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Inverse Spectral Problems for Spectral Data and Two Spectra of N by N Tridiagonal Almost-Symmetric Matrices

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Abstract

One way to study the spectral properties of Sturm-Liouville operators is difference equations. The coefficients of the second order difference equation which is equivalent Sturm-Liouville equation can be written as a tridiagonal matrix. One investigation area for tridiagonal matrix is finding eigenvalues, eigenvectors and normalized numbers. To determine these datas, we use the solutions of the second order difference equation and this investigation is called direct spectral problem. Furthermore, reconstruction of matrix according to some arguments is called inverse spectral problem. There are many methods to solve inverse spectral problems according to selecting the datas which are generalized spectral function, spectral data of the matrix and two spectra of the matrix. In this article, we study discrete form the Sturm-Liouville equation with generalized function potential and we will focus on the inverse spectral problems of second order difference equation for spectral data and two spectra. The examined difference equation is equivalent Sturm-Liouville equation which has a discontinuity in an interior point. First, we have written the investigated Sturm-Liouville equation in difference equation form and then constructed N by N tridiagonal matrix from the coefficients of this difference equation system. The inverse spectral problems for spectral data and two-spectra of N by N tridiagonal matrices which are need not to be symmetric are studied. Here, the matrix comes from the investigated discrete Sturm-Liouville equation is not symmetric, but almost symmetric. Almost symmetric means that the entries above and below the main diagonal are the same except two entries.

Keywords: Sturm-Liouville equation; Difference equation; Inverse problems; Spectral data; Two spectra

MSC 2010 No.: 39A10, 40A05

1. Introduction

The purpose of this article is twofold. The inverse spectral problem for spectral data will be investigated. Furthermore, inverse spectral problem for two spectra will be studied. This article is self-contained, but can also be considered as a sequel to an earlier publication in Bala et al. (2016). In this article we construct the $N \times N$ tridiagonal matrix

$$J = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_M & a_M & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & c_M & d_{M+1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & d_{N-3} & c_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & c_{N-3} & d_{N-2} & c_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & c_{N-2} & d_{N-1} \end{bmatrix}, \tag{1}$$

where

$$a_n,\ b_n \in \mathbb{C}, a_n \neq 0,$$
$$c_n = a_n/\alpha,\ n \in \{M, M+1, ..., N-2\},$$
$$d_n = b_n/\alpha,\ \ n \in \{M+1, M+2, ..., N-1\},$$

and $\alpha \neq 1$ is a positive real number. J is the almost symmetric matrix of the form (1). Almost symmetric here means that the entries above and below the main diagonal are the same except the entries a_M and c_M.

When $a_M = c_M$ in matrix J, we obtain a symmetric matrix and such a matrix is called Jacobi matrix. The most of first studies related to inverse problem for Jacobi matrices belong to Hochstadt (1974, 1979), (see also Gray and Wilson (1976) and Hald (1976)). Later, G. Guseinov has pioneered an inverse problem of infinite Jacobi matrices. He considered the inverse problems of spectral analysis for infinite Jacobi matrices in Gasymov and Guseinov (1990), the inverse spectral problems for the infinite non-selfadjoint Jacobi matrices from the generalized spectral function in Guseinov (1978) and from the spectral data and two spectra in Guseinov (2010, 2012a, 2012b, 2013). He has been also studied the inverse spectral problem for $N \times N$ tridiagonal symmetric matrix in Guseinov (2009) and than the inverse spectral problems with spectral parameter in the initial conditions are considered in Manafov and Bala (2013). Also, spectrum and scattering analysis have been investigated for discrete Schrödinger equations and Dirac systems in Bairamov et al. (2017) and Bairamov and Solmaz (2018). At the same time, Huseynov et al. (2017) studied inverse scattering problem for discrete Dirac system. Parseval equality of discrete Sturm-Liouville equation with periodic generalized function potentials has been considered in Manafov et al. (2018). Generally, inverse spectral problems for energy-dependent Sturm-Liouville equations with delta-interaction have been examined in Manafov (2016b, 2016a).

In this paper, we consider the eigenvalue problem $Jy = \lambda y$, where $y = \{y_n\}_{n=0}^{N-1}$ is a column vector. We can think this matrix eigenvalue problem as a matrix form of the following second order linear difference equation

$$\begin{gathered} a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda \rho_n y_n, \quad n \in \{0, 1, ..., M, ..., N-1\}, \\ a_{-1} = c_{N-1} = 1, \end{gathered} \tag{2}$$

for $\{y_n\}_{n=-1}^{N}$, with the boundary conditions

$$y_{-1} = y_N = 0, \tag{3}$$

where ρ_n is a constant defined by

$$\rho_n = \begin{cases} 1, & 0 \leq n \leq M, \\ \alpha, & M < n \leq N-1, \end{cases} \quad 1 \neq \alpha > 0.$$

On the other hand, the difference equation (2) with the boundary conditions (3) is a discrete form of the following Sturm-Liouville operator with discontinuous coefficients:

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}y(x)\right] + q(x)y(x) = \lambda\rho(x)y(x), \quad x \in [a, b],$$

$$y(a) = y(b) = 0,$$

where $\rho(x)$ is a piecewise function defined by

$$\rho(x) = \begin{cases} 1, & a \leq x \leq c, \\ \alpha^2, & c < x \leq b, \end{cases} \quad \alpha^2 \neq 1,$$

$[a, b]$ is a finite interval, α is a real number, and c is a discontinuity point in $[a, b]$. For some direct and inverse spectral investigations of such an equation, we refer to Akhmedova and Huseynov (2003, 2010). Also, the spectral properties of the equation given above are investigated for different cases in Mamedov and Cetinkaya (2015, 2017) and Menken et al. (2018).

2. Spectral Data

In this section, we introduce some new concepts and statements releated to spectral data of the matrix J. We will start this section with the following lemma. For the proof of this lemma, please see Bala et al. (2016). Let us denote by $\{P_n(\lambda)\}_{n=-1}^{N}$, the solution of Equation (2) together with the initial data

$$y_{-1} = 0, \quad y_0 = 1.$$

Lemma 2.1.

The following equality holds:

$$\det(J - \lambda I) = (-1)^N a_0 a_1 ... a_M c_{M+1} ... c_{N-1} P_N(\lambda). \tag{4}$$

Therefore, the roots of the polynomial $P_N(\lambda)$ and the eigenvalues of the matrix J are coincident.

Let $R(\lambda) = (J - \lambda I)^{-1}$ be resolvent of matrix J given in (1), and e_0 be an N dimensional column vector whose components are $(1, 0, ..., 0)$. Let's introduce a resolvent function $\omega(\lambda)$ of matrix J as a rational function

$$\omega(\lambda) = -\langle R(\lambda) e_0, e_0 \rangle = -\left\langle (J - \lambda I)^{-1} e_0, e_0 \right\rangle, \tag{5}$$

where $\langle \ , \ \rangle$ denotes standard inner product on $\mathbb{C}^N$.

Let $\lambda_1, \lambda_2, ..., \lambda_p$ be distinct eigenvalues of the matrix J and $m_1, m_2, ..., m_p$ are their multiplicites, respectively. These are also the roots of the polynomial $P_N(\lambda)$ from Lemma 2.1. Therefore $1 \leq p \leq N$ and $m_1 + m_2 + ... + m_p = N$. The rational function $\omega(\lambda)$ can be rewritten as a sum of partial fractions:

$$\omega(\lambda) = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{\beta_{kj}}{(\lambda - \lambda_k)^j}, \tag{6}$$

where $\beta_{kj} \in \mathbb{C}$ are uniquely determined by the matrix J.

Definition 2.2.

The collection of quantities

$$\left\{\lambda_k, \ \beta_{kj} \ \left(j = \overline{1, m_k}; \ k = \overline{1, p}\right)\right\}, \tag{7}$$

are called spectral data for the matrix J. For each $k \in \{1, 2, ..., p\}$ we call the numbers

$$\{\beta_{k1}, \beta_{k2}, ..., \beta_{km_k}\},$$

the normalizing chain (of the matrix J) associated with the eigenvalue λ_k.

When the matrix J is given, determining the spectral data of this matrix is called the direct spectral problem for this matrix. Let us show a way to calculate the spectral data of the matrices J. For this, we first describe the resolvent function $\omega(\lambda)$. Denote by $\{P_n(\lambda)\}_{n=-1}^{N}$ and $\{Q_n(\lambda)\}_{n=-1}^{N}$ the solutions of Equation (2) satisfying initial conditions

$$P_{-1}(\lambda)=0, \quad P_0(\lambda)=1, \tag{8}$$

$$Q_0(\lambda)=-1, \quad Q_1(\lambda)=0. \tag{9}$$

We can write the entries $R_{nm}(\lambda)$ of the matrix $R(\lambda)=(J-\lambda I)^{-1}$ are of the form by the same motivation in Guseinov (2009),

$$R_{nm}(\lambda)=\begin{cases}\rho_n P_n(\lambda)\left[Q_m(\lambda)+M(\lambda)P_m(\lambda)\right], & 0\le n\le m\le N-1,\\ \rho_n P_m(\lambda)\left[Q_n(\lambda)+M(\lambda)P_n(\lambda)\right], & 0\le m\le n\le N-1,\end{cases}$$

where $M(\lambda)=-Q_N(\lambda)/P_N(\lambda)$. Now, recalling (5) and using (8), (9), we obtain

$$\omega(\lambda)=-R_{00}(\lambda)=-M(\lambda)=\frac{Q_N(\lambda)}{P_N(\lambda)}. \tag{10}$$

Thus, we can define the resolvent function $\omega(\lambda)$ of matrix J by aid of the solutions $\{P_n(\lambda)\}_{n=-1}^{N}$ and $\{Q_n(\lambda)\}_{n=-1}^{N}$, and so spectral data can be determined.

In addition, a different way to define the resolvent function is as follows.

If we delete the first row and the first column of the matrix J, then we get

$$J_1=\begin{bmatrix}
b_0^* & a_0^* & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
a_0^* & b_1^* & a_1^* & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & a_1^* & b_2^* & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & b_{M-2}^* & a_{M-2}^* & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & 0 & \cdots & a_{M-2}^* & b_{M-1}^* & a_{M-1}^* & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & 0 & \cdots & 0 & c_{M-1}^* & d_M^* & c_M^* & \cdots & 0 & 0 & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & c_M^* & d_{M+1}^* & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & d_{N-4}^* & c_{N-4}^* & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & c_{N-4}^* & d_{N-3}^* & c_{N-3}^*\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & c_{N-3}^* & d_{N-2}^*
\end{bmatrix}, \tag{11}$$

where

$$\begin{aligned}
a_n^* &= a_{n+1}, \quad n\in\{0,1,...,M-1\},\\
b_n^* &= b_{n+1}, \quad n\in\{0,1,...,M\},\\
c_n^* &= c_{n+1}, \quad n\in\{M-1,...,N-3\},\\
d_n^* &= d_{n+1}, \quad n\in\{M,...,N-2\}.
\end{aligned}$$

The matrix J_1 is called *the first truncated matrix* of J.

Theorem 2.3.

The equality,

$$\omega(\lambda) = -\frac{\det(J_1 - \lambda I)}{\det(J - \lambda I)},$$

holds.

Proof:

Let us denote by the polynomials $P_n^*(\lambda)$ and $Q_n^*(\lambda)$ the solutions of corresponding difference equation of the matrix J_1:

$$a_{n-1}^* y_{n-1} + b_n^* y_n + a_n^* y_{n+1} = \lambda \rho_n y_n, \quad n \in \{0, 1, ..., M, ..., N-2\}, \quad c_{N-2}^* = 1.$$

It can be shown that

$$P_n^*(\lambda) = a_0 Q_{n+1}(\lambda),$$

$$Q_n^*(\lambda) = \tfrac{1}{a_0}\{(\lambda - b_0) Q_{n+1}(\lambda) - P_{n+1}(\lambda)\}, \tag{12}$$

$$n \in \{0, 1, ..., M, ..., N-1\}.$$

Taking into account (4) for the matrix J_1 instead of J and using (12), we obtain

$$\begin{aligned} \det(J_1 - \lambda I) &= (-1)^{N-1} a_0^* a_1^* ... a_{M-1}^* c_M^* ... c_{N-2}^* P_{N-1}^*(\lambda) \\ &= (-1)^{N-1} a_1 a_2 ... a_M c_{M+1} ... c_{N-1} a_0 Q_N(\lambda). \end{aligned} \tag{13}$$

Comparing this with (4), we get

$$\frac{Q_N(\lambda)}{P_N(\lambda)} = -\frac{\det(J_1 - \lambda I)}{\det(J - \lambda I)},$$

so that the statement of the theorem follows by (10). ■

3. Inverse Problem for Spectral Data

In spectral theory, the inverse spectral problem is stated as follows:

(i) To see if it is possible to reconstruct the matrix J given its spectral data (7). If it is possible, to describe the reconstruction procedure.

(ii) To find the necessary and sufficient conditions for a given collection (7) to be the spectral data for some matrix J.

The collection (7) is given. Let's define the quantities t_l by using this collection as below:

$$t_l = \sum_{k=1}^{p} \sum_{j=1}^{n_{kl}} \binom{l}{j-1} \beta_{kj} \lambda_k^{l-j+1}, \quad l = 0, 1, ..., \tag{14}$$

where $n_{kl} = \min\{m_k, l+1\}$, $\begin{pmatrix} l \\ j-1 \end{pmatrix}$ is a binomial coefficient and when $j-1 > l,$ we put $\begin{pmatrix} l \\ j-1 \end{pmatrix} = 0$. Let's present the determinants by using the numbers t_l

$$D_n = \begin{vmatrix} t_0 & t_1 & \cdots & t_n \\ t_1 & t_2 & \cdots & t_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ t_n & t_{+1} & \cdots & t_{2n} \end{vmatrix}, \quad n = \overline{0, N}. \tag{15}$$

Since the proof of the following theorem reproduces that of Theorem 6 in Guseinov (2009) with obvious modifications, we omit it.

Theorem 3.1.

Let an arbitrary collection (7) of complex numbers be given, where $\lambda_1, ..., \lambda_p$ $(1 \le p \le N)$ are distinct, $1 \le m_k \le N$ and $m_1 + ... + m_p = N$. In order for this collection to be the spectral data for some matrix J, it is necessary and sufficient that the following two conditions be satisfied:

(i) $\sum\limits_{k=1}^{p} \beta_{k1} = 1,$

(ii) $D_n \neq 0$ for $n = \left(\overline{1, N-1}\right)$, and $D_N = 0,$ where D_n is defined by (14) and (15).

Under the conditions of Theorem 3.1, the entries $a_n,$ $b_n,$ c_n and d_n of the matrix J for which the collection (7) is spectral data, are recovered by the formulas

$$a_n = \pm \frac{\sqrt{D_{n-1} D_{n+1}}}{D_n} \quad (0 \le n \le M-1), \; D_{-1} = 1, \tag{16}$$

$$a_M = \pm \frac{\sqrt{\alpha D_{M-1} D_{M+1}}}{D_M}, \quad c_M = \pm \frac{\sqrt{D_{M-1} D_{M+1}}}{\sqrt{\alpha} D_M}, \tag{17}$$

$$c_n = \pm \frac{\sqrt{D_{n-1} D_{n+1}}}{D_n} \quad (M < n \le N-2), \tag{18}$$

$$b_n = \frac{\triangle_n}{D_n} - \frac{\triangle_{n-1}}{D_{n-1}} \quad (0 \le n \le M), \quad \triangle_{-1} = 0, \tag{19}$$

$$d_n = \frac{\triangle_n}{D_n} - \frac{\triangle_{n-1}}{D_{n-1}} \; (M < n \le N-1), \; \triangle_0 = t_1, \tag{20}$$

where $D_n^{(n)} = \triangle_n$ is the determinant obtained from the determinant D_n by replacing in D_n the last column by the column with the components $t_{n+1}, t_{n+2}, ..., t_{2n+1}$.

Remark 3.2.

It follows from the equalities (16) and (18), the matrix (1) is not uniquely restored from the spectral data because of the sign $\pm$. To ensure that the inverse problem is uniquely solvable, we have to specify additionally a sequence of signs $+$ and $-$.

4. Inverse Problem for Two Spectra

Let J be a matrix which has the form (1), $\lambda_1, \lambda_2, ..., \lambda_p$ be distinct eigenvalues of the matrix J and $m_1, m_2, ..., m_p$ are their multiplicites, respectively. These are also the roots of the polynomial $P_N(\lambda)$ from Theorem 2 in Bala et al. (2016). Therefore $1 \leq p \leq N$ and $m_1 + m_2 + ... + m_p = N$. Furthermore, let J_1 be the $(N-1) \times (N-1)$ matrix which has the form (11), $\mu_1, \mu_2, ..., \mu_q$ be distinct eigenvalues of the matrix J_1 and $n_1, n_2, ..., n_q$ are their multiplicites, respectively. Also, these eigenvalues are roots of $\det(J_1 - \lambda I)$, so that $1 \leq q \leq N-1$ and $n_1+n_2+...+n_q = N-1$.

We assume that the collections

$$\left\{\lambda_k, m_k \ \left(k = \overline{1,p}\right)\right\}, \quad \text{and} \quad \left\{\mu_k, n_k \ \left(k = \overline{1,q}\right)\right\},$$

are the spectral data for the matrix J and J_1, respectively. These two collections are called *two-spectra* of the matrix J. The inverse spectral problem for two-spectra is defined as the reconstruction of the matrix J by aid of its two-spectra. In order to solve the inverse spectral problem for two-spectra we reduce this problem to the inverse problem for the spectral data discussed in previous section.

Let us give some properties of two-spectra of the matrix J which has (1) form. Denote by $\{P_n(\lambda)\}_{n=-1}^{N}$ and $\{Q_n(\lambda)\}_{n=-1}^{N}$ the solutions of Equation (2) satisfying initial conditions (8) and (9), respectively. Then, by (4) and (13), we get

$$\det(J - \lambda I) = (-1)^N a_0 a_1 ... a_M c_{M+1} ... c_{N-1} P_N(\lambda),$$
$$\det(J_1 - \lambda I) = (-1)^{N-1} a_0 a_1 ... a_M c_{M+1} ... c_{N-1} Q_N(\lambda),$$

where the eigenvalues of the matrix J and their multiplicities coincide with the roots of the polynomial $P_N(\lambda)$ and the eigenvalues of the matrix J_1 and their multiplicities coincide with the roots of the polynomial $Q_N(\lambda)$.

Lemma 4.1.

The equation

$$P_{N-1}(\lambda) Q_N(\lambda) - P_N(\lambda) Q_{N-1}(\lambda) = \frac{1}{\alpha}, \tag{21}$$

holds.

Proof:

Let's consider the equation

$$a_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda \rho_n y_n, \quad n \in \{0, 1, ..., M, ..., N-1\},$$
$$a_{-1} = c_{N-1} = 1,$$

Denote by $\{P_n(\lambda)\}_{n=-1}^{N}$ and $\{Q_n(\lambda)\}_{n=-1}^{N}$ the solutions of this equation satisfying initial conditions (8) and (9), respectively. We can write

$$a_{n-1} P_{n-1}(\lambda) + b_n P_n(\lambda) + a_n P_{n+1}(\lambda) = \lambda \rho_n P_n(\lambda),$$

$$a_{n-1}Q_{n-1}(\lambda)+b_nQ_n(\lambda)+a_nQ_{n+1}(\lambda)=\lambda\rho_nQ_n(\lambda).$$

If the first equality is multiplied by $P_n(\lambda)$ and the second equality is multiplied by $Q_n(\lambda)$, then the second result is substracted from the first, we get

$$W=a_{n-1}\left[P_{n-1}(\lambda)Q_n(\lambda)-P_n(\lambda)Q_{n-1}(\lambda)\right]=a_n\left[P_n(\lambda)Q_{n+1}(\lambda)-P_{n+1}(\lambda)Q_n(\lambda)\right],$$

and the Wronskian is independent from $n\in\{0,1,...,M,...,N-1\}$. By use of initial conditions (8) and (9), we have

$$\begin{gathered}a_{N-1}\left[P_{N-1}(\lambda)Q_N(\lambda)-P_N(\lambda)Q_{N-1}(\lambda)\right]\\=c_{N-1}.\alpha\left[P_{N-1}(\lambda)Q_N(\lambda)-P_N(\lambda)Q_{N-1}(\lambda)\right]=1.\end{gathered}$$

If we take $c_{N-1}=1$, we obtain Equation (21). ■

The matrices J and J_1 have no common eigenvalues, that is, $\lambda_k\neq\mu_j$ for all possible values of k and j. From (6) and Theorem 2.3, we get

$$\sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{\beta_{kj}}{(\lambda-\lambda_k)^j}=\frac{\prod\limits_{i=1}^{q}(\lambda-\mu_i)^{n_i}}{\prod\limits_{l=1}^{p}(\lambda-\lambda_l)^{m_l}}.$$

Hence,

$$\beta_{kj}=\frac{1}{(m_k-j)!}\lim_{\lambda\to\lambda_k}\frac{d^{m_k-j}}{d\lambda^{m_k-j}}\left[(\lambda-\lambda_k)^{m_k}\frac{\prod\limits_{i=1}^{q}(\lambda-\mu_i)^{n_i}}{\prod\limits_{l=1}^{p}(\lambda-\lambda_l)^{m_l}}\right].$$

We can write

$$\beta_{kj}=\frac{1}{(m_k-j)!}\lim_{\lambda\to\lambda_k}\frac{d^{m_k-j}}{d\lambda^{m_k-j}}\frac{\prod\limits_{i=1}^{q}(\lambda-\mu_i)^{n_i}}{\prod\limits_{\substack{l=1\\l\neq k}}^{p}(\lambda-\lambda_l)^{m_l}},\quad\left(j=\overline{1,m_k};\ k=\overline{1,p}\right).\tag{22}$$

The procedure of reconstruction of the matrix J from its two-spectra consist in the following.

If we have two-spectra

$$\left\{\lambda_k,m_k\ \left(k=\overline{1,p}\right)\right\},\quad\text{and}\quad\left\{\mu_k,n_k\ \left(k=\overline{1,q}\right)\right\},$$

we obtain the quantities β_{kj} from (22) and then solve the inverse spectral problem for the spectral data

$$\left\{\lambda_k,\beta_{kj}\left(j=\overline{1,m_k};k=\overline{1,p}\right)\right\},$$

to determine the entries $a_n,\ b_n,\ c_n$ and d_n of the matrix J we use the formulas (16)-(20).

5. Example

Now, we will work out an example to illustrate our formulas. In the following example, by using Theorem 2, it can be shown that the necessary and sufficient conditions for a given collection (7) hold, and the matrix J can be constructed from (16)-(20).

Let's take $N = 4,\ M = 2$ in Theorem 3.1, and $p = 3$, $m_1 = 1$, $m_2 = 1$, $m_3 = 2$ in the formulas (16)-(20). Let's choose the quantities in collection (7) as follows.

The eigenvalues are

$$\lambda_1 = -1,\ \lambda_2 = 0,\ \lambda_{3,4} = 1,$$

and the normalizing numbers which satisfy the condition $\sum_{k=1}^{p} \beta_{k1} = 1$ are

$$\beta_{11} = 1/4,\ \beta_{21} = 1/2,\ \beta_{31} = 1/4,\ \beta_{32} = 1/3.$$

Firstly, we find the numbers $t_l\ \left(l = \overline{0,8}\right)$ by using the formula (14)

$$t_0 = 1,\ t_1 = \frac{1}{3},\ t_2 = \frac{7}{6},\ t_3 = 1,\ t_4 = \frac{11}{6},\ t_5 = \frac{5}{3},\ t_6 = \frac{5}{2},\ t_7 = \frac{7}{3},\ t_8 = \frac{19}{6}.$$

Now, it follows from the numbers $t_l\ \left(l = \overline{0,8}\right)$, we can find the determinants $D_n\ (n = \overline{0,4})$ defined in (15),

$$D_{-1} = 1,\quad D_0 = 1,\quad D_1 = \frac{19}{18},\quad D_2 = \frac{1}{8},\quad D_3 = -\frac{2}{9},\quad D_4 = 0, \tag{23}$$

and the determinants $D_m^{(m)} = \triangle_m\ \left(m = \overline{0,3}\right)$,

$$\triangle_{-1} = 0,\quad \triangle_0 = \frac{1}{3},\quad \triangle_1 = \frac{11}{18},\quad \triangle_2 = -\frac{7}{18},\quad \triangle_3 = -\frac{2}{9}. \tag{24}$$

Then, by using the formulas (16)-(20) and from (23) and (24), we get

$$a_0 = \pm\sqrt{\frac{19}{18}},\ a_1 = \pm\frac{9}{19\sqrt{2}},\ a_2 = \pm\frac{8\sqrt{19\alpha}}{9}i,\ c_2 = \pm\frac{8\sqrt{19}}{9\sqrt{\alpha}}i,$$

and

$$b_0 = \frac{1}{3},\ b_1 = \frac{14}{57},\ b_2 = -\frac{631}{171},\ d_3 = \frac{37}{9}.$$

Consequently, we find the eight matrices $J_\pm$ for the spectral data given above, as follows:

$$J_\pm = \begin{bmatrix} \frac{1}{3} & \pm\sqrt{\frac{19}{18}} & 0 & 0 \\ \pm\sqrt{\frac{19}{18}} & \frac{14}{57} & \pm\frac{9}{19\sqrt{2}} & 0 \\ 0 & \pm\frac{9}{19\sqrt{2}} & -\frac{631}{171} & \pm\frac{8\sqrt{19\alpha}}{9}i \\ 0 & 0 & \pm\frac{8\sqrt{19}}{9\sqrt{\alpha}}i & \frac{37}{9} \end{bmatrix}.$$

The characteristic polynomials which are determined by the matrices $J_{\pm}$ are

$$\det\left(J_{\pm}-\lambda I\right)=\left(\lambda+1\right)\lambda\left(\lambda-1\right)^{2}.$$

The inverse spectral problem for two-spectra is solved uniquely up to signs of the off-diagonal elements of the recovered matrix from Remark 3.2.

6. Conclusion

In this article, the investigated tridiagonal matrix is the coefficient matrix of the discrete form Sturm-Liouville operator with discontinous coefficient. We studied the inverse spectral problem of almost-symmetric tridiagonal matrix for spectral data and two spectra. Spectral data is a collection of eigenvalues λ_k and normalized numbers β_{kj} of the matrix J. Two spectra of the matrix J occur the eigenvalues λ_k and μ_k of the matrices J and J_1, respectively. While solving the inverse spectral problem for two spectra, firstly we determine the normalized numbers β_{kj} of the matrix J by using the eigenvalues λ_k and μ_k of the matrices J and J_1, respectively. Thus, we reduce the problem from two spectra to spectral data. In the last section, we gave an example the inverse spectral problem and we reconstruct the matrix by using the given eigenvalues and normalized numbers set. For future research, this article will help solve the inverse spectral problems for the discrete form Sturm-Liouville equation which has discontinuity points at the internal points.

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