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Adjoint Appell-Euler and First Kind Appell-Bernoulli Polynomials

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Abstract

The adjunction property, recently introduced for Sheffer polynomial sets, is considered in the case of Appell polynomials. The particular case of adjoint Appell-Euler and Appell-Bernoulli polynomials of the first kind is analyzed.

Keywords: Appell polynomials; Generating functions; Monomiality principle; Differential equations; Combinatorial analysis

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1. Introduction

In recent articles (refer to Bretti et al. (2018a), Ricci et al. (2017)), new sets of Sheffer (refer to Sheffer (1939)) and Brenke (refer to Brenke (1945)) polynomials, based on higher order Bell numbers (refer to Bernardini et al. (2005), Natalini and Ricci (2004), Natalini and Ricci (2016), Natalini and Ricci (2018), Ricci et al. (2017)), have been studied. Furthermore, the notion of adjoint Sheffer polynomial sets has been introduced (refer to Ricci (2018)) and applied in several cases (refer to Bretti et al. (2018b), Natalini et al. (2018), Ricci et al. (2019)). In this article we consider a similar notion in the case of Appell polynomial sets, starting by the case of Euler and first kind Bernoulli polynomials.

2. Sheffer polynomials

The Sheffer polynomials $\{s_n(x)\}$ are introduced (refer to Sheffer (1939)) by means of the exponential generating function (refer to Srivastava and Manocha (1984)) of the type:

$$A(t)\exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x)\,\frac{t^n}{n!}\,, \tag{1}$$

where

$$\begin{aligned} A(t) &= \sum_{n=0}^{\infty} a_n\,\frac{t^n}{n!}\,, \qquad (a_0 \neq 0)\,,\\ H(t) &= \sum_{n=0}^{\infty} h_n\,\frac{t^n}{n!}\,, \qquad (h_0 = 0)\,. \end{aligned} \tag{2}$$

According to a different characterization (refer to Roman (1984), [p. 18]), the same polynomial sequence can be defined by means of the pair $(g(t), f(t))$, where $g(t)$ is an invertible series and $f(t)$ is a delta series:

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} g_n\,\frac{t^n}{n!}\,, \qquad (g_0 \neq 0)\,,\\ f(t) &= \sum_{n=0}^{\infty} f_n\,\frac{t^n}{n!}\,, \qquad (f_0 = 0,\ f_1 \neq 0)\,. \end{aligned} \tag{3}$$

Denoting by $f^{-1}(t)$ the compositional inverse of $f(t)$ (i.e. such that $f\left(f^{-1}(t)\right) = f^{-1}\left(f(t)\right) = t$), the exponential generating function of the sequence $\{s_n(x)\}$ is given by

$$\frac{1}{g[f^{-1}(t)]}\,\exp\left(xf^{-1}(t)\right) = \sum_{n=0}^{\infty} s_n(x)\,\frac{t^n}{n!}\,, \tag{4}$$

so that

$$A(t) = \frac{1}{g[f^{-1}(t)]}\,, \qquad H(t) = f^{-1}(t)\,. \tag{5}$$

When $g(t) \equiv 1$, the Sheffer sequence corresponding to the pair $(1, f(t))$ is called the associated Sheffer sequence $\{\sigma_n(x)\}$ for $f(t)$, and its exponential generating function is given by

$$\exp\left(xf^{-1}(t)\right) = \sum_{n=0}^{\infty} \sigma_n(x)\,\frac{t^n}{n!}\,. \tag{6}$$

A list of known Sheffer polynomial sequences and their associated ones can be found in Boas and Buck (1958b).

3. Adjointness for Sheffer polynomial sets

According to the above considerations, Sheffer polynomials are characterized both by the ordered couples $(A(t), H(t))$, or by $(g(t), f(t))$.

Definition 3.1.

Adjoint Sheffer polynomials are defined by interchanging the ordered couple $(A(t), H(t))$ with $(g(t), f(t))$, when writing the generating function.

Here and in the following, the *tilde* "$\sim$" above the symbol of a polynomial set stands for the adjective "*adjoint*".

3.1. Adjointness for Appell polynomial sets

In the case of Appell polynomials, since $H(t) \equiv t$, so that $f(t) = H^{-1}(t) = t$, and $A(t) = 1/g(t)$, the above definition becomes:

Definition 3.2.

Adjoint Appell polynomials are simply defined by changing $A(t)$ with $1/A(t)$ in the generating function.

The simplest examples are given by the Adjoint Appell-Euler and the Adjoint Appell-Bernoulli polynomials of the first kind:

1. Adjoint Appell-Euler polynomials

$$\begin{aligned} A(t) &= \frac{e^t + 1}{2} \,, \\ G(t, x) &= \frac{e^t + 1}{2}\, e^{xt} = \sum_{n=0}^{\infty} \tilde{\varepsilon}_n(x)\, \frac{t^n}{n!} \,. \end{aligned} \tag{7}$$

2. Adjoint Appell-Bernoulli polynomials of the first kind

$$\begin{aligned} A(t) &= \frac{e^t - 1}{t} \,, \\ G(t, x) &= \frac{e^t - 1}{t}\, e^{xt} = \sum_{n=0}^{\infty} \tilde{\beta}_n(x)\, \frac{t^n}{n!} \,. \end{aligned} \tag{8}$$

4. Adjoint Appell-Euler polynomials

We recall that a polynomial set $\{p_n(x)\}$ is called quasi-monomial if and only if there exist two operators $\hat{P}$ and $\hat{M}$ such that

$$\begin{aligned} \hat{P}\,(p_n(x)) &= n\,p_{n-1}(x)\,, \\ \hat{M}\,(p_n(x)) &= p_{n+1}(x)\,, \qquad (n=1,2,\dots). \end{aligned} \tag{9}$$

$\hat{P}$ is called the *derivative* operator and $\hat{M}$ the *multiplication* operator, as they act in the same way of classical operators on monomials.

This definition traces back to a paper by Steffensen (1941), recently improved by Dattoli (2000) and widely used in several applications (refer to Dattoli et al. (2003), Dattoli et al. (2009), and the references therein).

Since we are considering an Appell polynomial set, the derivative operator is simply given by the derivative with respect to x, that is

$$\hat{P} = D_x\,, \qquad D_x\,\tilde{\varepsilon}_n(x) = n\,\tilde{\varepsilon}_{n-1}(x)\,. \tag{10}$$

This can be easily checked by partially differentiating the generating function (7) with respect to x.

The relevant differential equation is given by

$$\frac{\partial G}{\partial x} = t\,\frac{e^t-1}{2}\,e^{xt}\,. \tag{11}$$

4.1. Recurrence relation

Theorem 4.1.

For any $k \geq 0$, the polynomials $\tilde{\varepsilon}_k(x)$ satisfy the recurrence relation:

$$\tilde{\varepsilon}_{k+1}(x) = (1+x)\,\tilde{\varepsilon}_k(x) - \frac{x^k}{2}\,. \tag{12}$$

Proof:

Differentiating equation (7) with respect to t we find:

$$\frac{\partial G}{\partial t} = \sum_{k=0}^{\infty} \tilde{\varepsilon}_{k+1}(x)\frac{t^k}{k!} = (1+x)G(t,x) - \frac{e^{xt}}{2} = (1+x)\sum_{k=0}^{\infty}\tilde{\varepsilon}_k(x)\frac{t^k}{k!} - \frac{1}{2}\sum_{k=0}^{\infty} x^k\,\frac{t^k}{k!}\,, \tag{13}$$

so that the recurrence (12) follows. ■

4.2. Generating function's PDE

Theorem 4.2.

The generating function (7) satisfies the linear PDE

$$t\,\frac{\partial G}{\partial t} = (1+x)\frac{\partial G}{\partial x} - \frac{1}{e^t+1}\frac{\partial G}{\partial x}\,. \tag{14}$$

Proof:

It is sufficient to eliminate the exponential function between the partial derivatives given in (11) and (13). ■

Note that the last addend becomes negligible when t increases.

4.3. Differential equation

We start noting that

$$\frac{A'(t)}{A(t)} = \frac{e^t}{e^t+1} = \sum_{k=1}^{\infty}(-1)^{k+1}t^k\,. \tag{15}$$

Now, we can apply a general result by Ben Cheikh (2003) (Corollary 3.2), in order to find the multiplication operator for the quasi-monomial set $\{\tilde{\varepsilon}_k(x)\}$, which at present is given by

$$\hat{M} = \frac{A'(D_x)}{A(D_x)} + x = \sum_{k=1}^{\infty}(-1)^{k+1}D_x^k + x\,, \tag{16}$$

According to the results of monomiality principle (refer to Dattoli (2000)), the quasi-monomial polynomials $\{\tilde{\varepsilon}_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}\,\tilde{\varepsilon}_n(x) = n\,\tilde{\varepsilon}_n(x)\,. \tag{17}$$

In the present case, we have

$$\hat{M}\hat{P} = \left[\sum_{k=1}^{\infty}(-1)^{k+1}D_x^k + x\right]D_x\,, \tag{18}$$

so that we have the following theorem.

Theorem 4.3.

The adjoint Appell-Euler polynomials $\{\tilde{\varepsilon}_n(x)\}$ satisfy the differential equation

$$\left[\sum_{k=2}^{n}(-1)^k D_x^k + xD_x\right]\tilde{\varepsilon}_n(x) = n\,\tilde{\varepsilon}_n(x)\,, \tag{19}$$

because, for any fixed n, the series expansion in Equation (18) reduces to a finite sum when it is applied to a polynomial of degree n.

Remark 4.4.

The first few values of the adjoint Appell-Euler polynomials are as follows:

$$\begin{aligned}
&\tilde{\varepsilon}_0(x) = 1,\\
&\tilde{\varepsilon}_1(x) = x + \tfrac{1}{2},\\
&\tilde{\varepsilon}_2(x) = x^2 + x + \tfrac{1}{2},\\
&\tilde{\varepsilon}_3(x) = x^3 + \tfrac{3}{2}x^2 + \tfrac{3}{2}x + \tfrac{1}{2},\\
&\tilde{\varepsilon}_4(x) = x^4 + 2x^3 + 3x^2 + 2x + \tfrac{1}{2},\\
&\tilde{\varepsilon}_5(x) = x^5 + \tfrac{5}{2}x^4 + 5x^3 + 5x^2 + +\tfrac{5}{2}x + \tfrac{1}{2},\\
&\tilde{\varepsilon}_6(x) = x^6 + 3x^5 + \tfrac{15}{2}x^4 + 10x^3 + \tfrac{15}{2}x^2 + 3x + \tfrac{1}{2},\\
&\tilde{\varepsilon}_7(x) = x^7 + \tfrac{7}{2}x^6 + \tfrac{21}{2}x^5 + \tfrac{35}{2}x^4 + \tfrac{35}{2}x^3 + \tfrac{21}{2}x^2 + \tfrac{7}{2}x + \tfrac{1}{2},\\
&\tilde{\varepsilon}_8(x) = x^8 + 4x^7 + 14x^6 + 28x^5 + 35x^4 + 28x^3 + 14x^2 + 4x + \tfrac{1}{2}.
\end{aligned}$$

Remark 4.5.

Note that the above scheme satisfy a particular symmetry. Writing the polynomials in the form

$$\tilde{\varepsilon}_n(x) = c_{n,0}x^n + c_{n,1}x^{n-1} + \cdots + c_{n,n-1}x + c_{n,n}\,, \quad (n \geq 0)\,,$$

the coefficients $c_{n,h}$, $(0 \leq h \leq n)$ verify the conditions:

$$\begin{aligned}
&c_{n,0} = 1\,, \quad \forall\, n \geq 0,\\
&c_{n,n} = \tfrac{1}{2}\,, \quad \forall\, n \geq 1,\\
&c_{h,k} = c_{h,h-k}\,, \quad \forall\, 1 \leq h \leq n-1\,,\ 1 \leq k \leq n-1\,.
\end{aligned}$$

5. Adjoint Appell-Bernoulli polynomials of the first kind

Even in this case, the derivative operator is simply given by the derivative with respect to x, that is

$$\hat{P} = D_x\,, \qquad D_x\,\tilde{\beta}_n(x) = n\,\tilde{\beta}_{n-1}(x)\,, \tag{20}$$

as it can be easily checked by differentiating the generating function (8) with respect to x.

The relevant differential equation is given by

$$\frac{\partial G}{\partial x} = (e^t - 1)\,e^{xt}\,. \tag{21}$$

5.1. Recurrence relation

Theorem 5.1.

For any $k \geq 0$, the polynomials $\tilde{\varepsilon}_k(x)$ satisfy the recurrence relation:

$$(k+1)\tilde{\beta}_k(x) = (1+x)\,k\,\tilde{\beta}_{k-1}(x) + x^k\,. \tag{22}$$

Proof:

Differentiating Equation (8) with respect to t we find:

$$\frac{\partial G}{\partial t} = \left(\frac{te^t - e^t + 1}{t^2} + x\,\frac{e^t - 1}{t}\right) e^{xt}\,,$$

and consequently

$$t\,\frac{\partial G}{\partial t} = (1+x)\,t\,G(t,x) - G(t,x) + e^{xt}\,,$$

that is

$$\sum_{k=0}^{\infty} k\tilde{\beta}_k(x)\frac{t^k}{k!} = (1+x)\sum_{k=0}^{\infty} k\,\tilde{\beta}_{k-1}(x)\frac{t^k}{k!} - \sum_{k=0}^{\infty}\tilde{\beta}_k(x)\frac{t^k}{k!} + \sum_{k=0}^{\infty} x^k\,\frac{t^k}{k!}\,,$$

so that the recurrence (21) follows. ∎

5.2. Generating function's PDE

Theorem 5.2.

The generating function (7) satisfies the linear PDE

$$t\,\frac{\partial G}{\partial t} = \left[(1+x) - \frac{1}{t}\right]\frac{\partial G}{\partial x} + \frac{1}{e^t - 1}\frac{\partial G}{\partial x}\,, \qquad (t > 0)\,. \tag{23}$$

Proof:

It is sufficient to eliminate the exponential function between the partial derivatives stated in (11) and (13). ∎

Note that the last addend becomes negligible when t increases.

5.3. Differential equation

First, note that the following expansion holds:

$$\frac{te^t - e^t + 1}{t(e^t - 1)} = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots\,, \tag{24}$$

where the coefficients a_k are solution of the following triangular system

$$\begin{aligned}
&a_0 = \tfrac{1}{2},\\
&a_1 = \tfrac{1}{3} - \tfrac{a_0}{2!},\\
&a_2 = \tfrac{1}{8} - \tfrac{a_1}{2!} - \tfrac{a_0}{3!},\\
&\dots\\
&a_k = \tfrac{k+1}{(k+2)!} - \sum_{\ell=0}^{k-1} \frac{a_\ell}{(k+1-\ell)!},
\end{aligned} \tag{25}$$

so that, for the first few values, we find:

$$\begin{aligned}
&a_0 = \tfrac{1}{2}, && a_1 = \tfrac{1}{12},\\
&a_2 = 0, && a_3 = -\tfrac{1}{720},\\
&a_4 = a_{2h} = 0\,, \qquad \forall h \geq 1 &&\\
&a_5 = \tfrac{1}{30240}, && a_7 = -\tfrac{1}{1209600},\\
&a_9 = \tfrac{1}{479001160}, && a_{11} = -\tfrac{1}{1307674368000},
\end{aligned}$$

By applying the same result by Ben Cheikh (2003) (Corollary 3.2), and using expansion (24), we find that the multiplication operator for the quasi-monomial set $\left\{\tilde{\beta}_k(x)\right\}$ is given by

$$\hat{M} = \frac{A'(D_x)}{A(D_x)} + x = \sum_{k=0}^{\infty} a_k D_x^k + x\,. \tag{26}$$

Therefore, we have the result:

Theorem 5.3.

The adjoint Appell-Bernoulli polynomials of the first kind $\{\tilde{\beta}_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}\tilde{\beta}_n(x) = \left(\sum_{k=0}^{\infty} a_k D_x^{k+1} + x\,D_x\right)\tilde{\beta}_n(x) = n\,\tilde{\beta}_n(x)\,,$$

that is

$$\left(\sum_{k=0}^{n-1} a_k D_x^{k+1} + x\,D_x\right)\tilde{\beta}_n(x) = n\,\tilde{\beta}_n(x)\,, \tag{27}$$

because, for any fixed n, the series expansion in Equation (26) reduces to a finite sum when it is applied to a polynomial of degree n.

Remark 5.4.

The first few values of the adjoint Appell-Bernoulli polynomials of the first kind are as follows:

$$\begin{aligned}
&\tilde{\beta}_0(x) = 1,\\
&\tilde{\beta}_1(x) = x + \tfrac{1}{2},\\
&\tilde{\beta}_2(x) = x^2 + x + \tfrac{1}{3},\\
&\tilde{\beta}_3(x) = x^3 + \tfrac{3}{2}x^2 + x + \tfrac{1}{4},\\
&\tilde{\beta}_4(x) = x^4 + 2x^3 + 2x^2 + x + \tfrac{1}{5},\\
&\tilde{\beta}_5(x) = x^5 + \tfrac{5}{2}x^4 + \tfrac{10}{3}x^3 + \tfrac{5}{2}x^2 + x + \tfrac{1}{6},\\
&\tilde{\beta}_6(x) = x^6 + 3x^5 + 5x^4 + 5x^3 + 3x^2 + x + \tfrac{1}{7},\\
&\tilde{\beta}_7(x) = x^7 + \tfrac{7}{2}x^6 + 7x^5 + \tfrac{35}{4}x^4 + 7x^3 + \tfrac{7}{2}x^2 + x + \tfrac{1}{8},\\
&\tilde{\beta}_8(x) = x^8 + 4x^7 + \tfrac{28}{3}x^6 + 14x^5 + 14x^4 + \tfrac{28}{3}x^3 + 4x^2 + x + \tfrac{1}{9}.
\end{aligned}$$

Remark 5.5.

Note that the above scheme satisfy a particular symmetry. Writing the polynomials in the form

$$\tilde{\beta}_n(x) = d_{n,0}x^n + d_{n,1}x^{n-1} + \cdots + d_{n,n-1}x + d_{n,n}\,, \quad (n \geq 0)\,,$$

the coefficients $d_{n,h}$, $(0 \leq h \leq n)$ verify the conditions:

$$\begin{aligned}
&d_{n,0} = 1\,, \qquad \forall\, n \geq 0,\\
&d_{n,n-1} = 1\,, \qquad \forall\, n \geq 2,\\
&d_{n,n} = \tfrac{1}{n+1}\,, \qquad \forall\, n \geq 1,\\
&d_{h,k} = d_{h,h-k-1}\,, \quad \forall\, 1 \leq h \leq n-1\,,\ 1 \leq k \leq n-1\,, \quad (n \geq 4)\,.
\end{aligned}$$

6. Conclusion

We have introduced two set of adjoint Appell polynomials, called adjoint Euler and adjoint Bernoulli polynomials, in the framework of a general technique associating a new Appell polynomial set starting from a given one. We have derived their main characteristics by using the monomiality property and we have found their shift operators by means of a preceding result by Y. Ben Cheihk. Therefore, the differential equation satisfied by these polynomials have been shown.

It is worth to note that the relevant differential equations are of infinite order, but actually they reduce to finite order when applied to each polynomial.

For the classical Euler and Bernoulli polynomials the same result was proved in He and Ricci (2002), where a similar phenomenon was found for the recursion satisfied by the classical Euler and Bernoulli polynomials. In fact, it was noticed that the order of these recurrence relations increases with polynomial degree.

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