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Series of divergence measures of type k, information inequalities and particular cases

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Abstract

Information and Divergence measures deals with the study of problems concerning information processing, information storage, information retrieval and decision making. The purpose of this paper is to find a new series of divergence measures and their applications, discuss the mathematical tools for finding convexity of the functions. Applications of convex functions in information theory, relationship between new and well-known divergence measures are discussed. Also some new bounds have been established for divergence measures using new *f* divergence measures and its properties.

Keywords: Directed Divergence; Convex functions; new f-divergence; Hellinger discrimination; Kullback-Leibler divergence measure; Entropy; Jensen-Shannon divergence measure

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1. Introduction

The first person; Harry Nyquist (1924) and (1928), Hartley (1928) discovered the logarithmic nature of measure of information. Harry Nyquist published the paper 'Certain Factors Affecting Telegraph Speed' in which discussed the relation $W = K \log m$ where W is the speed of transmission of intelligence, m is the number of difference voltage levels to choose from at each time step and K is a constant. Information Theory is the intersection of mathematics, statistics, computer science, physics, neurobiology, and electrical engineering. Its impact has been crucial to the success of the voyager missions to deep space, the invention of the compact disc, the feasibility of mobile phones, the development of the Internet, the study of linguistics and of human perception, the understanding of black holes and numerous other fields.

As a generalization of the uncertainty theory based on the notion of possibility; Hartley (1928), Information Theory consider the uncertainty of randomness perfectly. The concept of Shannon (1948) entropy is the central concept of Information Theory. Sometimes this measure or entropy is referred to as the measure of uncertainty. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. Shannon's model used the formalized language of the classical set theory, so it is only suitable to be used in limitation of classical set theory.

As to the divergence and inaccuracy of information; Kullback et al. (1951) studied a measure of information from statistical aspects of view involving two probability distributions associated with the same experiment, calling discrimination function, later different authors named as cross entropy, relative information etc. It is a non-symmetric measure of the difference between two probability distributions P and Q. Shannon (1948) introduced the following measure of information

$$H_n(P) = \sum_{i=1}^{n} p_i \log p_i \,, \tag{1.1}$$

for all $P \in \Gamma_n$, where

$$\Gamma_n = \left\{ P = (p_1, p_2, \ldots, p_n) \,\middle|\, p_i \geq 0,\ \sum_{i=1}^{n} p_i = 1, n \geq 2 \right\} .$$

Denote the set of all finite discrete complete and generalized probability distributions respectively. The equation (1.1) is Shannon's entropy. The function $H_n(P)$ represents the expected value of uncertainty associated with the given probability distributions. It is uniquely determined by some rather natural postulates.

Divergence measure is a distance or affinity between two probability distribution. Kullback et al. (1951) studied a measure of information from statistical of view and given by

$$K(P,Q) = \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right), \tag{1.2}$$

where $P, Q \in \Gamma_n$.

The measure (1.2) has many names given by different authors such as relative information, Directed Divergence, cross entropy, measure of discrimination etc. As a generalization of the uncertainty theory based on the notion of possibility, information theory considers the uncertainty of randomness perfectly. Jain et al. (2012 and 2013) introduced new f-divergence measure and its properties which is given by

$$S_f(P,Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i + q_i}{2q_i}\right), \tag{1.3}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a convex function and $P, Q \in \Gamma_n$. Here we have discussed well-known divergence measures like as symmetric chi-square divergence; Dragomir et al. (2001), relative

Jensen-Shannon divergence; Sibson (1969), relative arithmetic-geometric divergence; Taneja et al. (2000), Hellinger discrimination; Hellinger (1909). These divergence measure, we can also obtain by using new f-divergence measure which are as following:

(i) If $f(t)=\frac{t(t-1)^2}{(2t-1)}, \forall t>\frac{1}{2}$, then, the symmetric chi-square divergence is given by

$$S_f(P,Q)=\frac{1}{8}\left[\sum_{i=1}^{n}\frac{(p_i+q_i)(p_i-q_i)^2}{p_i q_i}\right]=\frac{1}{8}\Psi(P,Q). \tag{1.4}$$

(ii) If $f(t)=-\log(t)$, then, the relative Jensen-Shannon divergence measure is given by

$$S_f(P,Q)=\sum_{i=1}^{n} q_i \log\left(\frac{2q_i}{p_i+q_i}\right)=F(Q,P). \tag{1.5}$$

(iii) If $f(t)=t\log(t)$, then, the relative arithmetic-geometric divergence measure is given by

$$S_f(P,Q)=\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2q_i}\right)=G(Q,P) . \tag{1.6}$$

(iv) If $f(t)=(1-\sqrt{t})$, then, the Hellinger discrimination is given by

$$S_f(P,Q)=\left[1-B\left(\frac{P+Q}{2},Q\right)\right]=h\left(\frac{P+Q}{2},Q\right). \tag{1.7}$$

2. New Information Divergence Measure

Here, we will discuss about new convex function which is satisfied the properties of convex functions. Let us consider the function $f:(0.5,\infty)\to\mathbb{R}$ given by

$$f_k(t)=\frac{t(t-1)^{2k}}{(2t-1)^k},\ \forall t>\frac{1}{2}, \tag{2.1}$$

$$f'_k(t)=\frac{(t-1)^{2k-1}}{(2t-1)^{k+1}}\left[(2k+2)t^2-3t+1\right], \tag{2.2}$$

$$f_k''(t)=\frac{2k(t-1)^{2k-2}t}{(2t-1)^{k+2}}[(2k+2)t^2-6t+3]. \tag{2.3}$$

Therefore, the function $f_k(t)$ is always convex, $\forall\ t>1/2$ and normalized $f(1)=0$. Using new f-divergence measure, we get

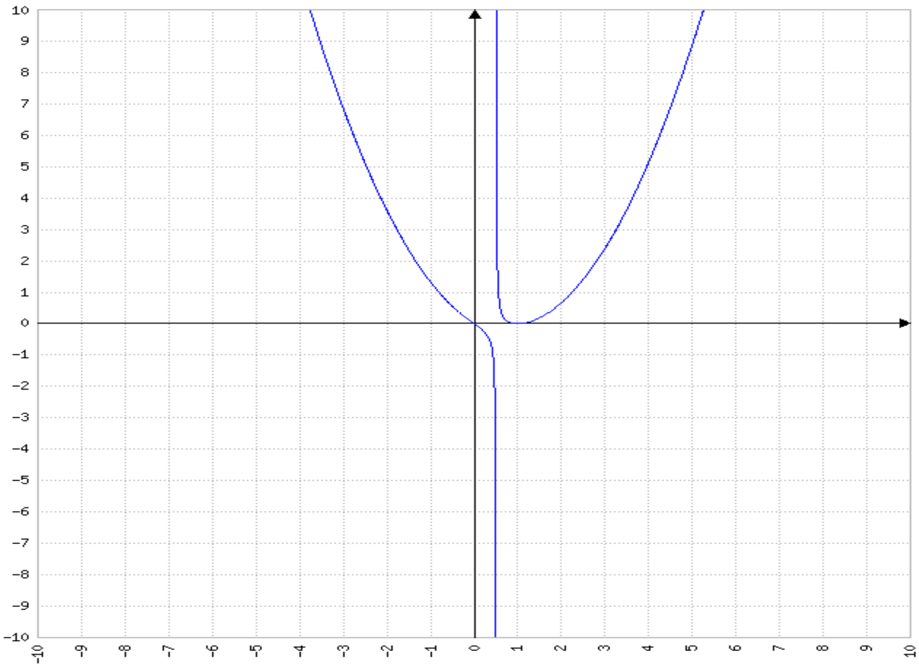

Figure 2.1. Graph of function $f_k(t) = \dfrac{t(t-1)^{2k}}{(2t-1)^k}$, $\forall t > 1/2$ when $k = 1$

$$S_f(P,Q) = \frac{1}{2^{2k+1}} \sum_{i=1}^{n} \frac{(p_i + q_i)(p_i - q_i)^{2k}}{(p_i q_i)^k} = N_k^*(P,Q), \tag{2.4}$$

If $k = 1$,

$$S_{f_1}(P,Q) = \frac{1}{8} \sum_{i=1}^{n} \frac{(p_i + q_i)(p_i - q_i)^2}{p_i q_i} = \frac{1}{8} \Psi(P,Q) = N_1^*(P,Q), \tag{2.5}$$

where $\Psi(P,Q)$ is known as symmetric chi-square divergence measure.

If $k = 2$,

$$S_{f_2}(P,Q) = \frac{1}{32} \sum_{i=1}^{n} \left(\frac{(p_i + q_i)(p_i - q_i)^4}{(p_i q_i)^2} \right) = N_2^*(P,Q). \tag{2.6}$$

If $k = 3$,

$$S_{f_3}(P,Q) = \frac{1}{128} \sum_{i=1}^{n} \frac{(p_i + q_i)(p_i - q_i)^6}{(p_i q_i)^3} = N_3^*(P,Q), \ldots . \tag{2.7}$$

3. Series of Divergence Measures

In this section, we will discuss about different new information divergence measure with the help of new f-divergence measure and properties of convex functions. From equation (2.1)

$$F(t)=\frac{t\,(t-1)^{2k}}{(2t-1)^{k}}, \tag{3.1}$$

is convex for $k=1,2,3,4,\ldots$.

$$F_1(t)=\frac{t\,(t-1)^{2}}{(2t-1)},\ \ k=1,$$

$$F_2(t)=\frac{t\,(t-1)^{4}}{(2t-1)^{2}},\quad k=2,$$

$$F_3(t)=\frac{t\,(t-1)^{6}}{(2t-1)^{3}},\quad k=3,$$

$$F_4(t)=\frac{t\,(t-1)^{8}}{(2t-1)^{4}},\ \ k=4,\ \ldots\ .$$

Further, we know that if,

$$F_1(t),\ F_2(t),\ F_3(t),\ F_4(t),\ \ldots$$

are convex functions. Then, the linear combination of these functions

$$c_1F_1(t)+c_2F_2(t)+c_3F_3(t)+c_4F_4(t)+,\ \ldots$$

is also convex functions, where $c_1, c_2, c_3, c_4, \ldots$ are positive constants such that at least one c_i is not equal to zero. Now taking

$$c_1=1,\ c_2=2,\ c_2=\frac{1}{2!},\ c_4=\frac{1}{3!},\ \ldots\ .,$$

$$F^*(t)=\left[\frac{t(t-1)^2}{(2t-1)}+\frac{t(t-1)^4}{(2t-1)^2}+\frac{1}{2!}\frac{t(t-1)^6}{(2t-1)^3}+\frac{1}{3!}\frac{t(t-1)^8}{(2t-1)^4},\ \ldots\ .\right],$$

$$F^*(t)=\frac{t\,(t-1)^2}{(2t-1)}\left[1+\frac{(t-1)^2}{(2t-1)}+\frac{(t-1)^4}{2!(2t-1)^2}+\frac{(t-1)^6}{3!(2t-1)^3},\ \ldots\ .\right],$$

$$F^*(t)=\frac{t\,(t-1)^2}{(2t-1)}\,exp\left\{\frac{(t-1)^2}{(2t-1)}\right\}, \tag{3.2}$$

from the properties of new f divergence measures, we get

$$S_{F^*}(P,Q)=\frac{1}{8}\sum\frac{(p_i+q_i)(p_i-q_i)^2}{p_i\,q_i}\exp\left\{\frac{(p_i-q_i)^2}{4p_i\,q_i}\right\}. \tag{3.3}$$

It may be said combination of Symmetric Chi-Square divergence and Chi-Square divergence measure. Again, if, we take $c_1=1,\ c_2=1,\ c_3=\frac{1}{2!},\ c_4=\frac{1}{3!},\ \ldots$, then

$$F_1=\frac{t(t-1)^4}{(2t-1)^2},\quad F_2=\frac{t(t-1)^6}{(2t-1)^3},\quad F_3=\frac{t(t-1)^8}{(2t-1)^4},\ F_4=\frac{t(t-1)^{10}}{(2t-1)^5},\ \ldots,$$

$$F^*(t)=\left[\frac{t(t-1)^4}{(2t-1)^2}+\frac{t(t-1)^6}{(2t-1)^3}+\frac{t(t-1)^8}{(2t-1)^4}+\frac{t(t-1)^{10}}{(2t-1)^5},\ \ldots.\right],$$

$$F^*(t)=\frac{t(t-1)^4}{(2t-1)^2}\left[1+\frac{(t-1)^2}{(2t-1)}+\frac{(t-1)^4}{2!(2t-1)^2}+\frac{(t-1)^6}{3!(2t-1)^3},\ \ldots.\right],$$

$$F^*(t)=\frac{t(t-1)^4}{(2t-1)^2}\exp\left\{\frac{(t-1)^2}{(2t-1)}\right\}.$$

Then, we obtain the following divergence measure of new f-divergences class

$$S_{F^*}(P,Q)=\frac{1}{32}\sum\frac{(p_i+q_i)(p_i-q_i)^4}{(p_i\,q_i)^2}\exp\left\{\frac{(p_i-q_i)^2}{4p_i\,q_i}\right\}.$$

Similarly, we get the following series of divergence measure

$$S_F^*(P,Q)=\frac{1}{2^{2k+1}}\sum\frac{(p_i+q_i)(p_i-q_i)^{2k}}{(p_iq_i)^k}\exp\left\{\frac{(p_i-q_i)^2}{4p_i\,q_i}\right\},\ k=1,2,3,4,\ldots.$$

(ii) If, we take $c_1=1,\ c_2=1,\ c_3=c_4=c_5=0,\ \ldots$, then, we have

$$F_{1,2}(t)=F_1(t)+F_2(t)=\frac{t(t-1)^2}{(2t-1)}+\frac{t(t-1)^4}{(2t-1)^2}=\frac{t^3(t-1)^2}{(2t-1)^2}. \tag{3.4}$$

Similarly, if, we take $c_1=1,\ c_2=1,\ c_3=c_4=c_5=0,\ \ldots$, then

$$F_{2,3}(t)=F_2(t)+F_3(t)=\frac{t(t-1)^4}{(2t-1)^2}+\frac{t(t-1)^6}{(2t-1)^3}=\frac{t^3(t-1)^4}{(2t-1)^3}, \tag{3.5}$$

In this way, we can write for $k=1,2,3,\ldots$

$$F_{k,k+1}(t)\ ,= F_k(t)+F_{k+1}(t)=\frac{t(t-1)^{2k}}{(2t-1)^k}+\frac{t(t-1)^{2k+2}}{(2t-1)^{k+1}}=\frac{t^3(t-1)^{2k}}{(2t-1)^{k+1}}, \tag{3.6}$$

$$S_{F_{k,k+1}}(P,Q)=\frac{1}{2^{2k+3}}\sum\frac{(p_i+q_i)^3(p_i-q_i)^{2k}}{(p_iq_i)^{k+1}}\ . \tag{3.7}$$

4. New information inequalities

Various mathematicians and researchers have been used different inequalities for a variety of purposes. Taneja has been used the applications of inequalities for obtaining bounds on symmetric and non-symmetric divergence measures in terms of relative information of type's; Taneja et al. (2000), for obtaining relationships among mean divergence measures; Taneja et al. (2004), for obtaining bounds on symmetric divergence measures in terms of non-symmetric divergence measures; Taneja et al. (2000 and 2004). Unified and generalized three theorems studied by Dragomir et al. (2001 and 2016) and Cerone et al. (2016) which provide bounds on $I_f(P,Q)$ (Csiszar's 1961 and 1978). Saraswat (2015) and Saraswat et al. (2018 and 2019) have provided different new information inequalities and applications. Here, we will discuss Theorem 4.1, 4.2 and 4.3 using new f- divergence measure in which Theorem 4.1 and 4.2 have proved by Jain et al. (2012) and (2013), Dragomir et al. (2016) respectively. Theorem 4.3, we have to prove here. It is may be interested in Information Theory and Statistics.

Theorem 4.1.

If, the function f is convex and normalized, i.e., $f''(t)\geq 0\ \forall t>0$ and $f(1)=0$ respectively, then the new f-divergence, $S_f(P,Q)$ and its adjoint $S_f(Q,P)$ are both non-negative and convex in the pair of probability distribution $(P,Q)\in\Gamma_n\times\Gamma_n$.

Theorem 4.2.

Let $f_1, f_2:(0.5,\infty)\subset R_+\to R$ be normalized i.e. $f_1(1)=f_2(1)=0$ and twice differentiable functions on $(r,\ R)$ and there are m and M such that

$$m\leq\frac{f_1''(t)}{f_2''(t)}\leq M,\quad f_2''(t)>0\,,\forall t\in(0.5,\infty)\ , \tag{4.1}$$

then, the following inequalities hold

$$mS_{f_2}(P,Q)\leq S_{f_1}(P,Q)\leq MS_{f_2}(P,Q). \tag{4.2}$$

Theorem 4.3.

Let $f:(0.5,\infty)\to R$ is normalized, i.e., $f(1)=0,$ and satisfies the following assumptions

(i) f is twice differentiable on $(r,\ R)$ where $0.5\leq r\leq 1\leq R\leq\infty$,

(ii) There exist constants m, M such that,

$$m \le \frac{(2t-1)^{k+2} f_k''(t)}{2k(t-1)^{2k-2} t\,[(2k+2)t^2 - 6t + 3]} \le M, \ \forall\, t \in (r, R). \tag{4.3}$$

If, P,Q are discrete probability distributions satisfying the assumption

$$r \le r_i = \frac{p_i + q_i}{2q_i} \le R, \quad \forall\, i \in \{1, 2, \dots n\}, \tag{4.4}$$

then, we have the inequality

$$m\ \Psi(P,Q) \le S_f(P,Q) \le M\ \Psi(P,Q). \tag{4.5}$$

Proof:

Define a mapping $F_m : (1/2, \infty) \to R$,

$$F_m(t) = f(t) - m\frac{t(t-1)^{2k}}{(2t-1)^k},$$

then, $F_m(.)$ is normalized, twice differentiable and since

$$F_m''(t) = f_k''(t) - m\frac{2k(t-1)^{2k-2} t((2k+2)t^2 - 6t + 3)}{(2t-1)^{k+2}},$$

$$F_m''(t) = \left(\frac{2k(t-1)^{2k-2} t((2k+2)t^2 - 6t + 3)}{(2t-1)^{k+2}}\right)\left[\frac{(2t-1)^{k+2} f_k''(t)}{2k(t-1)^{2k-2} t\,[(2k+2)t^2 - 6t + 3]} - m\right] \ge 0, \tag{4.6}$$

for all $t \in (r, R)$, implied by the first inequality in (4.1). It follows that $F_m(.)$ is convex on (r, R). Applying the non-negativity property of the new *f*-divergence functional for $F_m(.)$ and the linearity property (by proposition 1.2), we may state that

$$0 \le S_{F_m}(P,Q) = S_f(P,Q) - mS_{f_k}(P,Q) = S_f(P,Q) - m\Psi(P,Q), \tag{4.7}$$

from where we get the first inequality in (4.5). Now again, Define a mapping,

$$F_M : (0, \infty) \to R, F_M(t) = M\frac{t(t-1)^{2k}}{(2t-1)^k} - f(t), \tag{4.8}$$

which is obviously normalized, twice differentiable and by (4.1), convex on (r,R). Applying the non-negativity property and by proposition 1.2 of the new *f*-divergence for $F_M(.)$, we obtain the second part of (4.5).

5. Some particular results

Here, we will discussed the particular cases of Theorem 4.1, 4.2, 4.3 for the relationship between new and well-known divergence measure which are given in results 5.1, 5.2, 5.2, 5.3, 5.4 and 5.5 as following:

Result 5.1.

Let $\Psi(P,Q)$ and $h(P,Q)$ be defined as in (1.8), (2.5) and (4.2) respectively. For $P,Q \in \Gamma_n$, we have

(i) If, $0.5 < r \le 1$, then,

$$2h(P,Q) \le \frac{1}{8}\Psi(P,Q) \le \max\left[\frac{2r(4r^2-6r+3)}{(2r-1)^{\frac{3}{2}}}, \frac{2R(4R^2-6R+3)}{(2R-1)^{\frac{3}{2}}}\right] h(P,Q). \tag{5.1}$$

(ii) If, $1 < a < \infty$, then,

$$\frac{2r(4r^2-6r+3)}{(2r-1)^{\frac{3}{2}}} h(P,Q) \le \frac{1}{8}\Psi(P,Q) \le \frac{2R(4R^2-6R+3)}{(2R-1)^{\frac{3}{2}}} h(P,Q). \tag{5.2}$$

Proof:

Let us consider

$$f_2(t) = \frac{1}{2}\left(\sqrt{(2t-1)}-1\right)^2, \quad \forall t \in (0.5,\infty), \tag{5.3}$$

$$f_2'(t) = \frac{\sqrt{(2t-1)}-1}{\sqrt{(2t-1)}},$$

$$f_2''(t) = \frac{1}{(2t-1)^{\frac{3}{2}}},$$

since, $f_2''(t) > 0, \forall t > 0.5$ and $f_2(1) = 0$, So $f_2(t)$ is convex and normalized function respectively.

Now put $f_2(t)$ in (2.1), we get

$$S_{f_2}(P,Q) = \frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{p_i}-\sqrt{q_i}\right)^2 = h(P,Q), \tag{5.4}$$

$$g_1(t) = \frac{2t(4t^2-6t+3)}{(2t-1)^{\frac{3}{2}}},$$

$$g_1{}'(t) = \frac{(24t^3 - 36t^2 + 18t - 6)}{(2t - 1)^{\frac{5}{2}}},$$

$$g_1{}''(t) = \frac{(24t^3 - 36t^2 + 18t + 12)}{(2t - 1)^{\frac{7}{2}}},$$

where $f_1{}''(t)$ and $f_2{}''(t)$ are given by (2.3) and (5.10) respectively. If $g_1{}'(t) = 0 \Rightarrow t = 1$. It is clear that $g_1^{'}(t) < 0$ in $(0.5,1]$ and > 0 in $(1, \infty]$, i.e., $g_1(t)$ is decreasing in $(0.5,1]$ and increasing in $(1, \infty]$. So $g_1(t)$ has a minimum value at $t = 1$ because $g^{''}(1) = 18 > 0$. So,

(i) If, $0.5 < r \le 1$, then,

$$m = \inf_{t\in(r,R)} g_1(t) = g_1(1) = 2.000, \tag{5.5}$$

$$\begin{aligned} M &= \sup_{t\in(r,R)} g_1(t) = \max[g_1(r), g_1(R)] \\ &= \max\left[\frac{2r(4r^2 - 6r + 3)}{(2r - 1)^{\frac{3}{2}}}, \frac{2R(4R^2 - 6R + 3)}{(2R - 1)^{\frac{3}{2}}}\right]. \end{aligned} \tag{5.6}$$

(ii) If, $1 < r < \infty$, then,

$$m = \inf_{t\in(r,R)} g_1(t) = g_1(r) = \frac{2r(4r^2 - 6r + 3)}{(2r - 1)^2}, \tag{5.7}$$

$$M = \sup_{t\in(r,R)} g_1(t) = g_1(r) = \frac{2R(4R^2 - 6R + 3)}{(2R - 1)^2}. \tag{5.8}$$

The results (5.1) and (5.2) are obtained by using (2.5), (5.5), (5.6), (5.7), and (5.8).

Result 5.2.

Let $\Psi(P,Q)$ and $KL(P,Q)$ be defined as in (1.4), (2.5) and (4.2) respectively. For $P,Q \in \Gamma_n$, we have

(i) If, $0.5 < r \le 1.12$, then,

$$\begin{aligned} (0.472591)KL(P,Q) &\le \frac{1}{8}\Psi(P,Q) \\ &\le \max\left[\frac{r(4r^2 - 6r + 3)}{2(2r - 1)^2}, \frac{R(4R^2 - 6R + 3)}{2(2R - 1)^2}\right] KL(P,Q) . \end{aligned} \tag{5.9}$$

(ii) If, $1.12 < r < \infty$, then,

$$\frac{r(4r^2-6r+3)}{2(2r-1)^2}KL(P,Q)\le\frac{1}{8}\Psi(P,Q)\le\frac{R(4R^2-6R+3)}{2(2R-1)^2}KL(P,Q). \tag{5.10}$$

Proof:

Let us consider

$$f_2(t)=(2t-1)\log(2t-1), \tag{5.11}$$

$$f_2\,'(t)=2[1+\log(2t-1)],$$

$$f_2\,''(t)=\frac{4}{(2t-1)}>0, t\in(0.5,\infty).$$

Since, $f_2\,''(t)>0, \forall t>0.5$ and $f_2(1)=0$. So $f_2(t)$ is convex and normalized function respectively. Now put $f_2(t)$ in (1.1), we get from (1.7)

$$S_{f_2}(P,Q)=\sum_{i=1}^{n}p_i\log\left(\frac{p_i}{q_i}\right)=KL(P,Q),$$

$$g_1(t)=\frac{1}{2}\frac{t(4t^2-6t+3)}{(2t-1)^2},$$

$$g_1\,'(t)=\frac{(8t^3-12t^2+6t-3)}{2(2t-1)^3},\quad g_1\,''(t)=\frac{6}{(2t-1)^4},$$

where $f_1\,''(t)$ and $f_2\,''(t)$ are given by (2.3) and (5.10) respectively.

If $g_1\,'(t)=0\Rightarrow t=1.129960524947437\approx1.12$. It is clear $g_1\,'(t)$ that $g_1\,'(t)<0$ in $0.5, 1.12$ and >0 in $1.12,\infty$. That is, $g_1(t)$ is decreasing in $0.5, 1.12$ and increasing in $1.12, \infty$). So, $g_1(t)$ has a minimum value at $t=1.129960524947437\approx1.12$ because $g''\ 1.12\ =2.537842>\ 0$. So,

(i) If, $0.5<r\le1.12$, then,

$$m=\inf_{t\in(r,R)}g_1(t)=g_1(1.12)=0.472591, \tag{5.12}$$

$$M=\sup_{t\in(r,R)}g_1(t)=\max[g_1(r),g_1(R)]$$

$$= \max\left[\frac{r(4r^2-6r+3)}{2(2r-1)^2},\frac{R(4R^2-6R+3)}{2(2R-1)^2}\right]. \tag{5.13}$$

(ii) If, $1.12 < r < \infty$, then,

$$m = \inf_{t\in(r,R)} g_1(t) = g_1(r) = \frac{r(4r^2-6r+3)}{2(2r-1)^2}, \tag{5.14}$$

$$M = \sup_{t\in(r,R)} g_1(t) = g_1(R) = \frac{R(4R^2-6R+3)}{2(2R-1)^2}. \tag{5.15}$$

The results (5.9) and (5.10) are obtained by using (1.7), (2.5), (5.11), (5.12), (5.13), (5.14) and (5.15).

Result 5.3.

Let $\Psi(P,Q)$ and $KL(Q,P)$ be defined as in (1.4), (2.5) and (4.2) respectively. For $P,Q\in\Gamma_n$, we have

(i) If, $0.5 < r \le 0.89$, then,

$$(0.472613)KL(Q,P) \le \frac{1}{8}\Psi(P,Q)$$
$$\le \frac{1}{2}Max\left[\frac{r(4r^2-6r+3)}{(2r-1)},\frac{R(4R^2-6R+3)}{(2R-1)}\right]KL(Q,P). \tag{5.16}$$

(ii) If, $0.89 < r \le 1$, then,

$$\frac{r(4r^2-6r+3)}{2(2r-1)}KL(Q,P) \le \frac{1}{8}\Psi(P,Q) \le \frac{R(4R^2-6R+3)}{2(2R-1)}KL(Q,P). \tag{5.17}$$

Proof:

Let us consider

$$f_2(t) = -\log(2t-1), \tag{5.18}$$

$$f_2'(t) = -\frac{2}{(2t-1)},$$

$$f_2''(t) = \frac{4}{(2t-1)^2} > 0,\ \ \forall t\in(0.5,\infty).$$

Since, $f_2''(t) > 0, \forall t > 0.5$ and $f_2(1) = 0$. So $f_2(t)$ is convex and normalized function respectively. Now put $f_2(t)$ in (1.1), we get

$$S_{f_2}(P,Q) = \sum_{i=1}^{n} q_i \log\left(\frac{q_i}{p_i}\right) = KL(Q,P), \tag{5.19}$$

$$g_1(t) = \frac{1}{2}\frac{t(4t^2 - 6t + 3)}{(2t - 1)},$$

$$g_1'(t) = \frac{(16t^3 - 24t^2 + 12t - 3)}{2(2t-1)^2}, \quad g_1''(t) = \frac{4t(4t^2 - 6t + 3)}{(2t-1)^3},$$

where $f_1''(t)$ and $f_2''(t)$ are given by (2.3) when $k = 1$.

If, $g_1'(t) = 0 \Rightarrow t = 0.8968502629920499 \approx 0.89$. It is clear that $g_1'(t) < 0$ in $(0.5, 0.89)$ and > 0 in $(0.89, \infty)$, i.e., $g_1'(t)$ is decreasing in $(0.5, 0.89)$ and increasing in $(0.89, \infty)$. So $g_1(t)$ has a minimum value at $t = 0.8968502629920499 \approx 0.89$ because $g''(0.89) = 6.214501 > 0$. So,

(i) If, $0.5 < r \le 0.89$, then,

$$m = \inf_{t\in(r,R)} g_1(t) = g_1(0.89) = 0.472613, \tag{5.20}$$

$$\begin{aligned} M &= \sup_{t\in(r,R)} g_1(t) = \max[g_1(r), g_1(R)] \\ &= \max\left[\frac{r(4r^2 - 6r + 3)}{2(2r-1)}, \frac{R(4R^2 - 6R + 3)}{2(2R-1)}\right]. \end{aligned} \tag{5.21}$$

(ii) If, $0.89 < r \le 1$, then,

$$m = \inf_{t\in(r,R)} g_1(t) = g_1(r) = \frac{r(4r^2 - 6r + 3)}{2(2r-1)}, \tag{5.22}$$

$$M = \inf_{t\in(r,R)} g_1(t) = g_1(r) = \frac{R(4R^2 - 6R + 3)}{2(2R-1)}. \tag{5.23}$$

The results (5.16) and (5.17) are obtained by using (2.5), (5.18), (5.19), (5.20), (5.21), (5.22) and (5.23).

Result 5.4.

Let $\Psi(P,Q)$ and $G(Q,P)$ be defined as in (1.7), (2.5) and (4.2) respectively. For $P,Q \in \Gamma_n$, we have

(i) If, $0.5 < r \le 1.2$, then,

$$(1.617038)G(Q,P) \le \frac{1}{8}\Psi(P,Q)$$
$$\le \max\left[\frac{2r^2(4r^2-6r+3)}{(2r-1)^3}, \frac{2R^2(4R^2-6R+3)}{(2R-1)^3}\right]G(Q,P). \tag{5.24}$$

(ii) If, $1.2 < r < \infty$, then,

$$\frac{2r^2(4r^2-6r+3)}{(2r-1)^3}G(Q,P) \le \frac{1}{8}\Psi(P,Q) \le \frac{2R^2(4R^2-6R+3)}{(2R-1)^3}G(Q,P). \tag{5.25}$$

Proof:

Let us consider

$$f_2(t) = t\log(t), \tag{5.26}$$

$$f_2'(t) = (1+\log t),$$

$$f_2''(t) = \frac{1}{t} > 0, \ \ \forall\, t \in (0.5,\infty),$$

Since, $f_2''(t) > 0, \forall t > 0.5$ and $f_2(1) = 0$. So $f_2(t)$ is convex and normalized function respectively. Now put $f_2(t)$ in (1.1), we get

$$S_{f_2}(P,Q) = \sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2q_i}\right) = G(Q,P), \tag{5.27}$$

$$g_1(t) = \frac{2t^2(4t^2-6t+3)}{(2t-1)^3},$$

$$g_1'(t) = \frac{4t(4t^3-8t^2+6t-3)}{(2t-1)^4}, \quad g_1''(t) = \frac{12(2t+1)}{(2t-1)^5},$$

where $f_1''(t)$ and $f_2''(t)$ are given by (2.3) when $k=1$ and in (6.33), respectively.

If, $g_1'(t)=0 \Rightarrow t=1.287371536943511 \approx 1.2$. It is clear that $g_1'(t)<0$ in $(0.5,\ 1.2)$ and >0 in $(1.2,\ \infty)$, i.e., $g_1(t)$ is decreasing in $(0.5,\ 1.2)$ and increasing in $(1.2,\ \infty)$. So $g_1(t)$ has a minimum value at $t=1.287371536943511 \approx 1.2$ because $g''(1.2)=4.429733>0$. So,

(i) If, $0.5<r\le 1.2$, then,

$$m=\inf_{t\in(r,R)} g_1(t)=g_1(1.2)=1.617038, \tag{5.28}$$

$$\begin{aligned} M &= \sup_{t\in(r,R)} g_1(t)=\max[g_1(r),g_1(R)] \\ &=\max\left[\frac{2r^2(4r^2-6r+3)}{(2r-1)^3},\frac{2R^2(4R^2-6R+3)}{(2R-1)^3}\right]. \end{aligned} \tag{5.29}$$

(ii). If, $1.2<r<\infty$, then,

$$m=\inf_{t\in(r,R)} g_1(t)=g_1(r)=\frac{2r^2(4r^2-6r+3)}{(2r-1)^3}, \tag{5.30}$$

$$M=\inf_{t\in(r,R)} g_1(t)=g_1(R)=\frac{2R^2(4R^2-6R+3)}{(2R-1)^3}. \tag{5.31}$$

The results (5.24) and (5.25) are obtained by using (2.5), (5.26), (5.27), (5.28), (5.29), (5.30) and (5.31).

Result 5.5.

Let $\Psi(P,Q)$ and $F(Q,P)$ be defined as in (1.8), (2.5), (4.2), respectively. For $P,Q\in\Psi_n$, we have

(i) If, $0.5<r\le 1.1$, then,

$$\begin{aligned} (1.910203)F(Q,P) &\le \frac{1}{8}\Psi(P,Q) \\ &\le \max\left[\frac{2r^3(4r^2-6r+3)}{(2r-1)^3},\frac{2R^3(4R^2-6R+3)}{(2R-1)^3}\right]F(Q,P) \end{aligned} \tag{5.32}$$

(ii) If, $1.1<r<\infty$, then,

$$\frac{2r^3(4r^2-6r+3)}{(2r-1)^3}F(Q,P)\le\frac{1}{8}\Psi(P,Q)\le\frac{2R^3(4R^2-6R+3)}{(2R-1)^3}F(Q,P). \tag{5.33}$$

Proof:

Let us consider

$$f_2(t) = -\log(t), \tag{5.34}$$

$$f_2'(t) = \frac{-1}{t}, \quad f_2''(t) = \frac{1}{t^2} > 0, \quad t \in (0.5, \infty),$$

Since, $f_2''(t) > 0,\ \forall t > 0.5$ and $f_2(1) = 0$, So $f_2(t)$ *is* convex and Normalized function respectively. Now put $f_2(t)$ in (2.1), we get

$$S_{f_2}(P,Q) = \sum_{i=1}^{n} q_i \log\left(\frac{2q_i}{p_i + q_i}\right) = F(Q,P), \tag{5.35}$$

$$g_1(t) = \frac{2t^3(4t^2 - 6t + 3)}{(2t-1)^3},$$

$$g_1'(t) = \frac{2t^2(16t^3 - 32t^2 + 24t - 9)}{(2t-1)^4}, \quad g_1''(t) = \frac{4t(16t^4 - 40t^3 + 40t^2 - 18t + 9)}{(2t-1)^5},$$

where $f_1''(t)$ and $f_2''(t)$ are given by (2.3) when $k = 1$ and in (5.41) respectively.

If $g_1'(t) = 0 \Rightarrow t = 1.102047318427496 \approx 1.1$. It is clear that $g_1'(t) < 0$ in $(0.5,\ 1.1)$ and > 0 in $(1.1,\ \infty)$, i.e., $g_1(t)$ is decreasing in $(0.5,\ 1.1)$ and increasing in $(1.1,\ \infty)$. So, $g_1(t)$ has a minimum value at $t = 1.102047318427496 \approx 1.1$ because $g''(1.1) = 13.59545 > 0$. So,

(i) If, $0.5 < r \le 1.1$, then

$$m = \inf_{t\in(r,R)} g_1(t) = g_1(1.1) = 1.90203, \tag{5.36}$$

$$\begin{aligned} M &= \sup_{t\in(r,R)} g_1(t) = \max[g_1(r), g_1(R)] \\ &= \max\left[\frac{2r^3(4r^2 - 6r + 3)}{(2r-1)^3}, \frac{2R^3(4R^2 - 6R + 3)}{(2R-1)^3}\right]. \end{aligned} \tag{5.37}$$

(ii) If, $1.1 < r < \infty$, then,

$$m = \inf_{t\in(r,R)} g_1(t) = g_1(r) = \frac{2r^3(4r^2 - 6r + 3)}{(2r-1)^3}, \tag{5.38}$$

$$M = \inf_{t\in(r,R)} g_1(t) = g_1(R) = \frac{2R^3(4R^2 - 6R + 3)}{(2R-1)^3}. \tag{5.39}$$

The results (5.32) and (5.33) are obtained by using (2.5), (5.34), (5.35), (5.36), (5.37), (5.38) and (5.39).

6. Conclusion

The main purpose of the paper is to present k-series for different new and well-known divergence measures and may be useful in information theory and statistics. From the new established divergence measure, we can find applications in the computer science like pattern recognition, information retrieval, image processing, face detection techniques. Given new information inequalities helps to find relations between new and well-known divergence measures. Results 5.1, 5.2, 5.3, 5.4 and 5.5 are the particular cases of information inequalities theorem 4.1, 4.2 and 4.3. These inequalities are based on new f divergence measure and established k series of divergence measures.

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