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Certain Quadruple Hypergeometric Series and their Integral Representations

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Abstract

While investigating the Exton's list of twenty one hyper-geometric functions of four variables and the Sharma's and Parihar's list of eighty three hyper-geometric functions of four variables, we noticed existence of new hyper-geometric series of four variables. The principal object of this paper is to introduce new hyper-geometric series of four variables and present a natural further step toward the mathematical integral presentation concerning these new series of four variables. Integral representations of Euler type and Laplace type involving Appell's hyper-geometric functions and the Horn's series of two variables, Exton's and Lauricella's triple functions and Sharma and Parihar hyper-geometric functions of four variables are established.

Keywords: Quadruple hyper-geometric functions; Integrals of Euler type; Laplace integral; Exton’s hyper-geometric functions; Appell functions

MSC 2010 No.: 33C20, 33C65

1. Introduction

Admittedly, the first basic problem in the study of multiple Gaussian hyper-geometric series (in n variables) involves the construction of the set of all such distinct series. There are many papers on this subject in the literature (for example Ahmad (2013), Appell and Kampé de Fériet (1926), Erdélyi et al. (1953), Exton (1976), Niukkanen (1983), Saran (1954), Srivastava (1985), Srivastava and Karlsson (1985) and Srivastava and Manocha (1984)). The problems concerning the construction of the sets of all distinct Gaussian hypergeometric series when $n=4$, become more and more involved. In addition to the Lauricella series $F_A^{(4)},..., F_D^{(4)}$ some examples of quadruple Gaussian hypergeometric series are considered by Exton (1972), (1973) and Karlsson (1976) studies certain 2m-dimensional series which, for $m=2$, would belong to the set of Gaussian hypergeometric series when $n=4$. Exton (1982) introduced twenty distinct triple hyper-geometric functions namely X_i $(i=1, 2, ..., 20)$. By the motivation of double and triple hyper-geometric functions, Exton (1976) defined twenty one complete hyper-geometric functions of four variables by symbols $K_1, K_2, ..., K_{21}$. Sharma and Parihar (1989) introduced eighty three complete hyper-geometric $F_1^{(4)}, F_2^{(4)}, ..., F_{83}^{(4)}$ of four variables. Each quadruple hyper-geometric series in Exton (1976) and Sharma and Parihar (1989) is of the form

$$X^{(4)}(.) = \sum_{m,n,p,q=0}^{\infty} \Lambda(m,n,p,q) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!},$$

where $\Lambda(m,n,p,q)$ is a certain sequence of complex parameters and there are twelve parameters in each series $X^{(4)}(.)$ (eight a's and four c's). The 1st, 2nd, 3rd and 4th parameters in $X^{(4)}(.)$ are connected with the integers m,n,p and q, respectively. Each repeated parameter in the series $X^{(4)}(.)$ points out a term with double parameters in $\Lambda(m,n,p,q)$. For example, $X^{(4)}(a_1,a_1,a_2,a_2,a_3,a_3,a_4,a_5)$ means that $\Lambda(m,n,p,q)$ includes the term $(a_1)_{m+n}(a_2)_{p+q}(a_3)_{m+n}(a_4)_p(a_5)_q$. Similarly, $X^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4)$ points out the term $(a_1)_{2m+n+q}(a_2)_n(a_3)_p(a_4)_{p+q}$ and $X^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_2,a_2)$ shows the existence of the term $(a_1)_{2m+n+p}(a_2)_{p+n+2q}$. Thus, it is possible to form various combinations of indices. There seems to be no way of establishing independently the number of distinct Gaussian hyper-geometric series for any given integer $n \geq 2$ without stating explicitly all such series. Thus, in every situation with $n = 4$, one ought to begin by actually constructing the set just as in the case $n = 3$ (see Srivastava and Karlsson (1985)). Motivated by this fact the fact that only a comparatively small number of quadruple Gaussian hyper-geometric series have appeared in the literature Bin-Saad and Younis (2018a), (2018b) and Bin-Saad et al. (2018a), (2018b) introduced a number of new quadruple series together with their basic properties.

2. New quadruple series and their integral representations

By using the conventions and notations above, we introduce the following five new quadruple hypergeometric series:

$$X_{26}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_1,c_1,c_1,c_2;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_n(a_3)_p(a_4)_{p+q}}{(c_1)_{m+n+p}(c_2)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1}$$

$$X_{27}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_2,c_1,c_1,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_n(a_3)_p(a_4)_{p+q}}{(c_1)_{n+p}(c_2)_m(c_3)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{2}$$

$$X_{28}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_4;c_1,c_1,c_2,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p+q}(a_2)_n(a_3)_p(a_4)_q}{(c_1)_{m+n}(c_2)_p(c_3)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{3}$$

$$X_{29}^{(4)}(a_1,a_1,a_3,a_4,a_1,a_2,a_5,a_6;c_1,c_2,c_1,c_1;x,y,z,u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_n(a_3)_p(a_4)_q(a_5)_p(a_6)_q}{(c_1)_{m+p+q}(c_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{4}$$

$$X_{30}^{(4)}(a_1,a_1,a_3,a_4,a_1,a_2,a_5,a_6;c,c,c,c;x,y,z,u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_n(a_3)_p(a_4)_q(a_5)_p(a_6)_q}{(c)_{m+n+p+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{5}$$

where the numerator and the denominator parameters are separated via a semicolon.

This paper is devoted to obtain several integral representations for these new quadruple functions defined above. In subsection 2.1, we present five integral representations of Euler-type for each series $X_i^{(4)}$ (i = 26, 27, 28, 29, 30) in terms of Appell's functions of two variables F_2 and F_3, the Horn's functions H_3 and H_4 of two variables, the Gaussian hyper-geometric function ${}_2F_1$, the Exton's triple series X_2, X_3, X_6, X_8, X_{16}, X_{17}, X_{18}, X_{19} and X_{20} (see Exton (1982)), the Lauricella's triple series $F_B^{(3)}, F_P$ and F_R, and the quadruple series $F_{48}^{(4)}$, $F_{53}^{(4)}$, $F_{73}^{(4)}$, $X_{27}^{(4)}$ and $F_C^{(4)}$. In the subsection 2.2, Laplace-type integrals are obtained for each series $X_i^{(4)}$ (i = 26, 27, 28, 29, 30).

2.1. Integral Representations of Euler-Type

We recall Gaussian hyper-geometric function ${}_2F_1$ defined by (see Srivastava and Karlsson (1985))

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} \quad (|x|<1).$$

Appell's hypergeometric functions F_2 and F_3 of two variables and the Horn's series H_3 and H_4 of two variables are given by

$$F_2(a,b,c;d,e;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{(d)_m(e)_n}\frac{x^m}{m!}\frac{y^n}{n!}, \quad (|x|+|y|<1),$$

$$F_3(a,b,c,d;e;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(b)_n(c)_m(d)_n}{(e)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, \quad (\max\{|x|,|y|\}<1),$$

$$H_3(a,b;c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(b)_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, \quad \left(|x|<r, |y|<s, \ r+\left(s-\frac{1}{2}\right)^2=\frac{1}{4}\right),$$

and

$$H_4(a,b;c,d;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(b)_n}{(c)_m(d)_n}\frac{x^m}{m!}\frac{y^n}{n!}, \quad \left(|x|<r, |y|<s, \ 4r=(s-1)^2\right),$$

Respectively, (see Srivastava and Karlsson (1985)). Exton's triple functions X_2, X_3, X_6,

X_8, X_{16}, X_{17}, X_{18}, X_{19} and X_{20} (see Exton (1982)) are defined as follows:

$$X_2(a,b;c,d,e;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+2n+p}(b)_p}{(c)_m(d)_n(e)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$X_3(a,b;c,d;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b)_{n+p}}{(c)_{m+n}(d)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$X_6(a,b,c;d,e;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b)_n(c)_p}{(d)_{m+n}(e)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$X_8(a,b,c;d,e,f;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b)_n(c)_p}{(d)_m(e)_n(f)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$X_{16}(a,b,c;d,e;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+p}(c)_p}{(d)_{m+p}(e)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$X_{17}(a,b,c;d,e,f;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+p}(c)_p}{(d)_m(e)_n(f)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$X_{18}(a,b,c,d;e;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_n(c)_p(d)_p}{(e)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$X_{19}(a,b,c,d;e,f;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_n(c)_p(d)_p}{(e)_m(f)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

and

$$X_{20}(a,b,c,d;e,f;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_n(c)_p(d)_p}{(e)_{m+p}(f)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}.$$

Lauricella's triple functions $F_B^{(3)}, F_P$ and F_R (see Saran (1954)) are given by

$$F_B^{(3)}(a_1,a_2,a_3,b_1,b_2,b_3;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_n(a_3)_p(b_1)_m(b_2)_n(b_3)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

$$F_P(a,b,a,c,c,d;e,f,f;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+p}(b)_n(c)_{m+n}(d)_p}{(e)_m(f)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

and

$$F_R(a,b,a,c,d,c;e,f,f;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+p}(b)_n(c)_{m+p}(d)_n}{(e)_m(f)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}.$$

Sharma and Parihar hyper-geometric functions of four variables $F_{48}^{(4)}$, $F_{53}^{(4)}$ and $F_{73}^{(4)}$ are as follows (see Sharma and Parihar (1989)):

$$F_{48}^{(4)}(a_1,a_1,a_2,a_2,b_1,b_1,b_2,b_3;c_1,c_2,c_1,c_2;x,y,z,u)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{p+q}(b_1)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+p}(c_2)_{n+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

$$F_{53}^{(4)}(a_1,a_1,a_2,a_3,b_1,b_1,b_2,b_3;c_1,c_2,c_1,c_2;x,y,z,u)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_p(a_3)_q(b_1)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+p}(c_2)_{n+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

and

$$F_{73}^{(4)}(a_1,a_1,a_2,a_3,b_1,b_1,b_2,b_3;c_1,c_2,c_1,c_1;x,y,z,u)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_p(a_3)_q(b_1)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+p+q}(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}.$$

Lauricella hyper-geometric function of four variables $F_C^{(4)}$ is recalled (see Srivastava and Karlsson (1985)) :

$$F_C^{(4)}(a,b;c_1,c_2,c_3,c_4;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{m+n+p+q}(b)_{m+n+p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!},$$

where

$$\left(\sqrt{x}+\sqrt{y}+\sqrt{z}+\sqrt{u}<1\right).$$

Now, by means of the Gauss hyper-geometric function ${}_2F_1$, Appell hyper-geometric functions F_2 and F_3, Horn's functions H_3 and H_4 of two variables, Exton's triple series X_2, X_3, X_6, X_8, X_{16}, X_{17}, X_{18}, X_{19} and X_{20}, Lauricella's triple series $F_B^{(3)}, F_P$ and F_R, and quadruple series $F_{48}^{(4)}$, $F_{53}^{(4)}$, $F_{73}^{(4)}$, $X_{27}^{(4)}$ and $F_C^{(4)}$, we investigate some further integral representations of Euler-type for $X_i^{(4)}$ (i = 26, 27, 28, 29, 30) as follows:

$$X_{26}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_1,c_1,c_1,c_2;x,y,z,u)=\frac{\Gamma(c_1)}{\Gamma(a)\Gamma(c_1-a)}\times\int_0^{\infty}\left(e^{-\alpha}\right)^a\left(1-e^{-\alpha}\right)^{c_1-a-1}$$
$$\times X_{27}^{(4)}\left(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_1-a,a,a,c_2;x\left(1-e^{-\alpha}\right),y\,e^{-\alpha},z\,e^{-\alpha},u\right)d\alpha, \tag{6}$$
$$\left(\mathrm{Re}(a)>0,\ \mathrm{Re}(c_1-a)>0\right);$$

$$X_{26}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_1,c_1,c_1,c_2;x,y,z,u)=\frac{4\,\Gamma(a_1+a_3)\,\Gamma(a_2+a_4)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}$$
$$\times\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left(\sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(\cos^2\alpha\right)^{a_3-\frac{1}{2}}\left(\sin^2\beta\right)^{a_2-\frac{1}{2}}\left(\cos^2\beta\right)^{a_4-\frac{1}{2}}$$
$$\times X_3\left(a_1+a_3,a_2+a_4;c_1,c_2;x\sin^4\alpha,\ y\sin^2\alpha\sin^2\beta+z\cos^2\alpha\cos^2\beta,u\sin^2\alpha\cos^2\beta\right)d\alpha\,d\beta, \tag{7}$$
$$\left(\mathrm{Re}(a_i)>0,\ i=1,2,3,4\right);$$

where X_3 is Exton function of three variables;

$$X_{26}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_1,c_1,c_1,c_2;x,y,z,u)=\frac{\Gamma(c_1)(S-T)^{a_2}(R-T)^{c_1-a_2}}{\Gamma(a_2)\Gamma(c_1-a_2)(S-R)^{c_1-a_1-1}}$$
$$\times\int_R^S(\alpha-R)^{a_2-1}(S-\alpha)^{c_1-a_2-1}(\alpha-T)^{a_1-c_1}\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)y\right]^{-a_1}$$
$$\times X_{16}(a_1,a_4,a_3;c_1,c_2;\lambda_1 x,\lambda_2 u,\lambda_3 z)\,d\alpha,$$
$$\left(\lambda_1=\frac{(S-R)(R-T)(S-\alpha)(\alpha-T)}{\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)y\right]^2},\ \lambda_2=\frac{(S-R)(\alpha-T)}{\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)y\right]},\right. \tag{8}$$
$$\left.\lambda_3=\frac{(R-T)(S-\alpha)}{\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)y\right]}\right),$$
$$\left(\mathrm{Re}(a_2)>0,\ \mathrm{Re}(c_1-a_2)>0, T<R<S\right);$$

where X_{16} is Exton function of three variables;

$$X_{26}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_1,c_1,c_1,c_2;x,y,z,u)=\frac{\Gamma(c_2)(1+M)^{a_4}}{\Gamma(a_4)\Gamma(c_2-a_4)}$$
$$\times\int_0^1\alpha^{a_4-1}(1-\alpha)^{c_2-a_4-1}(1+M\alpha)^{a_1-c_2}\left[(1+M\alpha)-(1+M)\alpha u\right]^{-a_1}$$
$$\times X_{18}\left(a_1,a_2,a_3,1+a_4-c_2;c_1;(\lambda_1)^2x,\lambda_1 y,\lambda_2 z\right)d\alpha \tag{9}$$
$$\left(\lambda_1=\frac{(1+M\alpha)}{\left[(1+M\alpha)-(1+M)\alpha u\right]},\ \lambda_2=-\frac{(1+M)\alpha}{(1-\alpha)}\right),$$
$$\left(\mathrm{Re}(a_4)>0, \mathrm{Re}(c_2-a_4)>0, M>-1\right);$$

where X_{18} is Exton function of three variables;

$$X_{26}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_1,c_1,c_1,c_2;x,y,z,u)=\frac{\Gamma(c_1)}{\Gamma(a_2)\Gamma(a_3)\Gamma(c_1-a_2-a_3)}$$
$$\times\int_0^1\int_0^1\alpha^{a_2-1}(1-\alpha)^{c_1-a_2-1}\beta^{a_3-1}(1-\beta)^{c_1-a_2-a_3-1}(1-\alpha y)^{-a_1}\left[1-(1-\alpha)\beta z\right]^{-a_4}$$
$$\times H_4\left(a_1,a_4;c_1-a_2-a_3,c_2;\frac{(1-\alpha)(1-\beta)x}{(1-\alpha y)^2},\frac{u}{(1-\alpha y)\left[1-(1-\alpha)\beta z\right]}\right)d\alpha\,d\beta, \tag{10}$$
$$\left(\mathrm{Re}(a_2)>0, \mathrm{Re}(a_3)>0, \mathrm{Re}(c_1-a_2)>0, \mathrm{Re}(c_1-a_2-a_3)>0\right),$$

where H_3 is Exton function of two variables;

$$X_{27}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_2,c_1,c_1,c_3;x,y,z,u)=\frac{\Gamma(a_1+a_2+a_3+a_4)\Gamma(c_1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)\Gamma(a)\Gamma(c_1-a)}$$
$$\times\int_0^1\int_0^1\int_0^1\int_0^1\alpha^{a_1-1}(1-\alpha)^{a_2-1}\beta^{a_1+a_2-1}(1-\beta)^{a_3-1}\gamma^{a_1+a_2+a_3-1}(1-\gamma)^{a_4-1}\zeta^{a-1}(1-\zeta)^{c_1-a-1}$$
$$\times F_C^{(4)}\left(\frac{a_1+a_2+a_3+a_4}{2},\frac{a_1+a_2+a_3+a_4+1}{2};c_2,a,c_1-a,c_3;\lambda_1 x,\lambda_2 y,\lambda_3 z,\lambda_4 u\right)d\alpha\,d\beta\,d\gamma\,d\zeta, \tag{11}$$
$$\left(\lambda_1=4\alpha^2\beta^2\gamma^2,\ \lambda_2=4\alpha\beta^2\gamma^2\zeta(1-\alpha),\ \lambda_3=4\gamma(1-\beta)(1-\gamma)(1-\zeta),\ \lambda_4=4\alpha\beta\gamma(1-\gamma)\right),$$
$$\left(\mathrm{Re}(a_i)>0\ (i=1,2,3,4), \mathrm{Re}(a)>0, \mathrm{Re}(c_1-a)>0\right),$$

where $F_C^{(4)}$ is Lauricella function of four variables;

$$X_{27}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_2,c_1,c_1,c_3;x,y,z,u)=\frac{\Gamma(a_1+a_4)\Gamma(a_2+a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}$$
$$\times\int_0^\infty\int_0^\infty \frac{\alpha^{a_1-1}}{(1+\alpha)^{a_1+a_4}}\frac{\beta^{a_2-1}}{(1+\beta)^{a_2+a_3}}$$
$$\times X_2\left(a_1+a_4,a_2+a_3;c_2,c_3,c_1;\frac{\alpha^2 x}{(1+\alpha)^2},\frac{\alpha u}{(1+\alpha)^2},\frac{\alpha\beta y}{(1+\alpha)(1+\beta)}+\frac{z}{(1+\alpha)(1+\beta)}\right)d\alpha\, d\beta, \tag{12}$$
$$\left(\mathrm{Re}(a_i)>0,\ i=1,2,3,4\right),$$

where X_2 is Exton function of three variables;

$$X_{27}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_2,c_1,c_1,c_3;x,y,z,u)=\frac{\Gamma(c_1)(S-T)^{a_2}(R-T)^{c_1-a_2}}{\Gamma(a_2)\Gamma(c_1-a_2)(S-R)^{c_1-a_1-1}}$$
$$\times\int_R^S(\alpha-R)^{a_2-1}(S-\alpha)^{c_1-a_2-1}(\alpha-T)^{a_1-c_1}\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)y\right]^{-a_1}$$
$$\times X_{17}\left(a_1,a_4,a_3;c_2,c_3,c_1-a_2;\lambda_1 x,\lambda_2 u,\lambda_3 z\right)d\alpha,.$$
$$\left(\lambda_1=\frac{(S-R)^2(\alpha-T)^2}{\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)y\right]^2},\ \lambda_2=\frac{(S-R)(\alpha-T)}{\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)y\right]},\right. \tag{13}$$
$$\left.\lambda_3=\frac{(R-T)(S-\alpha)}{\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)y\right]}\right),$$
$$\left(\mathrm{Re}(a_2)>0,\ \mathrm{Re}(c_1-a_2)>0, T<R<S\right);$$

$$X_{27}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_2,c_1,c_1,c_3;x,y,z,u)=\frac{\Gamma(c_3)}{\Gamma(a_4)\Gamma(c_3-a_4)}$$
$$\times\int_0^\infty\left(e^{-\alpha}\right)^{a_4}\left(1-e^{-\alpha}\right)^{c_3-a_4-1}\left(1-u\,e^{-\alpha}\right)^{-a_1}$$
$$\times X_{19}\left(a_1,a_2,a_3,1+a_4-c_3;c_2,c_1;\frac{x}{\left(1-u\,e^{-\alpha}\right)^2},\frac{y}{\left(1-u\,e^{-\alpha}\right)},-\frac{z\,e^{-\alpha}}{\left(1-e^{-\alpha}\right)}\right)d\alpha, \tag{14}$$
$$\left(\mathrm{Re}(a_4)>0,\ \mathrm{Re}(c_3-a_4)>0\right);$$

where X_{19} is Exton function of three variables;

$$X_{27}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_2,c_1,c_1,c_3;x,y,z,u)=\frac{2\,\Gamma(c_2)\,M^{a_1}}{\Gamma(a_1)\Gamma(c_2-a_1)}$$
$$\times\int_0^{\frac{\pi}{2}}\left(\sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(\cos^2\alpha\right)^{c_2-a_1-\frac{1}{2}}\left(\cos^2\alpha+M\sin^2\alpha\right)^{1+a_1-2c_2}$$
$$\times\left[\left(\cos^2\alpha+M\sin^2\alpha\right)+M^2x\sin^2\alpha\tan^2\alpha\right]^{c_2-a_1-1}$$
$$\times F_P\left(1+a_1-c_2,a_3,1+a_1-c_2,a_4,a_4,a_2;c_3,c_1,c_1;\lambda u,z,\lambda y\right)d\alpha,$$
$$\left(\lambda=-\frac{M\left(\cos^2\alpha+M\sin^2\alpha\right)\tan^2\alpha}{\left[\left(\cos^2\alpha+M\sin^2\alpha\right)+M^2x\sin^2\alpha\tan^2\alpha\right]}\right), \tag{15}$$
$$\left(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(c_2-a_1)>0,\ M>0\right);$$

where F_P is Saran function of three variables;

$$X_{28}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_4;c_1,c_1,c_2,c_3;x,y,z,u)=\frac{\Gamma(c_2)(S-T)^{a_3}(R-T)^{c_2-a_3}}{\Gamma(a_3)\Gamma(c_2-a_3)(S-R)^{c_2-a_1-1}}$$
$$\times\int_R^S(\alpha-R)^{a_3-1}(S-\alpha)^{c_2-a_3-1}(\alpha-T)^{a_1-c_2}\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)z\right]^{-a_1}$$
$$\times X_6(a_1,a_2,a_4;c_1,c_3;\lambda_1x,\lambda_2y,\lambda_2u)\,d\alpha,$$
$$\left(\lambda_1=\frac{(S-R)^2(\alpha-T)^2}{\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)z\right]^2},\lambda_2=\frac{(S-R)(\alpha-T)}{\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)z\right]}\right), \tag{16}$$
$$\left(\mathrm{Re}(a_3)>0,\ \mathrm{Re}(c_2-a_3)>0,T<R<S\right),$$

where X_6 is Exton function of three variables;

$$X_{28}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_4;c_1,c_1,c_2,c_3;x,y,z,u)=\frac{2\,\Gamma(c_1)(1+M)^{a_2}}{\Gamma(a_2)\Gamma(c_1-a_2)}$$
$$\times\int_0^{\frac{\pi}{2}}(\sin^2\alpha)^{a_2-\frac{1}{2}}(\cos^2\alpha)^{c_1-a_2-\frac{1}{2}}(1+M\sin^2\alpha)^{a_1-c_1}$$
$$\times\left[(1+M\sin^2\alpha)-(1+M)y\sin^2\alpha\right]^{-a_1}$$
$$\times X_8(a_1,a_3,a_4;c_1-a_2,c_2,c_3;\lambda_1x,\lambda_2z,\lambda_2u)\,d\alpha,$$
$$\left(\lambda_1=\frac{(1+M\sin^2\alpha)\cos^2\alpha}{\left[(1+M\sin^2\alpha)-(1+M)y\sin^2\alpha\right]^2},\lambda_2=\frac{(1+M\sin^2\alpha)}{\left[(1+M\sin^2\alpha)-(1+M)y\sin^2\alpha\right]}\right), \tag{17}$$
$$\left(\mathrm{Re}(a_2)>0,\mathrm{Re}(c_1-a_2)>0,M>-1\right);$$

where X_8 is Saran function of three variables;

$$X_{28}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_4;c_1,c_1,c_2,c_3;x,y,z,u)=\frac{\Gamma(c_1)(1+M)^{a_1}}{\Gamma(a_1)\Gamma(c_1-a_1)}$$
$$\times\int_0^1\alpha^{a_1-1}(1+M\alpha)^{1+a_1+a_2-2c_1}\left[(1-\alpha)(1+M\alpha)+(1+M)^2\alpha^2x\right]^{c_1-a_1-1}$$
$$\times\left[(1+M\alpha)-(1+M)\alpha y\right]^{-a_2}$$
$$\times F_2(1+a_1-c_1,a_3,a_4;c_2,c_3;\lambda z,\lambda u)\,d\alpha,$$
$$\left(\lambda=-\frac{(1+M)(1+M\alpha)\alpha}{\left[(1-\alpha)(1+M\alpha)+(1+M)^2\alpha^2x\right]}\right), \tag{18}$$
$$\left(\mathrm{Re}(a_1)>0,\mathrm{Re}(c_1-a_1)>0,M>-1\right);$$

where F_2 is Appell function of two variables;

$$
\begin{aligned}
&X_{28}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_4;c_1,c_1,c_2,c_3;x,y,z,u)\\
&=\frac{\Gamma(c_2)\,\Gamma(c_3)}{\Gamma(a_3)\,\Gamma(a_4)\,\Gamma(c_2-a_3)\,\Gamma(c_3-a_4)}\\
&\times\int_0^\infty\int_0^\infty \left(e^{-\alpha}\right)^{a_3}\left(1-e^{-\alpha}\right)^{c_2-a_3-1}\left(e^{-\beta}\right)^{a_4}\left(1-e^{-\beta}\right)^{c_3-a_4-1}\left(1-z\,e^{-\alpha}-u\,e^{-\beta}\right)^{-a_1}\\
&\times H_3\left(a_1,a_2;c_1;\frac{x}{\left(1-z\,e^{-\alpha}-u\,e^{-\beta}\right)^2},\frac{y}{\left(1-z\,e^{-\alpha}-u\,e^{-\beta}\right)}\right)d\alpha\,d\beta,\\
&\qquad\left(\mathrm{Re}(a_3)>0,\ \mathrm{Re}(a_4)>0,\ \mathrm{Re}(c_2-a_3)>0,\ \mathrm{Re}(c_3-a_4)>0\right);
\end{aligned}
\tag{19}
$$

where H_3 is Horn function of two variables;

$$
\begin{aligned}
&X_{28}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_4;c_1,c_1,c_2,c_3;x,y,z,u)\\
&=\frac{\Gamma(c_1)\,\Gamma(c_2)\,\Gamma(c_3)}{\Gamma(a_2)\,\Gamma(a_3)\,\Gamma(a_4)\,\Gamma(c_1-a_2)\,\Gamma(c_2-a_3)\,\Gamma(c_3-a_4)}\\
&\times\int_0^1\int_0^1\int_0^1 \alpha^{a_2-1}(1-\alpha)^{c_1-a_2-1}\,\beta^{a_3-1}(1-\beta)^{c_2-a_3-1}\,\gamma^{a_4-1}(1-\gamma)^{c_3-a_4-1}(1-\alpha\,y-\beta\,z-\gamma\,u)^{-a_1}\\
&\times{}_2F_1\left(\frac{a_1}{2},\frac{a_1+1}{2};c_1-a_2;\frac{4(1-\alpha)\,x}{(1-\alpha\,y-\beta\,z-\gamma\,u)^2}\right)d\alpha\,d\beta\,d\gamma,\\
&\qquad\left(\mathrm{Re}(a_i)>0\,(i=2,3,4)>0,\ \mathrm{Re}(c_1-a_2)>0,\ \mathrm{Re}(c_2-a_3)>0,\ \mathrm{Re}(c_3-a_4)>0\right);
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&X_{29}^{(4)}(a_1,a_1,a_3,a_4,a_1,a_2,a_5,a_6;c_1,c_2,c_1,c_1;x,y,z,u)=\frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\,\Gamma(a_2)}\times\int_0^1\alpha^{a_1-1}(1-\alpha)^{a_2-1}\\
&\times F_{73}^{(4)}\left(\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},a_3,a_4,\frac{a_1+a_2+1}{2},\frac{a_1+a_2+1}{2},a_5,a_6;c_1,c_2,c_1,c_1;4\,\alpha^2x,\ 4\,\alpha\,(1-\alpha)\,y,\ z,u\right)d\alpha\\
&\qquad\left(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_2)>0\right);
\end{aligned}
\tag{21}
$$

where $F_{73}^{(4)}$ is function of four variables;

$$
\begin{aligned}
&X_{29}^{(4)}(a_1,a_1,a_3,a_4,a_1,a_2,a_5,a_6;c_1,c_2,c_1,c_1;x,y,z,u)=\frac{2\,\Gamma(c_1)}{\Gamma(a_4)\,\Gamma(c_1-a_4)}\\
&\times\int_0^{\frac{\pi}{2}}\left(\sin^2\alpha\right)^{a_4-\frac{1}{2}}\left(\cos^2\alpha\right)^{c_1-a_4-\frac{1}{2}}\left(1-u\sin^2\alpha\right)^{-a_6}\\
&\times X_{20}\left(a_1,a_2,a_3,a_5;c_1-a_4,c_2;x\cos^2\alpha,\ y,\ z\cos^2\alpha\right)d\alpha,\\
&\qquad\left(\mathrm{Re}(a_4)>0,\ \mathrm{Re}(c_1-a_4)>0\right);
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
&X_{29}^{(4)}(a_1,a_1,a_3,a_4,a_1,a_2,a_5,a_6;c_1,c_2,c_1,c_1;x,y,z,u)=\frac{\Gamma(c_2)\,(S-T)^{a_2}\,(R-T)^{c_2-a_2}}{\Gamma(a_2)\,\Gamma(c_2-a_2)\,(S-R)^{c_2-a_1-1}}\\
&\times\int_R^S(\alpha-R)^{a_2-1}(S-\alpha)^{c_2-a_2-1}(\alpha-T)^{a_1-c_2}\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)\,y\right]^{-a_1}\\
&\times F_B^{(3)}\left(\frac{a_1}{2},a_3,a_4,\frac{a_1+1}{2},a_5,a_6;c_1;\lambda x,z,u\right)d\alpha,\\
&\left(\lambda=\frac{4(S-R)^2(\alpha-T)^2}{\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)\,y\right]^2}\right),\\
&\qquad\left(\mathrm{Re}(a_2)>0,\ \mathrm{Re}(c_2-a_2)>0,T<R<S\right);
\end{aligned}
\tag{23}
$$

$$X_{29}^{(4)}(a_1,a_1,a_3,a_4,a_1,a_2,a_5,a_6;c_1,c_2,c_1,c_1;x,y,z,u)=\frac{\Gamma(a_1+a_2)\Gamma(a_3+a_4)\Gamma(a_5+a_6)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)}$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty \left(e^{-\alpha}\right)^{a_1}\left(e^{-\beta}\right)^{a_3}\left(e^{-\gamma}\right)^{a_5}\left(1-e^{-\alpha}\right)^{a_2-1}\left(1-e^{-\beta}\right)^{a_4-1}\left(1-e^{-\gamma}\right)^{a_6-1}$$
$$\times F_R\left(\frac{a_1+a_2}{2},a_3+a_4,\frac{a_1+a_2}{2},\frac{a_1+a_2+1}{2},a_5+a_6,\frac{a_1+a_2+1}{2};c_2,c_1,c_1;\lambda_1 y,\lambda_2 z+\lambda_3 u,\lambda_4 x\right)d\alpha\,d\beta\,d\gamma, \tag{24}$$
$$\left(\lambda_1=4e^{-\alpha}\left(1-e^{-\alpha}\right),\lambda_2=e^{-(\beta+\gamma)},\lambda_3=\left(1-e^{-\beta}\right)\left(1-e^{-\gamma}\right),\lambda_4=4e^{-2\alpha}\right),$$
$$\left(\mathrm{Re}(a_i)>0, i=1,2,3,4,5,6\right);$$

$$X_{29}^{(4)}(a_1,a_1,a_3,a_4,a_1,a_2,a_5,a_6;c_1,c_2,c_1,c_1;x,y,z,u)$$
$$=\frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_2)\Gamma(a_5)\Gamma(a_6)\Gamma(c_2-a_2)\Gamma(c_1-a_5-a_6)}$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty \frac{\alpha^{a_2-1}}{(1+\alpha)^{c_2-a_1}}\frac{\beta^{a_5-1}}{(1+\beta)^{c_1-a_3-a_4}}\frac{\gamma^{a_6-1}}{(1+\gamma)^{c_1-a_4-a_5}}$$
$$\times\left[(1+\alpha)-\alpha y\right]^{-a_1}\left[(1+\beta)-\beta z\right]^{-a_3}\left[(1+\beta)(1+\gamma)-\gamma u\right]^{-a_4} \tag{25}$$
$$\times{}_2F_1\left(\frac{a_1}{2},\frac{a_1+1}{2};c_1-a_5-a_6;\frac{4(1+\alpha)^2 x}{(1+\beta)(1+\gamma)\left[(1+\alpha)-\alpha y\right]^2}\right)d\alpha\,d\beta\,d\gamma,$$
$$\left(\mathrm{Re}(a_i)>0\ (i=2,5,6),\mathrm{Re}(c_1-a_5)>0,\mathrm{Re}(c_2-a_2)>0,\mathrm{Re}(c_1-a_5-a_6)>0\right);$$

$$X_{30}^{(4)}(a_1,a_1,a_3,a_4,a_1,a_2,a_5,a_6;c,c,c,c;x,y,z,u)=\frac{8\Gamma(a_1+a_2)\Gamma(a_3+a_4)\Gamma(c)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)\Gamma(a)\Gamma(c-a)}$$
$$\times\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left(\sin^2\alpha\right)^{a_1-\frac12}\left(\cos^2\alpha\right)^{a_2-\frac12}\left(\sin^2\beta\right)^{a_3-\frac12}\left(\cos^2\beta\right)^{a_4-\frac12}\left(\sin^2\gamma\right)^{a-\frac12}\left(\cos^2\gamma\right)^{c-a-\frac12}$$
$$\times F_{48}^{(4)}\left(\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},a_3+a_4,a_3+a_4,\frac{a_1+a_2+1}{2},\frac{a_1+a_2+1}{2},a_5,a_6;\right. \tag{26}$$
$$\left.a,c-a,a,c-a;\lambda_1 x,\lambda_2 y,\lambda_3 z,\lambda_4 u\right)\ d\alpha\,d\beta\,d\gamma,$$
$$\left(\lambda_1=4\sin^4\alpha\sin^2\gamma,\ \lambda_2=\sin^2 2\alpha\cos^2\gamma,\ \lambda_3=\sin^2\beta\sin^2\gamma,\lambda_4=\cos^2\beta\cos^2\gamma\right),$$
$$\left(\mathrm{Re}(a_i)>0\ (i=1,2,3,4),\mathrm{Re}(a)>0,\mathrm{Re}(c-a)>0\right);$$

$$X_{30}^{(4)}(a_1,a_1,a_3,a_4,a_1,a_2,a_5,a_6;c,c,c,c;x,y,z,u)=\frac{\Gamma(a_1+a_2)\Gamma(c)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a)\Gamma(c-a)}$$
$$\times\int_0^1\int_0^1\alpha^{a_1-1}(1-\alpha)^{a_2-1}\beta^{a-1}(1-\beta)^{c-a-1}$$
$$\times F_{53}^{(4)}\left(\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},a_3,a_4,\frac{a_1+a_2+1}{2},\frac{a_1+a_2+1}{2},a_5,a_6;a,c-a,a,c-a;\lambda_1 x,\lambda_2 y,\lambda_3 z,\lambda_4 u\right)d\alpha\,d\beta \tag{27}$$
$$\left(\lambda_1=4\alpha^2\beta,\ \lambda_2=4\alpha(1-\alpha)(1-\beta),\lambda_3=\beta,\lambda_4=(1-\beta)\right),$$
$$\left(\mathrm{Re}(a_1)>0,\mathrm{Re}(a_2)>0,\mathrm{Re}(a)>0,\mathrm{Re}(c-a)>0\right);$$

$$X_{30}^{(4)}(a_1,a_1,a_3,a_4,a_1,a_2,a_5,a_6;c,c,c,c;x,y,z,u)=\frac{\Gamma(c)(1+M)^{a_4}}{\Gamma(a_4)\Gamma(c-a_4)}$$
$$\times\int_0^1\alpha^{a_4-1}(1-\alpha)^{c-a_4-1}(1+M\alpha)^{a_6-c}\left[(1+M\alpha)-(1+M)\alpha u\right]^{-a_6} \tag{28}$$
$$\times X_{18}\left(a_1,a_2,a_3,a_5;c-a_4;\lambda x,\lambda y,\lambda z\right)d\alpha,\ \left(\lambda=\frac{(1-\alpha)}{(1+M\alpha)}\right),$$
$$\left(\mathrm{Re}(a_4)>0,\mathrm{Re}(c-a_4)>0,M>-1\right);$$

$$X_{30}^{(4)}(a_1,a_1,a_3,a_4,a_1,a_2,a_5,a_6;c,c,c,c;x,y,z,u)=\frac{\Gamma(a_1+a_2+a_3+a_5)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_5)}$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty \left(e^{-\alpha}\right)^{a_1}\left(1-e^{-\alpha}\right)^{a_2-1}\left(e^{-\beta}\right)^{a_1+a_2}\left(1-e^{-\beta}\right)^{a_3-1}\left(e^{-\gamma}\right)^{a_1+a_2+a_3}\left(1-e^{-\gamma}\right)^{a_5-1}$$
$$\times F_3\left(\frac{a_1+a_2+a_3+a_5}{2},a_4,\frac{a_1+a_2+a_3+a_5+1}{2},a_6;c;\lambda_1 x+\lambda_2 y+\lambda_3 z,u\right)d\alpha\, d\beta\, d\gamma \tag{29}$$
$$\left(\lambda_1=4e^{-2(\alpha+\beta+\gamma)},\lambda_2=4e^{-(\alpha+2\beta+2\gamma)}\left(1-e^{-\alpha}\right),\lambda_3=4e^{-\gamma}\left(1-e^{-\beta}\right)\left(1-e^{-\gamma}\right)\right),$$
$$\left(\mathrm{Re}(a_i)>0,\ i=1,2,3,5\right);$$

$$X_{30}^{(4)}(a_1,a_1,a_3,a_4,a_1,a_2,a_5,a_6;c,c,c,c;x,y,z,u)=\frac{\Gamma(c)}{\Gamma(a_5)\Gamma(a_6)\Gamma(c-a_5-a_6)}$$
$$\times\int_0^\infty\int_0^\infty\frac{\alpha^{c-a_5-1}}{(1+\alpha)^{c-a_3-a_4}}\frac{\beta^{c-a_5-a_6-1}}{(1+\beta)^{c-a_4-a_5}}\times\left[(1+\alpha)-z\right]^{-a_3}\left[(1+\alpha)(1+\beta)-\alpha u\right]^{-a_4} \tag{30}$$
$$\times H_3\left(a_1,a_2;c-a_5-a_6;\frac{\alpha\beta x}{(1+\alpha)(1+\beta)},\frac{\alpha\beta y}{(1+\alpha)(1+\beta)}\right)d\alpha\, d\beta,$$
$$\left(\mathrm{Re}(a_5)>0,\ \mathrm{Re}(a_6)>0,\ \mathrm{Re}(c-a_5)>0,\ \mathrm{Re}(c-a_5-a_6)>0\right).$$

***Proof of formulas* (6) to (30) :**

Once substituting the series definition of the special function in each integrand and then, changing the order of integral and summation, and finally taking into account the following integral representations of Beta function and their various associated Eulerian integrals (see, e.g., Erdélyi et al. (1953), Srivastava and Choi (2001), (2012) and Srivastava and Manocha (1984)), we derive each of integral representations from (6) to (30).

$$B(a,b)=\begin{cases}\int_0^1 t^{a-1}(1-t)^{b-1}dt & \left(\mathrm{Re}(a)>0,\mathrm{Re}(b)>0\right),\\ \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} & \left(a,b\in C\setminus Z_0^-\right),\end{cases} \tag{31}$$

$$B(a,b)=\int_0^1\alpha^{a-1}(1-\alpha)^{b-1}d\alpha=\int_0^\infty\left(e^{-\alpha}\right)^a\left(1-e^{-\alpha}\right)^{b-1}d\alpha,$$
$$\left(\mathrm{Re}(a)>0,\mathrm{Re}(b)>0\right), \tag{32}$$

$$B(a,b)=2\int_0^{\frac{\pi}{2}}(\sin\alpha)^{2a-1}(\cos\alpha)^{2b-1}d\alpha=\int_0^\infty\frac{\alpha^{a-1}}{(1+\alpha)^{a+b}}d\alpha, \tag{33}$$
$$\left(\mathrm{Re}(a)>0,\mathrm{Re}(b)>0\right),$$

$$B(a,b)=\frac{(S-T)^a(R-T)^b}{(S-R)^{a+b-1}}\int_R^S\frac{(\alpha-R)^{a-1}(S-\alpha)^{b-1}}{(\alpha-T)^{a+b}}d\alpha\quad(T<R<S)$$
$$=(1+M)^a\int_0^1\frac{\alpha^{a-1}(1-\alpha)^{b-1}}{(1+M\alpha)^{a+b}}d\alpha, \tag{34}$$
$$(M>-1),\left(\mathrm{Re}(a)>0,\mathrm{Re}(b)>0\right).$$

2.2. Integrals of Laplace-Type

We present integral representations of Laplace transform type for $X_{26}^{(4)}$, $X_{27}^{(4)}$, $X_{28}^{(4)}$, $X_{29}^{(4)}$, $X_{30}^{(4)}$. The Laplace integral representations of these quadruple series are given as follows:

$$\begin{aligned} &X_{26}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_1,c_1,c_1,c_2;x,y,z,u) \\ &=\frac{1}{\Gamma(a_1)\Gamma(a_4)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}t^{a_4-1} \\ &\times\Phi_3^{(3)}\left(a_2,a_3;c_1;sy,tz,s^2x\right){}_0F_1\left(-;c_2;stu\right)ds\,dt\,, \\ &\left(\operatorname{Re}(a_1)>0,\operatorname{Re}(a_4)>0\right); \end{aligned} \tag{35}$$

$$\begin{aligned} &X_{27}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_2,c_1,c_1,c_3;x,y,z,u) \\ &=\frac{1}{\Gamma(a_1)\Gamma(a_4)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}t^{a_4-1} \\ &\times\Phi_2\left(a_2,a_3;c_1;sy,tz\right)\times{}_0F_1\left(-;c_2;s^2x\right) \\ &\times{}_0F_1\left(-;c_3;stu\right)ds\,dt, \\ &\left(\operatorname{Re}(a_1)>0,\operatorname{Re}(a_4)>0\right); \end{aligned} \tag{36}$$

$$\begin{aligned} &X_{28}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_4;c_1,c_1,c_2,c_3;x,y,z,u) \\ &=\frac{1}{\Gamma(a_1)}\int_0^\infty e^{-s}s^{a_1-1}\times\Phi_3\left(a_2;c_1;sy,s^2x\right)\times{}_1F_1\left(a_3;c_2;sz\right), \\ &\times{}_1F_1\left(a_4;c_3;su\right)ds,\quad\left(\operatorname{Re}(a_1)>0\right); \end{aligned} \tag{37}$$

$$\begin{aligned} &X_{29}^{(4)}(a_1,a_1,a_3,a_4,a_1,a_2,a_5,a_6;c_1,c_2,c_1,c_1;x,y,z,u) \\ &=\frac{1}{\Gamma(a_1)\Gamma(a_3)\Gamma(a_4)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}s^{a_1-1}t^{a_3-1}v^{a_4-1} \\ &\times\Phi_3^{(3)}\left(a_5,a_6;c_1;tz,vu,s^2x\right)\times{}_1F_1\left(a_2;c_2;sy\right)ds\,dt\,dv, \\ &\left(\operatorname{Re}(a_1)>0,\operatorname{Re}(a_3)>0,\operatorname{Re}(a_4)>0\right); \end{aligned} \tag{38}$$

$$\begin{aligned} &X_{30}^{(4)}(a_1,a_1,a_3,a_4,a_1,a_2,a_5,a_6;c,c,c,c;x,y,z,u) \\ &=\frac{1}{\Gamma(a_1)\Gamma(a_5)\Gamma(a_6)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}s^{a_1-1}t^{a_5-1}v^{a_6-1} \\ &\times\Phi_3^{(4)}\left(a_2,a_3,a_4;c;sy,tz,vu,s^2x\right)\,ds\,dt\,dv, \\ &\left(\operatorname{Re}(a_1)>0,\operatorname{Re}(a_5)>0,\operatorname{Re}(a_6)>0\right); \end{aligned} \tag{39}$$

where ${}_0F_1$, ${}_1F_1$, Φ_2, Φ_3, $\Phi_3^{(3)}$ and $\Phi_3^{(4)}$ are the confluent hyper-geometric functions defined, respectively, by

$${}_0F_1(-;c;x)=\sum_{m=0}^\infty\frac{1}{(c)_m}\frac{x^m}{m!},$$

$${}_1F_1(a;c;x)=\sum_{m=0}^\infty\frac{(a)_m}{(c)_m}\frac{x^m}{m!},$$

$$\Phi_2(a,b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$\Phi_3(a;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$\Phi_3^{(3)}(a,b;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_m(b)_n}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}$$

and

$$\Phi_3^{(4)}(a,b,d;c;x,y,z)=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_m(b)_n(d)_p}{(c)_{m+n+p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}.$$

***Proof of formulas* (35) to (39) :**

It is noted that each of the integral representations (35) to (39) can be proved mainly by expressing the series definition of the involved special functions in each integrand and changing the order of the integral sign and the summation, and finally using the following well-known integral formula (see Erdélyi et al. (1953)).

3. Conclusion

We have considered the problem of introducing new hyper-geometric series of four variables and the establishing of integral representations of Euler and Laplace types for these new series. The study of hyper-geometric series of four variables for applications as well as for its connections with various other hyper-geometric series is an interesting problem for further research. We conclude this investigation by remarking that the schema suggested in the derivation of the results in this work can be applied to find other new hyper-geometric series of four variables and study their properties integral representations, generating functions, operational relations, expansions and so on, and discuss the link with various hyper-geometric series.

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