_1K_0^2 + \frac{1}{2}a_1x'^2(t) + K_0^2\int_0^t |a'(s)|ds,\\ I_3(t) &\le b_1K_0^2 + K_0^2\int_0^t |b'(s)|ds + b_1\int_0^t x'^2(s)ds,\\ I_4(t) &\le \frac{1}{2}c_1K_0^2 + \frac{1}{2}K_0^2\int_0^t |c'(s)|ds,\\ I_5(t) &\le K_0\int_0^t |e(s)|ds.\end{aligned}$$

It follows that

$$\begin{aligned}
&\lim_{t\to+\infty} I_1(t) \le (2+\rho)q_1K_0^2 + (1+\rho)K_0^2\eta_1 + \rho q_1 K_2 + \frac{1}{2}\rho q_1 K_0^2 r = L_1,\\
&\lim_{t\to+\infty} I_2(t) \le \frac{3}{2}a_1K_0^2 + K_0^2\eta_1 = L_2,\\
&\lim_{t\to+\infty} I_3(t) \le K_0^2(b_1+\eta_1) + b_1K_2 = L_3,\\
&\lim_{t\to+\infty} I_4(t) \le \frac{1}{2}c_1K_0^2 + \frac{1}{2}K_0^2\eta_1 = L_4, \text{ and}\\
&\lim_{t\to+\infty} I_5(t) \le K_0\eta_2 = L_5.
\end{aligned}$$

Thus,

$$\lim_{t\to+\infty} \big(I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t)\big) \le \sum_{i=1}^{5} L_i < \infty. \tag{17}$$

Consequently, (16), (17) and condition iii) give

$$\begin{aligned}
\int_0^\infty x^2(s)ds &\le \frac{1}{d_0\delta}\int_0^\infty d(s)x(s)h(x(s))ds\\
&\le \frac{1}{d_0\delta}\sum_{i=0}^{5} L_i < \infty,
\end{aligned}$$

which completes the proof of the theorem. ■

Remark 2.3.

If $p(t,x,y,z,w)=0$, similar to above proof, then inequality (8) becomes

$$\dot{V}_{(3)} \le -\,D_3(y^2+z^2+W^2) + \frac{1}{\eta}\gamma\,(t)\,V. \tag{18}$$

From H1, (7), (18) and the Cauchy-Schwartz inequality, we get

$$\begin{aligned}
\dot{U}_{(3)} &= \left(\dot{V}_{(3)} - \frac{1}{\eta}\gamma\,(t)\,V\right) e^{-\frac{1}{\eta}\int_0^t \gamma\,(s)\,ds}\\
&\le -D_3\left(y^2+z^2+W^2\right) e^{-\frac{1}{\eta}\int_0^t \gamma\,(s)\,ds}\\
&\le -\mu\left(y^2+z^2+W^2\right),
\end{aligned}$$

where $\mu = D_3 e^{-\frac{\eta_1}{\eta}}$. It follows that the only solution of system (3) for which $\dot{U}_{(3)}(t,x,y,z,W)=0$ is the solution $x=y=z=w=0$. The above discussion guarantees that the trivial solution of Equation (1) is uniformly asymptotically stable, and the same conclusion as in the proof of Theorem 2.2 can be drawn for square integrability of solutions of Equation (1).

3. Example

We consider the following fourth order non-autonomous differential equation of neutral type

$$\begin{aligned}&\left(\left(\frac{e^t}{2e^{2t}+1}+\frac{2}{5}\right)\left(x'''\left(t\right)+\frac{1}{322}x'''\left(t-r\right)\right)\right)'+\left(e^{-t}\sin t+2\right)x'''\\&+\left(\frac{\sin\left(t\right)+7e^t+7e^{-t}}{e^t+e^{-t}}\right)x''+\left(e^{-2t}\sin^3 t+2\right)x'\\&+\left(\frac{1}{20\cosh t}+\frac{1+2\left(1+t^2\right)}{20\left(1+t^2\right)}\right)\left(\frac{x}{x^2+1}+\frac{x}{10}\right)\\&=\frac{2\sin t}{t^2+\left(x\left(t\right)+x'\left(t\right)\right)^2+\left(x''\left(t\right)x'''\left(t\right)\right)^2+1}.\end{aligned}$$

By taking

$$\begin{aligned}p(t,x(t),x'(t),x''(t),x'''(t))&=\frac{2\sin t}{t^2+\left(x\left(t\right)+x'\left(t\right)\right)^2+\left(x''\left(t\right)x'''\left(t\right)\right)^2+1}\\&\leq e\left(t\right)=\frac{2\sin t}{t^2+1},\\h\left(x\right)&=\frac{x}{x^2+1}+\frac{x}{10},\end{aligned}$$

$$h_0-\frac{a_0\delta_0}{d_1}=-\frac{53}{10}\leq h'(x)=\frac{1-x^2}{(1+x^2)^2}+\frac{1}{10}\left(x\right)\leq\frac{h_0}{2}=\frac{11}{10},$$
$$a_0=1\leq a\left(t\right)=e^{-t}\sin t+2\leq a_1=3,$$

$$b_0=\frac{13}{2}\leq b\left(t\right)=\frac{\sin\left(t\right)+7e^t+7e^{-t}}{e^t+e^{-t}}\leq b_1=\frac{15}{2},$$
$$c_0=1\leq c\left(t\right)=e^{-2t}\sin^3 t+2\leq c_1=3,$$

$$d_0=\frac{1}{10}\leq d\left(t\right)=\frac{1}{20\cosh t}+\frac{1+2\left(1+t^2\right)}{20\left(1+t^2\right)}\leq d_1=\frac{1}{5},$$
$$q_0=\frac{2}{5}\leq q(t)=\frac{e^t}{2e^{2t}+1}+\frac{2}{5}\leq q_1=\frac{4}{5},$$

and by taking

$$\begin{aligned}b_0&=\frac{13}{2}>\kappa=\frac{d_1h_0a_1}{c_0}+\frac{c_1+\delta_0}{a_0}=\frac{291}{50},\quad\text{for}\quad\delta_0=\frac{3}{2},\\\varepsilon&=\frac{1}{20}<\min\left\{\frac{1}{a_0},\ \frac{d_1h_0}{c_0},\ \frac{b_0-\kappa}{a_1+c_1}\right\},\\\lambda_0&=\frac{53}{10}=\max\left\{\frac{h_0}{2},\left|h_0-\frac{a_0\delta_0}{d_1}\right|\right\},\end{aligned}$$

we find

$$\alpha = \frac{21}{20} = \frac{1}{a_0} + \varepsilon,$$
$$\beta = \frac{49}{100} = \frac{d_1 h_0}{c_0} + \varepsilon,$$
$$\rho = \frac{1}{322}$$
$$< \min\left\{1, \frac{2\varepsilon}{\alpha h_0}, \frac{2\varepsilon c_0}{\alpha(c_1 + d_1\lambda_0)}, 2\frac{b_0 - \kappa - \varepsilon\,(a_1 + c_1)}{\alpha(b_1 + d_1\lambda_0 + d_1) + \beta}, \frac{2\varepsilon a_0}{\alpha(2a_1 + b_1 + c_1 + d_1) + 5 + \beta}\right\}.$$

It follows easily that

$$\int_0^{+\infty} |e\,(t)|\,dt = \int_0^{+\infty} \left|\frac{2\sin t}{t^2+1}\right| dt \le \int_0^{+\infty} \frac{2}{t^2+1} dt = \pi,$$
$$\int_0^{+\infty} |a'\,(t)|\,dt = \int_0^{+\infty} \left|\,(\cos t)\,e^{-t} - (\sin t)\,e^{-t}\right| dt \le \int_0^{+\infty} 2e^{-t} dt = 2,$$
$$\int_0^{+\infty} |b'\,(t)|\,dt = \int_0^{+\infty} \left|\frac{(e^t + e^{-t})\cos t - (e^t - e^{-t})\sin t}{(e^t + e^{-t})^2}\right| dt$$
$$\le \int_0^{+\infty} \left(\frac{1}{e^t + e^{-t}} + \frac{e^t - e^{-t}}{(e^t + e^{-t})^2}\right) dt \le \frac{\pi}{2},$$
$$\int_0^{+\infty} |c'\,(t)|\,dt = \int_0^{+\infty} \left|\,3\left(\cos t \sin^2 t\right) e^{-2t} - 2\left(\sin^3 t\right) e^{-2t}\right| dt$$
$$\le \int_0^{+\infty} 5e^{-2t} dt = \frac{5}{2},$$

and

$$\int_0^{+\infty} (-d'\,(t))\,dt = \int_0^{+\infty} \frac{1}{20}\left(\frac{\sinh t}{\cosh^2 t} + \frac{2t}{(1+t^2)^2}\right) dt = \frac{1}{10},$$
$$\int_0^{+\infty} |q'\,(t)|\,dt = \int_0^{+\infty} \left|\frac{e^t}{2e^{2t}+1} - \frac{4e^{3t}}{(2e^{2t}+1)^2}\right| dt = \frac{1}{3}.$$

Therefore,

$$\int_0^{+\infty} \left(|a'\,(t)| + |b'\,(t)| + |c'\,(t)| - d'\,(t) + |q'\,(t)|\right) dt < +\infty.$$

Thus all the assumptions of Theorem 2.2 hold, so solutions of (19) are bounded and square integrable.

4. Conclusion

It is well known that the problem of asymptotic behavior of solutions for neutral differential equations is very important in the theory and applications of differential equations. In the present work, conditions were obtained for the stability, boundedness and square integrability of solutions for certain fourth-order neutral differential equations with delay. Using Lyapunov second or direct method, a Lyapunov functional was defined and used to obtain our results.

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