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Solutions of the Generalized Abel's Integral Equation using Laguerre Orthogonal Approximation

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Abstract

In this paper, a numerical approximation is drafted for solving the generalized Abel's integral equation by practicing Laguerre orthogonal polynomials. The proposed approximation is framed for the first and second kinds of the generalized Abel's integral equation. We have utilized the properties of fractional order operators to interpret Abel's integral equation as a fractional integral equation. It offers a new approach by employing Laguerre polynomials to approximate the integrand of a fractional integral equation. Given examples demonstrate the simplicity and suitability of the method. The graphical representation of exact and approximate solutions helps in visualizing a solution at discrete points, together with the absolute error function. We have also carried out a numerical comparison with Chebyshev polynomials to display less error in the posed formulation.

Keywords: Fractional calculus; Generalized Abel's integral equations; Laguerre polynomials; Chebyshev polynomials; Orthogonal approximation

MSC 2010 No.: 26A33, 41A10, 45A05

1. Introduction

The integral equation is considered to be the most effective tool for modeling various engineering and physical phenomena, having enormous applications in pure and applied mathematics. These can be broadly categorized into Volterra, Fredholm, homogeneous and inhomogeneous integral equations. A variety of integral equations consistently arises in many physical problems of science and engineering such as electronics, optimization, microscopy, plasma spectroscopy, mechanics, etc. (see Cremers and Birkebak (1966), De et al. (2010), Gorenflo and Vessella (1991), Fleurier and Chapelle (1974), Jakeman and Anderssen (1975), Mirzaee and Alipour (2018)). The applicability of this topic in most of the applied areas leads to deep exploration and investigation.

Earlier, Abel practiced $1/2$-order derivative in the solution of an integral equation that appears in the formulation of tautochrone problem, and is assumed to be the most remarkable application in the development of fractional calculus. One may note that integral equations may not always possess an analytical solution. The absence of analytical solution motivates the researchers to enquire new approaches for solving these type of integral equations numerically. Different type of methods for solving the generalized Abel's integral equation already exist in literature (see Avazzadeh et al. (2011a), Avazzadeh et al. (2011b), Dixit et al. (2011), Saleh et al. (2014), Li and Zhao (2013), and references therein). For instance, Avazzadeh et al. (2011a) and Saleh et al. (2014) presented the solution by Chebyshev polynomials. Legendre polynomials and Bernstein polynomials have also been used in Avazzadeh et al. (2011b) and Dixit et al. (2011). Solutions of a system of the generalized Abel's integral equation using fractional calculus is given by Mandal et al. (1996).

This work intends to propose a new solution scheme for the generalized Abel's integral equations of the first and second kind, with the approximation by Laguerre orthogonal polynomials. Fractional calculus performs a significant role as Abel's integral equations can be described nicely with respect to fractional integrals. The primary motive of this method is to take advantage of the properties of fractional integrals and Laguerre polynomials, as the orthogonal property of Laguerre polynomials scales down the complexity of the system. The adoption of Laguerre polynomials assists in approximating the fractional integral, and thus, the original problem reduces into a simpler system of algebraic equations. Given numerical examples and evaluated errors describe the applicability and efficiency of the method.

2. Preliminaries

In this section, we present the necessary details of Abel's integral equation, fractional order operators and Laguerre polynomials, which will be needed in the sequel.

2.1. Abel's Integral Equation

Intregral equation refers to an equation which involves an unknown function and integral of that function to be solved. Mathematically, for an unknown function $u(x)$,

$$f(x) = A(x)\,u(x) + \int_{a(x)}^{b(x)} k(x,t)\,u(t)\,dt,$$

represents a general form of integral equation. Here, $f(x), A(x), a(x), b(x)$ and $k(x,t)$ are given functions. $k(x,t)$ is called the kernel of integral equation. These can be broadly categorized into Volterra, Fredholm, homogeneous and inhomogeneous integral equations.

- Volterra integral equations of the first kind:

$$f(x) = \int_{a(x)}^{b(x)} k(x,t)\, u(t)\, dt.$$

- If a and b are constants, it reduces to Fredholm integral equations of first kind.
- Volterra integral equations of second kind:

$$f(x) = u(x) + \int_{a(x)}^{b(x)} k(x,t)\, u(t)\, dt.$$

- If a and b are constants, it reduces to Fredholm integral equations of second kind.
- Clearly, $f(x) \equiv 0$ is known as homogeneous integral equation.

We primarily focus on generalized Abel's integral equation, usually emerges in two different kinds:

- First Kind,

$$f(x) = \int_0^x \frac{u(t)}{(x-t)^\alpha}\, dt. \tag{1}$$

- Second Kind,

$$u(x) = f(x) + \int_0^x \frac{u(t)}{(x-t)^\alpha}\, dt, \tag{2}$$

  where $0 < \alpha < 1$, $f(x) \in C[0,T]$, $0 \leq x, t \leq T$, T is a constant. $C[0,T]$ denotes the class of all continuous function defined over the closed interval $[0,T]$.

The term “Generalized" is introduced due to the presence of α, where $0 < \alpha < 1$. For $\alpha = \frac{1}{2}$, above integral equations are simply called Abel's integral equations, introduced and investigated by Abel in (1823). Taking $a(x) = 0$, $b(x) = x$, $k(x,t) = (x-t)^{-\alpha}$, Equations (1)-(2) can be regarded as Volterra integral equations of first and second kind, respectively.

2.2. Fractional Derivatives/Integrals

The concept of fractional calculus is associated with integrals and derivative of a function of non-integer order. The restriction on the order of integral (or derivative) to be an integer is removed, eventually, it is allowed to be any real or complex number. Nowadays several definitions of a fractional derivative are available in literature which includes Riemann-Liouville, Grunwald-Letnikov, Weyl, Caputo, and Riesz fractional derivatives (see Kilbas et al. (2006), Miller and Ross (1993), Oldham and Spanier (1974), Podlubny (1999)). We briefly give the definitions of fractional order derivatives and integrals used throughout the work.

Let $f \in C[a,b]$, where $C[a,b]$ is the space of all continuous functions defined over $[a,b]$.

Definition 2.1.

For all $t \in [a,b]$ and $\alpha > 0$, Left Riemann-Liouville Fractional Integral (LRLFI) of order α is defined as

$$_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau, \quad t > a.$$

Definition 2.2.

For all $t \in [a,b]$ and $\alpha > 0$, Right Riemann-Liouville Fractional Integral (RRLFI) of order α is defined as

$$_tI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\,d\tau, \quad t < b.$$

Let us consider $f \in C^n[a,b]$, where $C^n[a,b]$ is the space of n times continuously differentiable functions defined over $[a,b]$.

Definition 2.3.

For all $t \in [a,b]$, $n-1 \leq \alpha < n$, Left Riemann-Liouville Fractional Derivative (LRLFD) of order α is defined as

$$_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,d\tau.$$

Definition 2.4.

For all $t \in [a,b]$, $n-1 \leq \alpha < n$, Right Riemann-Liouville Fractional Derivative (RRLFD) of order α is defined as

$$_tD_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau) d\tau.$$

Definition 2.5.

For all $t \in [a,b]$, $n-1 \leq \alpha < n$, Left Caputo Fractional Derivative (LCFD) of order α is defined as

$${}_a^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,d\tau.$$

Definition 2.6.

For all $t \in [a,b]$, $n-1 \leq \alpha < n$, Right Caputo Fractional Derivative (RCFD) of order α is defined as

$${}_t^cD_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} f^{(n)}(\tau)\,d\tau.$$

2.3. Laguerre Polynomials

The Laguerre polynomials denoted by $L_n(x)$ are solutions of a second-order linear differential equation $xy'' + (1-x)y' + ny = 0$, known as Laguerre's equation (see Agarwal and Regan (2009), Aizenshtadt et al. (1966)). One may observe that it has non-singular solutions only if n is a non-negative integer. We list some of the useful properties when working with Laguerre polynomials.

- The power series representation for Laguerre polynomials
$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} C_{\{n,k\}}\, x^k,$$
where
$$C_{\{n,k\}} = \frac{n!}{k!(n-k)!}.$$
- The Rodrigue's representation for Laguerre polynomials
$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n\, e^{-x}).$$
- Generating function of Laguerre polynomial is
$$\frac{e^{\frac{-xt}{1-t}}}{1-t} = \sum_{n=0}^{\infty} L_n(x)t^n.$$
- The first few Laguerre polynomials are
$$\begin{aligned} L_0(x) &= 1, \\ L_1(x) &= 1 - x, \\ L_2(x) &= 1 - 2x + \frac{x^2}{2}, \\ L_3(x) &= 1 - 3x + \frac{3x^2}{2} - \frac{x^3}{6}. \end{aligned}$$
- Recurrence relation for Laguerre polynomials, for any $n \geq 1$
$$L_{n+1}(x) = \frac{(2n+1-x)L_n(x) - nL_{n-1}(x)}{n+1}.$$
- Laguerre polynomials are orthogonal with respect to an inner product defined as
$$< f, g >= \int_0^{\infty} f(x)g(x)e^{-x}\, dx.$$

Remark 2.7.

One may mark with above inner product that Laguerre polynomials are orthogonal in $[0, \infty)$ with e^{-x} as the weight function, that is,
$$\int_0^{\infty} L_n(x)L_m(x)\, e^{-x}\, dx = \delta_{nm} = \begin{cases} 1, \text{ if } n = m, \\ 0, \text{otherwise.} \end{cases}$$

3. Main Results: Laguerre Orthogonal Approximation Scheme

In this section, we use Laguerre orthogonal polynomials to approximate the generalized Abel's integral equations with fractional order operators. One of the most significant point is to use Laguerre polynomials over any desired interval. We explicitly focus on the interval $(0,1)$ as we wish to consider the fractional derivative of order α, where $0<\alpha<1$. It is to be noted that we are not concerned about the end points ($\alpha = 0,1$). For such values of α we reach a constant and a first order derivative (both are classical cases).

In order to match our requirements, we choose the transformation of variable x to variable z as

$$x = \frac{z}{1-z}\,;\; z \neq 1,$$
$$dx = \frac{1}{(1-z)^2}\,dz\;;\; z\neq 1.$$

As $x \to 0 \Rightarrow z \to 0$, and $x\to\infty \Rightarrow z \to 1$. Thus,

$$\int_0^1 L_n\left(\frac{z}{1-z}\right) L_m\left(\frac{z}{1-z}\right) e^{-\frac{z}{1-z}} \frac{1}{(1-z)^2}\,dz = \delta_{nm} = \begin{cases} 1, \text{ if } n=m, \\ 0, \text{otherwise.}\end{cases}$$

To simplify, we further transform the variable z to variable t by taking

$$e^{-\frac{z}{1-z}} = t\,;\; z\neq 1,$$
$$e^{-\frac{z}{1-z}}\,\frac{1}{(1-z)^2}\,dz = -dt\,;\; z \neq 1.$$

As $z\to 0 \Rightarrow t\to 1$, and $z \to 1 \Rightarrow t \to 0$. Thus, we can write

$$\int_0^1 L_n\left(-\ln(t)\right) L_m\left(-\ln(t)\right)\, dt = \delta_{nm} = \begin{cases} 1, \text{ if } n=m, \\ 0, \text{otherwise.}\end{cases} \tag{3}$$

The orthogonality relation (3) makes sense as the end points $0,1$ are excluded. Collectively, we rewrite Laguerre polynomials in $(0,1)$ as

$$\begin{aligned} L_n(-\ln(x)) &= \sum_{k=0}^{n} \frac{(-1)^k}{k!}\, C_{\{n,k\}}\,[-\ln(x)]^k, \\ &= \sum_{k=0}^{n} \frac{C_{\{n,k\}}}{k!}\,[\ln(x)]^k, \end{aligned}$$

together with the orthogonality relation

$$\int_0^1 L_n\left(-\ln(x)\right) L_m\left(-\ln(x)\right)\, dx = \delta_{nm} = \begin{cases} 1, \text{ if } n=m, \\ 0, \text{otherwise.}\end{cases}$$

This conversion of interval from $[0,\infty)$ to $(0,1)$ is not a restriction of the approach. One may arrange the change of variable to obtain any convenient interval accordingly. We now proceed with constructing a numerical scheme for generalized Abel's integral equation.

3.1. Approximating Generalized Abel's Integral Equations of the First Kind

For $0 < \alpha < 1$, consider the generalized Abel's integral equation of first kind as

$$f(x) = \int_0^x \frac{u(t)}{(x-t)^\alpha}\, dt, \tag{4}$$

$$= \Gamma(1-\alpha)\, {}_0I_x^{1-\alpha} u(x), \tag{5}$$

where ${}_0I_x^{1-\alpha}u(x)$ is the left Riemann-Liouville integral of $u(x)$ of order $(1-\alpha)$.

The purpose of the present analysis is to find an approximate solution of $u(x)$ for Equation (5). In order to fulfil the objective, we employ Laguerre polynomials for approximating $u(x)$,

$$u(x) = \sum_{i=0}^{\infty} a_i\, L_i(x), \tag{6}$$

where $L_i(x)$ denote the Laguerre polynomial of degree i. One may view the expression (6) as the linear combination of Laguerre polynomials with the constants a_i to be determined.

The proposed approximation (6) makes sense as we have already devised the change of variable to obtain the required interval $(0, 1)$. Further, we write $u(x)$ as a truncated series with n terms,

$$u_n(x) = \sum_{i=0}^{n} a_i\, L_i(x), \tag{7}$$

$$\text{and} \quad L_i(x) = \sum_{k=0}^{i} b_k^i\, x^k, \tag{8}$$

where b_k^i are the coefficients of x^k in Laguerre polynomial of degree i. Thus, we rewrite $u_n(x)$ as

$$\begin{aligned} u_n(x) &= \sum_{i=0}^{n} a_i \sum_{k=0}^{i} b_k^i\, x^k, \\ &= a_0\, b_0^0\, x^0 + a_1\,(b_0^1\, x^0 + b_1^1\, x^1) + ... + a_n\,(b_0^n\, x^0 + b_1^n\, x^1 + ... + b_n^n\, x^n), \\ &= \sum_{i=0}^{n} c_i\, x^i, \end{aligned}$$

where

$$c_k = \sum_{i=k}^{n} a_i\, b_k^i \quad ;\ k = 0, 1, 2..., n. \tag{9}$$

We observe that $c_i's$ are the linear combination of undetermined constants $a_i's$, for $i = 0, 1, 2, ..., n$.

At last, we arrive at

$$\begin{aligned} f(x) &= \Gamma(1-\alpha)_0 I_x^{1-\alpha}\, u_n(x), \\ &= \Gamma(1-\alpha) \sum_{i=0}^{n} c_i\, {}_0I_x^{1-\alpha}\, x^i, \\ &= \Gamma(1-\alpha) \sum_{i=0}^{n} c_i\, \frac{\Gamma(i+1)}{\Gamma(i-\alpha+2)}\, x^{i-\alpha+1}. \end{aligned}$$

The constants $a_i's$ can now be obtained by solving the above system of linear equations. Substituting the value of constants in $u_n(x) = \sum_{i=0}^{n} a_i\, L_i(x)$ leads to the approximate solution of the generalized Abel's integral equation of first kind.

3.2. Approximating Generalized Abel's Integral Equations of the Second Kind

For $0 < \alpha < 1$, consider the generalized Abel's integral equation of second kind as

$$u(x) = f(x) + \int_0^x \frac{u(t)}{(x-t)^\alpha}\, dt, \tag{10}$$

$$= f(x) + \Gamma(1-\alpha)\, {}_0I_x^{1-\alpha} u(x), \tag{11}$$

where ${}_0I_x^{1-\alpha} u(x)$ is the left Riemann-Liouville integral of $u(x)$ of order $(1-\alpha)$.

Following the same formulation (as in first kind) we approximate (11) as

$$\begin{aligned} u(x) &= f(x) + \Gamma(1-\alpha)\, {}_0I_x^{1-\alpha} u(x), \\ f(x) &= \sum_{i=0}^{n} c_i\, \left[x^i - \Gamma(1-\alpha)\, {}_0I_x^{1-\alpha} x^i\right], \\ f(x) &= \sum_{i=0}^{n} c_i\, \left[x^i - \Gamma(1-\alpha)\, \frac{\Gamma(i+1)}{\Gamma(i-\alpha+2)} x^{i-\alpha+1}\right], \end{aligned}$$

where c_i is described by Equation (9).

4. Numerical Computations

In this section, we illustrate the working mechanism of the proposed method with the help of some examples. We have also performed comparison and error estimation of the formulated approximation with Chebyshev polynomials (see Saleh et al. (2014)).

Example 4.1.

Consider the Abel's integral of first kind

$$\int_0^x \frac{u(t)}{\sqrt{x-t}}\, dt = \frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3),$$

with the exact solution $x^3 - x^2 + 1$.

Note: It is important to mark that the designed method is by virtue of n-degree approximation. Thus, n should be fixed while attaining solution. Higher accuracy of solution can be reached by taking larger values of n.

In order to find the third degree approximation for Example 4.1, choose $n = 3$. Here $\alpha = \frac{1}{2}$, $f(x) = \frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3)$, and

$$u(x) \approxeq u_3(x) = \sum_{i=0}^{3} a_i L_i(x). \tag{12}$$

$u(x)$ is approximated using Laguerre polynomials $\{L_i, i = 0, 1, 2, 3\}$ as described earlier. We intend to find the constants $\{a_i,\ i = 0, 1, 2, 3\}$ by employing the formulation for first kind

$$f(x) = \Gamma(1-\alpha) \sum_{i=0}^{3} c_i \frac{\Gamma(i+1)}{\Gamma(i-\alpha+2)} x^{i-\alpha+1},$$

where c_i's are given by (9). Substitute $\alpha = \frac{1}{2}$

$$f(x) = 2c_0\, x^{\frac{1}{2}} + \frac{4}{3} c_1\, x^{\frac{3}{2}} + \frac{16}{15} c_2\, x^{\frac{5}{2}} + \frac{32}{35} c_3\, x^{\frac{7}{2}}.$$

Using Equation (9), we get

$$\begin{aligned} c_0 &= a_0 b_0^0 + a_1 b_0^1 + a_2 b_0^2 + a_3 b_0^3, \\ c_1 &= a_1 b_1^1 + a_2 b_1^2 + a_3 b_1^3, \\ c_2 &= a_2 b_2^2 + a_3 b_2^3, \\ c_3 &= a_3 b_3^3, \end{aligned}$$

where b_k^i are the coefficients of x^k in $L_i(x)$.

Placing $b_0^0 = b_0^1 = b_0^2 = b_0^3 = 1,\ b_1^1 = -1, b_1^2 = -2,\ b_1^3 = -3,\ b_2^2 = \frac{1}{2},\ b_2^3 = \frac{3}{2},\ b_3^3 = -\frac{1}{6}$, we obtain

$$\begin{aligned} f(x) = {} & 2(a_0 + a_1 + a_2 + a_3)x^{\frac{1}{2}} + \frac{4}{3}(-a_1 - 2a_2 - 3a_3)x^{\frac{3}{2}} \\ & + \frac{16}{15}\left(\frac{a_2}{2} + \frac{3\,a_3}{2}\right) x^{\frac{5}{2}} + \frac{32}{35}\left(-\frac{a_3}{6}\right) x^{\frac{7}{2}}. \end{aligned}$$

Given $f(x) = \frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3)$. On comparing the coefficients of like powers of x,

$$\begin{aligned} 2(a_0 + a_1 + a_2 + a_3) &= 2, \\ -\frac{4}{3}(a_1 + 2a_2 + 3a_3) &= 0, \\ \frac{16}{30}(a_2 + 3a_3) &= -\frac{16}{15}, \\ -\frac{16}{105}(a_3) &= \frac{96}{105}. \end{aligned}$$

After simplification, we arrive at

$$a_0 = 5, \quad a_1 = -14, \quad a_2 = 16, \quad a_3 = -6.$$

Using Equation (12), the third degree approximate solution is given by

$$\begin{aligned} u(x) &\approxeq u_3(x) \\ &= \sum_{i=0}^{3} a_i\, L_i(x) \\ &= 5(1) - 14(1-x) + 16(1-2x) + \frac{x^2}{2} - 6\left(1 - 3x + \frac{3}{2}x^2 - \frac{x^3}{6}\right) \\ &= 1 - x^2 + x^3 = \text{Exact Solution}. \end{aligned}$$

In this example, both the Laguerre polynomial and Chebyshev polynomial (see Saleh et al. (2014)) yields the same solution which is exact. One may note that the orthogonality property of Laguerre polynomials can also be used (if needed) to simplify the computations involved in the formulated solution scheme.

Example 4.2.

Consider the Abel's integral of second kind,

$$u(x) = x^2 + \frac{16}{15}x^{\frac{5}{2}} - \int_0^x \frac{u(t)}{\sqrt{x-t}}\, dt,$$

with the exact solution x^2.

As computed previously, a second order approximation to $u(x)$ using Laguerre polynomials give the approximate solution $u(x) \approxeq u_2(x) = x^2$, which is same as the exact solution.

Note: If the exact solution is a polynomial of degree k, the solution by Laguerre polynomial or Chebyshev polynomial (see Saleh et al. (2014)) is the same as the exact solution for all $n \geq k$.

Example 4.3.

Consider the Abel's integral of first kind,

$$\int_0^x \frac{u(t)}{\sqrt{x-t}} dt = \frac{\pi x}{2},$$

and Abel's integral of second kind,

$$u(x) = \frac{\pi x}{2} + \sqrt{x} - \int_0^x \frac{u(t)}{\sqrt{x-t}} dt.$$

Both the integral equations have the same exact solution given by $\sqrt{x}$.

After applying the 5^{th}-order approximation with Laguerre polynomials, $a_0 = \frac{\sqrt{\pi}}{2}$, $a_1 = -\frac{\sqrt{\pi}}{4}$, $a_2 = -\frac{\sqrt{\pi}}{16}$, $a_3 = -\frac{\sqrt{\pi}}{32}$, $a_4 = -\frac{5\sqrt{\pi}}{256}$, $a_5 = -\frac{7\sqrt{\pi}}{512}$, for both integral equations, the same approximate solution $u(x) = \sum_{i=0}^{5} a_i L_i(x)$ is obtained for both integral equations (first and second kind).

Note: One may note that Chebyshev polynomials (see Saleh et al. (2014)) give distinct approximate solutions for above Abel's integral equations of first and second kind having exact solution $\sqrt{x}$.

Table 1. Comparison of approximate solutions by Laguerre polynomials with Chebyshev Polynomials (Example 4.3)

x	Exact Sol.	App. Sol. (Laguerre)	Abs. Error (Laguerre)	App. Sol. (Chebyshev)	Abs. Error (Chebyshev)
0.1	0.316228	0.323579	0.007352	0.326228	0.010001
0.2	0.447214	0.422221	0.024992	0.437091	0.010122
0.3	0.547723	0.514434	0.033288	0.551481	0.003759
0.4	0.632456	0.600615	0.031841	0.650405	0.017949
0.5	0.707107	0.681147	0.025959	0.727583	0.020477
0.6	0.774597	0.756399	0.018197	0.784353	0.009756
0.7	0.836660	0.826727	0.009933	0.827773	0.008887
0.8	0.894427	0.892469	0.001958	0.869568	0.024859
0.9	0.948683	0.953951	0.005268	0.927327	0.021355
1.0	1.000000	1.011691	0.011691	1.032311	0.032311

The numerical computations by Laguerre orthogonal polynomials is given in Table 1, one may compare the approximate solution using Laguerre polynomials by one with Chebyshev polynomials. One can also observe the action of $u(x)$ in Figure 1, that shows the exact and approximate solution for $x \in [0, 1]$. Additionally, Figure 2 represents the absolute error function for $u(x)$.

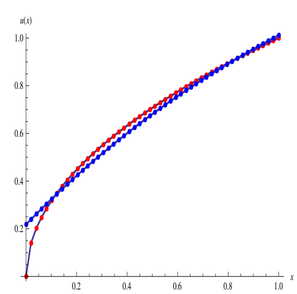

Figure 1. Plot of exact and approximate solutions of Example 4.3 (Note: Red dotted line represents the exact solution whereas blue corresponds to approximate solution of $u(x)$)

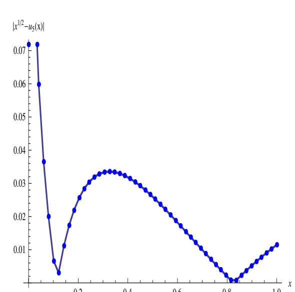

Figure 2. Absolute error function for $u(x)$, Example 4.3

Example 4.4.

Consider the generalized Abel's integral of first kind,

$$\int_0^x \frac{u(t)}{(x-t)^{\frac{1}{3}}}\, dt = \frac{2}{3\sqrt{3}}\pi x,$$

with the exact solution $x^{\frac{1}{3}}$.

After applying the orthogonal approximation by Laguerre polynomials for $n = 5$, we get $a_0 = 0.892979$, $a_1 = -0.297659$, $a_2 = -0.09922$, $a_3 = -0.055122$, $a_4 = -0.036748$, and $a_5 = -0.026949$.

Table 2. Comparison of approximate solutions by Laguerre polynomials with Chebyshev Polynomials (Example 4.4)

x	Exact Sol.	App. Sol. (Laguerre)	Abs. Error (Laguerre)	App. Sol. (Chebyshev)	Abs. Error (Chebyshev)
0.1	0.464159	0.467906	0.003747	0.456999	0.007159
0.2	0.584804	0.551447	0.033356	0.564806	0.019997
0.3	0.669433	0.628351	0.041082	0.674115	0.004682
0.4	0.736806	0.699046	0.037860	0.756718	0.019912
0.5	0.793701	0.763947	0.029754	0.809109	0.015408
0.6	0.843433	0.823451	0.019982	0.843098	0.000334
0.7	0.887904	0.877937	0.009967	0.872747	0.015156
0.8	0.928318	0.927773	0.000545	0.903743	0.024574
0.9	0.965489	0.973309	0.007819	0.961009	0.004481
1.0	1.000000	1.014881	0.014881	1.199736	0.199736

The numerical results are listed in Table 2 and can be compared with Chebyshev polynomial approximation. Figure 3 gives the exact and approximate solution for $u(x)$, followed by the absolute error function provided by Figure 4.

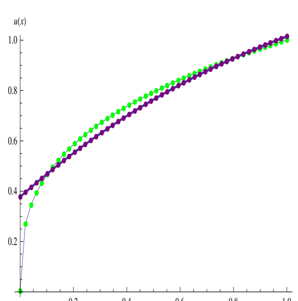

Figure 3. The plot of approximate and exact solution of Example 4.4 (Note: Green dotted line represents the exact solution whereas purple corresponds to approximate solution of $u(x)$)

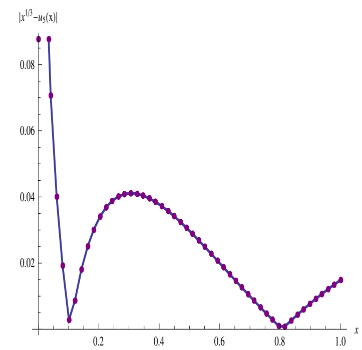

Figure 4. Absolute error function for $u(x)$, Example 4.4

Example 4.5.

Consider the Abel's integral equation of second kind,

$$u(x) = \frac{\pi x}{2\sqrt{2}} + x^{\frac{1}{4}} - \int_0^x \frac{u(t)}{\sqrt{x-t}}\, dt,$$

with the exact solution $x^{\frac{1}{4}}$.

For $n = 5$, the approximation by Laguerre polynomials give $a_0 = 0.906402$, $a_1 = -0.226601$, $a_2 = -0.084975$, $a_3 = -0.049569$, $a_4 = -0.0340787$, $a_5 = -0.025559$.

Table 3. Comparison of approximate solutions by Laguerre polynomials with Chebyshev Polynomials (Example 4.5)

x	Exact Sol.	Approx. Sol. (Laguerre)	Abs. Error (Laguerre)	Approx. Sol. (Chebyshev)	Abs. Error (Chebyshev)
0.1	0.562341	0.563159	0.000818	0.641434	0.008421
0.2	0.668740	0.634195	0.034544	0.641434	0.027306
0.3	0.740083	0.699143	0.040939	0.733927	0.006690
0.4	0.795271	0.758405	0.038654	0.803974	0.008703
0.5	0.840896	0.812368	0.028528	0.850477	0.009580
0.6	0.880112	0.861402	0.018710	0.884577	0.004465
0.7	0.914691	0.905863	0.008828	0.915611	0.000919
0.8	0.945742	0.946095	0.000353	0.938756	0.006986
0.9	0.974004	0.982424	0.008420	0.974074	0.000071
1.0	1.000000	1.015162	0.015162	1.218950	0.218950

The numerical results are listed in Table 3 and can be compared by one with Chebyshev polynomials. We plot the exact and approximate solutions for $u(x)$ in Figure 5, followed by the absolute error function in Figure 6.

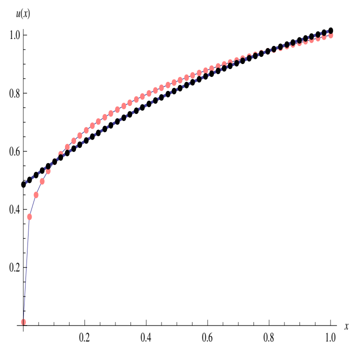

Figure 5. The plot of approximate and exact solution of Example 4.5 (Note: Pink dotted line represents the exact solution whereas black corresponds to approximate solution of $u(x)$)

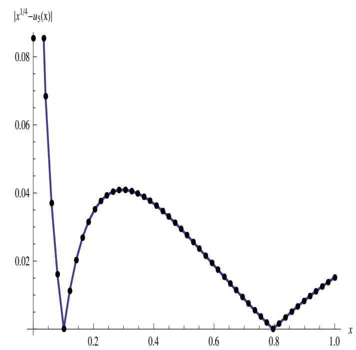

Figure 6. Absolute error function for $u(x)$, Example 4.5

One may study the comparison of approximate solutions of generalized Abel's integral equations obtained by Legendre polynomials, Chebyshev polynomials or by other methods available in the literature. We have used Mathematica software to plot the exact and approximate solutions obtained by using Laguerre polynomials.

5. Conclusion

We have drafted an orthogonal approximation technique to solve the generalized Abel's integral equation, with the assistance of fractional calculus and Laguerre polynomials. It has been observed that the proposed method is conveniently accessible, as well as efficient. The great efficiency is described with the help of numerical examples and graphical plots. Further, a comparison of the solutions obtained by distinct polynomials and techniques can be carried out in the future.

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