Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 14, Issue 2 (December 2019), pp. 1123 – 1131

An Efficient Computational Method for Solving a System of FDEs via Fractional Finite Difference Method

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Received: March 18, 2019; Accepted: July 8, 2019

Abstract

This paper aims to provide a numerical method for solving systems of fractional (Caputo sense) differential equations (FDEs). This method is based on the fractional finite difference method (FDM), where we implemented the Grünwald-Letnikov's approach. This method is computationally very efficient and gives very accurate solutions. In this study, the stability of the obtained numerical scheme is given. The numerical results show that the proposed approach is easy to be implemented and are accurate when applied to system of FDEs. The method introduces promising tool for solving many systems of FDEs. Two examples are given to demonstrate the applicability and the effectiveness of our method.

Keywords: System of FDEs; Fractional FDM; Grünwald-Letnikov's approach; Stability analysis

MSC 2010 No.: 41A30; 65N12

1. Introduction

FDEs are introduced during various phenomena, such as finance, applied mathematics, bioengineering, fluid mechanics, viscoelasticity, biology, physics and engineering (Podlubny (1999)). However, many researchers remain unaware of this field. Consequently, considerable attention has been given to the solutions of FDEs of physical interest. Most FDEs do not have exact solutions, therefore, it is important to develop efficient numerical schemes to approximate solutions (Gupta and Gupta (2008)-He (1999), Khader (2011), Khader (2018)-Meerschaert and Tadjeran (2006), Smith (1965), and Sweilam et al. (2011)).

The Caputo fractional derivative operator D^{α} of order α is defined in the following form:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \quad \alpha > 0, \quad x > 0.$$

For more details on the field of fractional derivatives definitions and its properties see Diethelm (2010), Samko et al. (1993).

The main aim of this work is to apply the Legendre collocation method and the fractional FDM to solve numerically systems of FDEs. The proposed method discretizes the introduced problem to a system of algebraic equations thus greatly simplifying the problem.

In this article, we consider the following general form of the non-linear system of FDEs:

$$D^{\nu_i} x_i(t) = f_i(t; x_1, x_2, ..., x_n), \qquad i = 1, 2, ..., n, \quad 0 \leq t \leq 1, \tag{1}$$

with the following initial conditions:

$$x_i^{(r)}(0) = c_i^r, \qquad 0 \leq r \leq \lceil \nu \rceil. \tag{2}$$

The existence and uniqueness of this initial value problem for the system of FDEs (1) have been proved in details via Daftardar and Babakhani (2004). Many authors considered this system to solve it using different numerical methods, for example, the differential transform method (Ert$\ddot{u}$rk and Momani (2008)) and the Adomain decomposition method (Jafari and Daftardar (2006)).

2. Grünwald-Letnikov's approach to Caputo's fractional derivative

In this section, we will introduce the definition of Grünwald-Letnikov fractional derivative (Podlubny (1999)).

Definition 2.1.

The Grünwald-Letnikov's approach can be defined as:

$$\frac{d^{\alpha}y(t)}{dt^{\alpha}} = \lim_{h\longrightarrow 0}\frac{1}{h^{\alpha}}\sum_{i=0}^{[\frac{t}{h}]}(-1)^{i}\begin{pmatrix}\alpha\\ i\end{pmatrix}y(t-ih), \tag{3}$$

and the shifted Grünwald-Letnikov fractional derivative is defined as:

$$\frac{d^{\alpha}y(t)}{dt^{\alpha}} = \lim_{h\longrightarrow 0}\frac{1}{h^{\alpha}}\sum_{i=0}^{[\frac{t}{h}]+1}(-1)^{i}\begin{pmatrix}\alpha\\ i\end{pmatrix}y(t-(i-1)h), \tag{4}$$

where $[t/h]$ means the integer part of t/h.

Lemma 2.2. (Jafari et al. (2012))

Assume that $y(t)$ satisfies some smoothness conditions e.g., $y(t)$ can be written in the form of a power series for $\mid t \mid < \rho$, with arbitrary constant ρ. The Grünwald-Letnikov formula holds for each $0 < r < \rho$ and a series of step size h with $\frac{\tau}{h} \in \mathbf{N}$,

$$D_{R}^{\alpha}y(\tau) = \frac{1}{h^{\alpha}}\Delta_{h}^{\alpha}y(nh) + O(h), \qquad (h \longrightarrow 0),$$

where,

$$\Delta_{h}^{\alpha}y(nh) = \sum_{i=0}^{n}(-1)^{i}\begin{pmatrix}\alpha\\ i\end{pmatrix}y(t_{n-i}). \tag{5}$$

In case of Caputo's operator, we have (for $0 < \alpha \leq 1$):

$$D^{\alpha}y(\tau) = \frac{1}{h^{\alpha}}\Delta_{h}^{\alpha}y(nh) - \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)}y(0) + O(h), \qquad (h \longrightarrow 0). \tag{6}$$

The most favorable case here is the initial values for Caputo's differential equation are given to be zero. The numerical formula (6) is used to solve numerically more problems, for example. This formula is implemented to solve the Bagley-Torvik equation (Podlubny (1999)). It also has been applied to solve the fractional-order heat equation (Scherer and Kalla (2008)). In this paper, we intended to extend the implementation of this formula to solve a system of FDEs.

3. Numerical implementation

In this section, we will solve numerically a system of FDEs using the Grünwald-Letnikov FDM. To achieve this task we consider the following two examples.

Example 3.1.

We consider the following system of two linear FDEs:

$$\begin{aligned} D^{\alpha}\, x(t) &= x(t) + y(t), \\ D^{\beta}\, y(t) &= -x(t) + y(t), \end{aligned} \tag{7}$$

the parameters α and β refer to the fractional orders of time derivative with $0 < \alpha,\ \beta \leq 1$. We also assume the following initial conditions:

$$x(0) = x_0 = 0, \qquad y(0) = y_0 = 1. \tag{8}$$

Now, we will implement the fractional FDM by using the approximate formula (6) to discretize the considered system (7). First, we rewrite the system (7) as follows:

$$\begin{aligned} D^{\alpha} x(t_n) &= x(t_n) + y(t_n), \\ D^{\beta} y(t_n) &= -x(t_n) + y(t_n). \end{aligned} \tag{9}$$

Secondly and more importantly, we use the uniform grid $t_n = nh$, where $n = 0, 1, ..., M, Mh = T$ and use the abbreviations x_n and y_n for approximation of the true solutions $x(t_n)$ and $y(t_n)$ in the grid point t_n. By applying the definition of the shifted Grünwald-Letnikov fractional derivative (6) to our system (9), we obtain:

$$\begin{aligned} &\frac{1}{h^{\alpha}} \sum_{i=0}^{n+1} (-1)^i\, \theta_i^{\alpha}\, x_{n+1-i} - \frac{(nh)^{-\alpha}}{\Gamma(1-\alpha)} x_0 = x_n + y_n, \\ &\frac{1}{h^{\beta}} \sum_{i=0}^{n+1} (-1)^i\, \theta_i^{\beta}\, y_{n+1-i} - \frac{(nh)^{-\beta}}{\Gamma(1-\beta)} y_0 = -x_n + y_n, \end{aligned} \tag{10}$$

where $\theta_i^p = \binom{p}{i}$, $p = \alpha$ or β.

To study the stability of the numerical scheme (10), we state and prove the following two theorems.

Theorem 3.2. (Cai and Liu (2007))

The numerical approximation (10) is consistent with the fractional differential equation (9) and $x_m - x(t_m) = O(h^{1+\alpha})$ and $y_m - y(t_m) = O(h^{1+\beta})$.

Theorem 3.3.

The numerical scheme (10) for the system of FDEs (9) is stable.

Proof:

Let $(x_n,\, y_n)$ and $(\bar{x}_n,\, \bar{y}_n)$ be two solutions for the numerical scheme (10). Let $e_n = x_n - \bar{x}_n$ and $E_n = y_n - \bar{y}_n$, we have:

$$(1-h^{\alpha})e_n = -\sum_{i=1}^{n+1}(-1)^i\theta_i^{\alpha}\, e_{n-i} + h^{\alpha}\, E_n \le -\sum_{i=0}^{\infty}(-1)^i\theta_i^{\alpha}\, e_{n-i}, \tag{11}$$

$$(1-h^{\beta})E_n = -\sum_{i=1}^{n+1}(-1)^i\theta_i^{\beta}\, E_{n-i} - h^{\beta}\, e_n \le -\sum_{i=0}^{\infty}(-1)^i\theta_i^{\beta}\, E_{n-i}. \tag{12}$$

We have $(-1)^0\theta_0^p = 1$, $(-1)^i\theta_i^p < 0,\ \ i = 1, 2, ...$ and $\sum_{i=0}^{\infty}(-1)^i\theta_i^p = 1$, $p = \alpha$ or β.
Thus, from (11) and (12), we can find:

$$\|e_n\| \le \max(\|e_0\|, \|e_1\|, ..., \|e_{n-1}\|) \le ... \le \|e_0\|,$$

$$\|E_n\| \le \max(\|E_0\|, \|E_1\|, ..., \|E_{n-1}\|) \le ... \le \|E_0\|.$$

Therefore, the numerical approximation (10) for solving the system of FDEs (9) is stable. ■

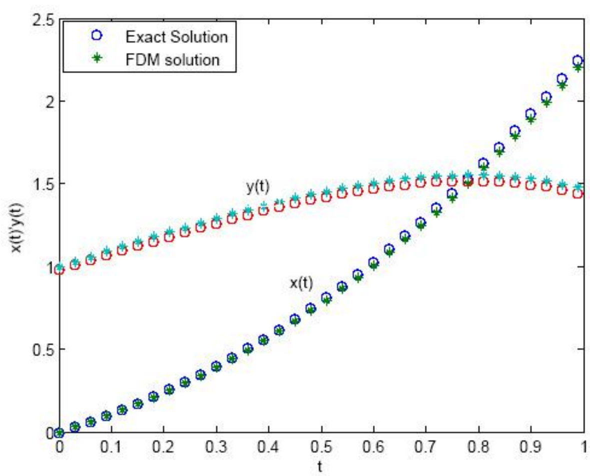

Figure 1. The behavior of the exact solution (at $\alpha = 1, \beta = 1$) and the numerical solution using the fractional FDM with $h = 0.1$

The obtained numerical results of this example by using the proposed method are presented in Figures 1 and 2. In Figure 1, we presented the behavior of the exact solution $(\alpha = \beta = 1)$ with the numerical solution by using the fractional FDM with $h = 0.1$. Whereas, in Figure 2, we presented the behavior of the numerical solutions by using the proposed method at $(\alpha = 0.7,\ \beta = 0.9)$. From these figures, we can see that our numerical results are in good agreement with the exact solution. This gives us indication that the proposed method is good to be implemented for solving such system of FDEs.

Example 3.4.

We consider the following system of non-linear FDEs:

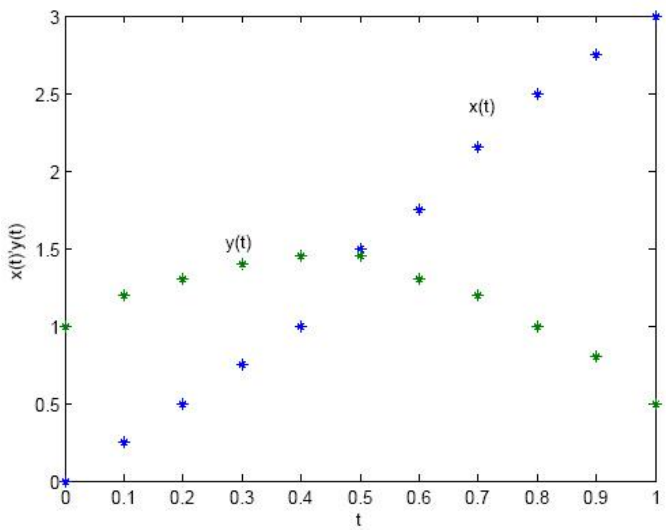

Figure 2. The behavior of the numerical solution (at $\alpha = 0.7, \beta = 0.9$) using the fractional FDM with $h = 0.1$

$$\begin{aligned} &D^{\alpha}\, x(t) = 2y^2, \qquad && 0 < \alpha \leq 1, \\ &D^{\beta}\, y(t) = tx, \qquad && 0 < \beta \leq 1, \\ &D^{\gamma}\, z(t) = yz, \qquad && 0 < \gamma \leq 1, \end{aligned} \tag{13}$$

with initial conditions:

$$x(0) = x_0 = 0, \qquad y(0) = y_0 = 1, \qquad z(0) = z_0 = 1. \tag{14}$$

Here, we will implement the fractional FDM by using the approximate formula (6) to discretize the considered system (13) as follows. First, we rewrite the system (13) as follows:

$$\begin{aligned} &D^{\alpha} x(t_n) = 2y^2(t_n), \\ &D^{\beta} y(t_n) = t_n x(t_n), \\ &D^{\gamma} z(t_n) = y(t_n) z(t_n). \end{aligned} \tag{15}$$

Secondly, we use the uniform grid $t_n = nh$, where $n = 0, 1, ..., M, Mh = T$ and use the abbreviations x_n, y_n and z_n for approximation of the true solutions $x(t_n)$, $y(t_n)$ and $z(t_n)$ in the grid point t_n.

By applying the definition of the shifted Grünwald-Letnikov fractional derivative (6) to system (15), we obtain:

$$
\begin{aligned}
&\frac{1}{h^{\alpha}}\sum_{i=0}^{n+1}(-1)^i\,\theta_i^{\alpha}\,x_{n+1-i}-\frac{(nh)^{-\alpha}}{\Gamma(1-\alpha)}x_0=2y_n^2,\\
&\frac{1}{h^{\beta}}\sum_{i=0}^{n+1}(-1)^i\,\theta_i^{\beta}\,y_{n+1-i}-\frac{(nh)^{-\beta}}{\Gamma(1-\beta)}y_0=t_nx_n,\\
&\frac{1}{h^{\gamma}}\sum_{i=0}^{n+1}(-1)^i\,\theta_i^{\gamma}\,z_{n+1-i}-\frac{(nh)^{-\gamma}}{\Gamma(1-\gamma)}z_0=y_nz_n,
\end{aligned}
\tag{16}
$$

where $\theta_i^p = \binom{p}{i}$, $p = \alpha,\ \beta, \gamma$.

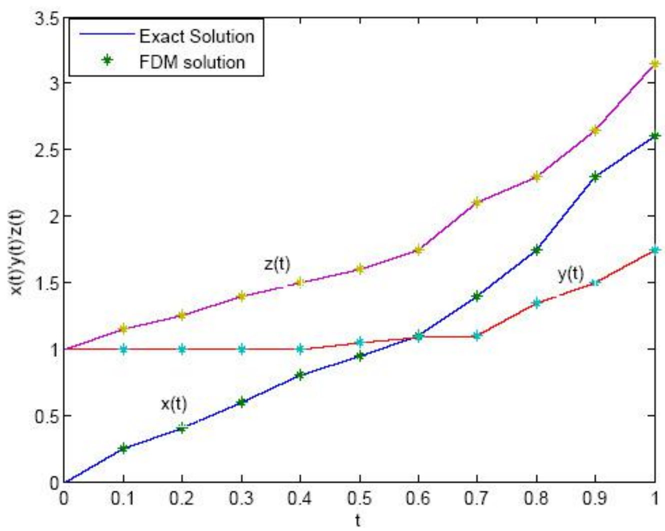

Figure 3. The behavior of the exact solution (at $\alpha = \beta = \gamma = 1$) and the numerical solution using the fractional FDM with $h = 0.1$

The obtained numerical results of this example by using the proposed method are presented in Figures 3 and 4. In Figure 3, we presented the behavior of the exact solution $(\alpha = \beta = \gamma = 1)$ with the numerical solutions by using the fractional FDM with $h = 0.1$. Whereas, in Figure 4, we presented the behavior of the numerical solutions by using the proposed method at $(\alpha = 0.8,\ \beta = 0.7,\ \gamma = 0.6)$. From these figures, we can see that our numerical results are matching well with the exact solution. This gives us indication that these two methods are good to be implemented for solving such system of FDEs.

4. Conclusions

We proposed an efficient computational method for calculating numerical solutions of systems of FDEs by performing the fractional FDM. The work emphasized our confidence that the method is reliable technique to handle the linear and non-linear systems of FDEs. On the other hand, we studied the stability of the numerical scheme which was obtained from the fractional FDM

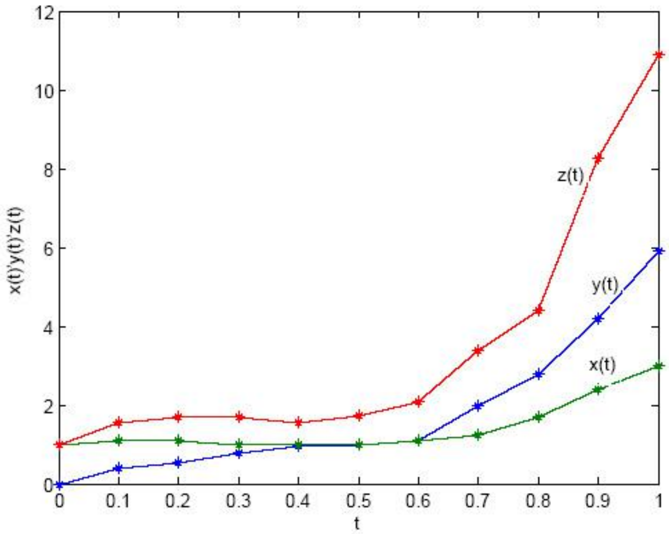

Figure 4. The behavior of the numerical solution (at $\alpha = 0.8, \beta = 0.7, \gamma = 0.6$) using the fractional FDM with $h = 0.1$

using the Grünwald-Letnikov's approach. Two illustrative examples are given to demonstrate the effectiveness of our method. From the solutions obtained by using the suggested method, we can conclude that these solutions are matching well with the exact solution and show that this approach can be solved the problem effectively.

Acknowledgment:

The authors are very grateful to the editor and the referees for carefully reading the paper and for their comments and suggestions, which have improved the paper.

REFERENCES

Cai, X. and Liu, F. (2007). Numerical simulation of the fractional-order control system, J. Appl. Math. Comput., Vol. 23, pp. 229-241.

Daftardar-Gejji, V. and Babakhani, A. (2004). Analysis of a system of fractional differential equations, J. Math. Anal. Appl., Vol. 293, pp. 511-522.

Diego Murio, A. (2008). Implicit finite difference approximation for time fractional diffusion equations, Comput. Math. Appl., Vol. 56, pp. 1138-1145.

Diethelm, K. (1997). An algorithm for the numerical solution of differential equations of fractional order, Electron. Trans. Numer. Anal., Vol. 5, pp. 1-6.

Diethelm, K. (2010). *The Analysis of Fractional Differential Equations*, Springer, Berlin.

Ertürk, V. S. and Momani, S. (2008). Solving systems of fractional differential equations using differential transform method, J. of Comput. and Appl. Maths., Vol. 215, pp. 142-151.

Hashim, I., Abdulaziz, O. and Momani, S. (2009). Homotopy analysis method for fractional IVPs, Commun. Nonlinear Sci. Numer. Simul., Vol. 14, pp. 674-684.

He, J. H. (1999). Variational iteration method-a kind of non-linear analytical technique: Some examples, International Journal of Non-Linear Mechanics, Vol. 34, pp. 699-708.

Jafari, H. and Daftardar-Gejji, V. (2006). Solving a system of nonlinear fractional differential equations using Adomian decomposition, J. of Comput. and Applied Math., Vol. 196, pp. 644-651.

Jafari, H. Khalique, C. M. and Nazari, M. (2012). An algorithm for the numerical solution of nonlinear fractional-order Van der Pol oscillator equation, Math. and Comput. Modelling, Vol. 55, pp. 1782-1786.

Khader, M. M. (2011). On the numerical solutions for the fractional diffusion equation, Commun. Nonlinear Sci. Numer. Simul., Vol. 16, pp. 2535-2542.

Khader, M. M. (2018). Quasi-linearization method with rational Legendre collocation method for solving MHD flow over a stretching sheet with variable thickness and slip velocity which embedded in a porous medium, Applications and Applied Mathematics: An International Journal, Vol. 13, No. 2, pp. 925-932.

Khader, M. M. (2017). Analytical solution for determination the control parameter in the inverse parabolic equation using HAM, Applications and Applied Mathematics: An *International Journal*, Vol. 12, No. 2, pp. 1072-1087.

Lubich, C. (1986). Discretized fractional calculus, SIAM J. Math. Anal., Vol. 17, pp. 704-719.

Meerschaert, M. M. and Tadjeran, C. (2006). Finite difference approximations for two-sided space-fractional partial differential equations, Appl. Numer. Math., Vol. 56, pp. 80-90.

Podlubny, I. (1999). *Fractional Differential Equations*, Academic Press, New York.

Samko, S., Kilbas, A. and Marichev, O. (1993). *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, London.

Scherer, R. and Kalla, S. L. (2008). Numerical treatment of fractional heat equation, Appl. Numer. Math., Vol. 58, pp. 1212-1223.

Smith, G. D. (1965). *Numerical Solution of Partial Differential Equations*, Oxford University Press.

Sweilam, N. H., Khader, M. M. and Nagy, A. M. (2011). Numerical solution of two-sided space-fractional wave equation using finite difference method, J. of Comput. Appl. Maths., Vol. 235, pp. 2832-2841.