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Bifurcation Analysis for Prey-Predator Model with Holling Type III Functional Response Incorporating Prey Refuge

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Abstract

In this paper, we carried out the bifurcation analysis for a Lotka-Volterra prey-predator model with Holling type III functional response incorporating prey refuge protecting a constant proportion of the preys. We study the local bifurcation considering the refuge constant as a parameter. From the center manifold equation, we establish a transcritical bifurcation for the boundary equilibrium. In addition, we prove the occurrence of Hopf bifurcation for the homogeneous equilibrium. Moreover, we give the radius and period of the unique limit cycle for our system.

Keywords: Lotka-Volterra prey-predator model; Holling type III functional response; Prey refuge; Transcritical bifurcation; Hopf bifurcation

MSC 2010 No.: 37G10, 34C23, 92B05

1. Introduction

In last decades, there have been many achievements in the theory of prey-predator interactions (Britton (2012)). The first definitive theoretical treatment of population dynamics was Thomas Malthus's essay in 1798, on the Principle of Population. Then, it comes Lotka's Elements of Physical Biology in 1925, which was the next major advance in population dynamics theory. Lotka proposed the first model of predator-prey interactions, we refer to (Sauer and Scholl (2016)) for more details about the philosophy behind this studies. Afterwards, Volterra (Anisi (2014)) assumed that the response of the populations would be proportional to the product of their biomass densities, which had led to the original Lotka-Volterra model (Berryman (1992); Murray (2011)).

$$\begin{cases} \frac{dN}{dt} & = aN - bNP, \\ \frac{dP}{dt} & = cNP - dP. \end{cases}$$

In this model, the prey population grows infinitely in the absence of predators. Hence, to correct this unreasonable assumption, a logistic self- limitation term is often added to the prey equation.

$$\frac{dN}{dt} = aN(1 - NK) - bNP.$$

The next major contribution to this theory was the addition of a predator functional response. Solomon and then Holling pointed out that there is a nonlinear function of prey density describes the prey death, because predators can only handle a finite number of prey in a unit of time (Holling (1965)).

$$\frac{dN}{dt} = aN\left(1 - e\frac{N}{K}\right) - b(N)P \ ,$$

where $b(N)$ is the functional response of the predator to prey density.

Lastly, many research emphasized that, in several situations, there was a constant proportion of prey which were protected from predation by "refugia", which occur in the interaction between predator-prey shall impact the stability of the model by vanishing equilibrium point, setting up instability, oscillation, and chaos phenomena (Haque et al. (2014); Ma et al. (2017); Wang and Wang (2012)).

In their article, Yunjin Huang, Fengde Chen and Li Zhong (Huang et al. (2006)) introduced a predator-prey model with Holling type III functional response incorporating prey refuge (noted m). They studied the existence and local stability of three different equilibria (origin, boundary equilibrium and homogeneous (coexistence) equilibrium). Furthermore, they have shown the existence of a unique asymptotically stable limit cycle.

Many manuscripts, in recent years, were interested to understand the behavior of prey-predator interaction by the study of stability and bifurcation as (Sarkar et al. (2017); Almanza-Vasquez et al. (2018); Xiao et al. (2018); Zhou et al. (2019)). While, we decide to investigate the bifurcation of the model presented in (Huang et al. (2006)) because the Holling type III functional response is used to show that the predators develop their capture's efficiency by learning as they hunt, where incorporate the prey refuge to the functional response can be defined in order to include decreasing strategies of predation risk, prey aggregations, or reduced search activity by prey.

In this paper, our main objective is to provide an explanation to the shifting in the behavior of the dynamic, relatively to the prey refuge parameter, as shown in paragraph (2.2.2). The theoretical process for this purpose is the analysis of local bifurcation with respect to m, for the equilibrium resulting the extinction of predator population (i.e, Boundary equilibrium) and the equilibrium representing coexistence of predators and preys (i.e, Homogeneous equilibrium). Our motivation is to prove the major impact of refuge parameter on the stability of the dynamic, deducing that the existence of refuge can clearly have important effects on the coexistence of the two species.

This worksheet is organized as follows. In Section 2, for our suitable, we reconvene the main results of existence and stability without using the change of variables involved in (Huang et al. (2006)), and we give some numerical simulations. In Section 3, we announce the main theorems of transcritical and Hopf bifurcation. In Section 4, we give bifurcation diagrams. Some essential conclusions are laying out in Section 5.

2. Modeling and Dynamical Behavior

2.1. Modeling

The basic model considered in this worksheet is based on the Lotka-Volterra prey-predator model with Holling type III functional response that was defined in (Jun-ping and Hong-de (1986)):

$$\begin{cases} \frac{dx}{dt} = ax - bx^2 - \frac{\alpha x^2 y}{\beta^2 + x^2}, \\ \frac{dy}{dt} = -cy + \frac{k\alpha x^2 y}{\beta^2 + x^2}, \end{cases} \tag{1}$$

where $x, \ y$ denote prey and predator population respectively at the time t. The constants $\alpha, \ \beta, \ a, \ b, \ c \ and \ k$ are all positive. The coefficient a represents the intrinsic growth rate and a/b the carrying capacity of the prey; c is the death rate of predator; k is the conversion factor denoting the number of newly born predators for each captured prey. Unless otherwise stated, we consider in the following that $c < k\alpha < 2c$. The left hand side of this assumption assures the sustainability of predators. Meanwhile, the right hand side of this inequality keeps the growth of predator under the given bound of twice the predator's death rate. Otherwise, the population of predators will be greater than the supply (preys).

The previous model is extended by (Huang et al. (2006)) to incorporate a refuge protecting mx of the prey, allowing $(1-m)x$ of the prey reachable to predation, where m is a constant in $[0.1)$, this modifies (1) into the following:

$$\begin{cases} \frac{dx}{dt} = ax - bx^2 - \frac{\alpha(1-m)^2x^2y}{\beta^2+(1-m)^2x^2}, \\ \frac{dy}{dt} = -cy + \frac{k\alpha(1-m)^2x^2y}{\beta^2+(1-m)^2x^2}. \end{cases} \tag{2}$$

To ensure the existence and uniqueness of solution for the system, we search for a solution in $\mathbb{R}^2_+ = \{(x, y) : x \geq 0, y \geq 0\}$. Therefore, we have the following well posedness result:

Theorem 2.1.

For each initial condition (x_0, y_0), the dynamical system (2) admits a unique solution $(x(t), y(t))$ defined on $\mathbb{R}^+$ and starting at (x_0, y_0). Furthermore, the domain $\mathbb{R}^2_+ = \{(x, y) : x \geq 0, y \geq 0\}$ is invariant by (2).

Proof:

The proof of this theorem is given in (Huang et al. (2006)). ■

2.2. Stability analysis

The equilibrium points and local stability analysis investigated here are similar to the ones in (Huang et al. (2006)). The only difference is that, they compute the equilibria in a new model with change of variables, while we expressed them in the original model without the change of variables to simplify the local bifurcation analysis in next section.

2.2.1. *Equilibrium points*

In this paragraph, we provide the positive equilibria of the system (2), which are:

- The trivial equilibrium $P_0(0, 0)$.
- The boundary equilibrium, which can be interpreted by the absence of predators $P_1(\frac{a}{b}, 0)$.
- The interior (the coexistence) equilibrium denoted $P_2 = (x_0, y_0)$, where

$$x_0 = \frac{\beta}{1-m}\sqrt{\frac{c}{k\alpha - c}}, \quad y_0 = \frac{\beta k}{(1-m)\sqrt{c(k\alpha - c)}}\left(a - \frac{b\beta}{1-m}\sqrt{\frac{c}{k\alpha - c}}\right).$$

  To ensure the positivity of P_2, we need to assume that m satisfies $0 < m < 1 - B$, with $B = 1 - \frac{b\beta}{a}\sqrt{\frac{c}{k\alpha - c}}$.

2.2.2. *Local stability analysis*

In the next table, we draw the nature of equilibrium points relatively to the parameter m achieved in (Huang et al. (2006)):

Table 1. The impact of refuge parameter on the stability of equilibria

Parameter	P_0	P_1	P_2	phase portrait
$m \in [0, A)$	Unstable	Unstable	Unstable, limit cycle occurs	Figure 1
$m = A$	Unstable	Unstable	Unstable, limit cycle occurs	Figure 2
$m \in (A, B)$	Unstable	Unstable	Asymptotically stable	Figure 3
$m = B$	Unstable	Unstable	Doesn't exist	Figure 4
$m \in (B, 1]$	Unstable	stable	Doesn't exist	Figure 5

Those results are supported by the simulations presented in the next figures, illustrating the existence and stability properties for the equilibria of the system (2) with the parameter values $a = 0.5$, $b = 0.2$, $\alpha = 0.3$, $k = 0.1$, $c = 0.024$, $\beta = 0.3$, and the initial conditions $(u_0, v_0) = (3, 1)$ [Red], $(u_0, v_0) = (1, 3)$ [Blue], $(u_0, v_0) = (3, 3)$ [Green].

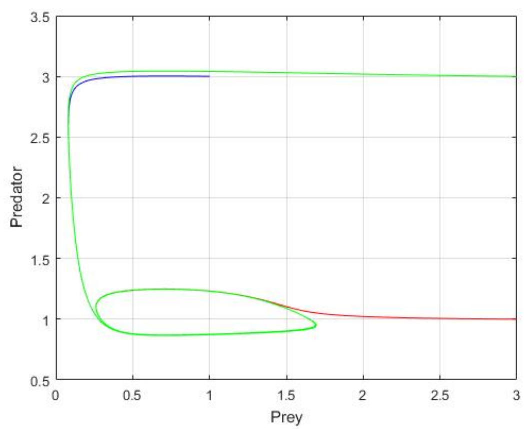

Figure 1. The phase portrait of system (2) for $m = 0, 15$

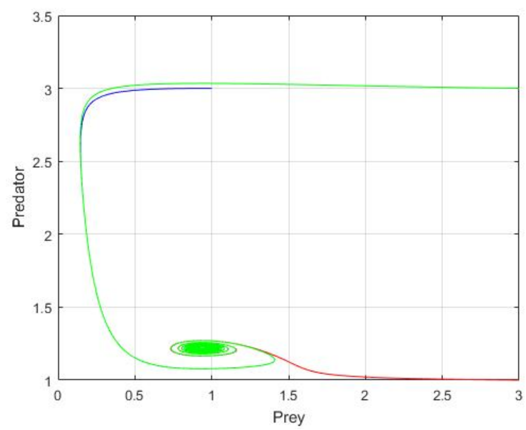

Figure 2. The portrait phase of the system (2) for $m = 0, 36$

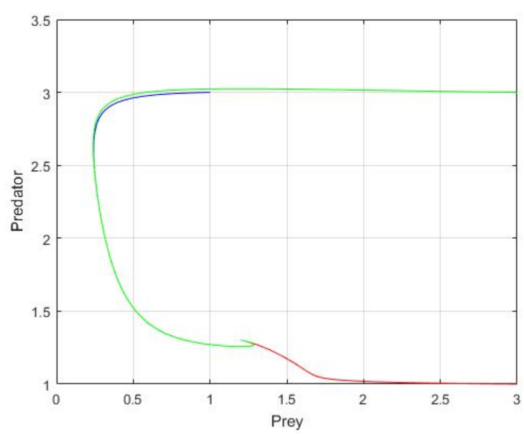

Figure 3. The portrait phase of the system (2) for $m = 0, 5$

In Figure (1), where $m = 0.15$, we clearly notice that the three distinct solutions of the system converge to an attractive limit cycle, which is an outcome of the existence of a unique asymptotically stable limit cycle. The same conclusion is true in Figure (3) where $m = 0.36$, the only

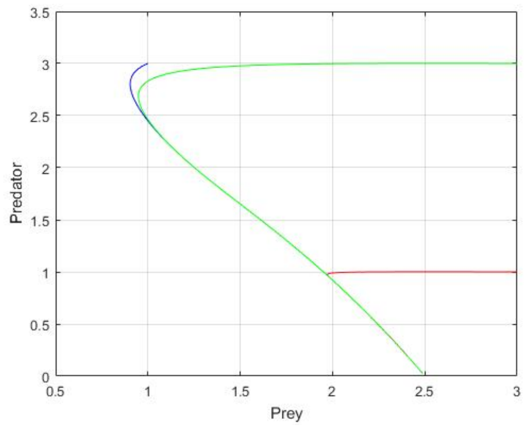

Figure 4. The portrait phase of the system (2) for $m = 0,76$

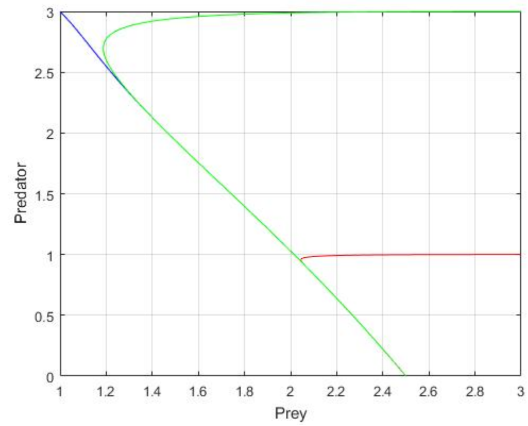

Figure 5. The portrait phase of the system (2) for $m = 0,8$

difference is that the radius of the limit cycle shrinks. Hence, from Figure (2) with $m = 0.5$, we can notice that all the curves converge asymptotically to the coexistence equilibrium. Lastly, Figures (4)($m = 0.76$) and (5)($m = 0.8$) show that the coexistence equilibrium disappears for the three initial values, while the boundary equilibrium becomes asymptotically stable.

3. Local Bifurcation

As previous figures show, it is obvious that there is a change in the local behavior of the dynamic for certain parameter values of m. Therefore, in this section, we review the local bifurcation depending on the refuge parameter m for boundary and interior equilibrium. For more details about the local bifurcation theory, we refer to (Guckenheimer and Holmes (1983); Dang-Vu and Delcarte (2000); Wiggins (2003)).

Firstly, we will prove that the dynamic switch from the local stability of the coexistence equilibrium to the stability of the equilibrium representing the extinction of predators, which indicates that a transcritical bifurcation occurs at the point $(P_1, m = B)$. Next, we can notice the occurrence of Hopf bifurcation in $(P_2, m = A)$, which can be explained by the convergence of the dynamic's population to an asymptotically stable limit cycle when the equilibria are unstable.

3.1. Transcritical bifurcation

Under this heading, we review the local bifurcation for the boundary equilibrium, proving that adding a large refuge parameter m to the model phased-out the oscillatory behavior in favor of a stable equilibrium.

Theorem 3.1.

The system (2) undergoes a transcritical bifurcation at the equilibrium P_1 when $m = B$.

Proof:

First of all, we consider the parameter m as a variable in the system (2), which can be written as follow:

$$\dot{X} = F(X, m), \quad X = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3}$$

The first step is to translate the equilibrium P_1 of the system (2) into the origin by the following variable change

$$\bar{x} = x - a, \;\; \bar{y} = y, \;\; \mu = m - B,$$

but, for the sake of simplicity, we will keep the same old variables notations, i.e, we will note

$$\bar{x} = x \text{ and } \bar{y} = y.$$

Then, the system becomes

$$\begin{cases} \frac{dx}{dt} = -b(x + \frac{a}{b})x - \frac{\alpha(1-\mu-B)^2\left(x+\frac{a}{b}\right)^2 y}{\beta^2+(1-\mu-B)^2\left(x+\frac{a}{b}\right)^2}, \\ \frac{dy}{dt} = -cy + \frac{k\alpha(1-\mu-B)^2\left(x+\frac{a}{b}\right)^2 y}{\beta^2+(1-\mu-B)^2\left(x+\frac{a}{b}\right)^2}. \end{cases} \tag{4}$$

The Jacobian matrix at the point $(0, 0, \mu), \;\; \mu \in [-B, 1 - B)$ for this system is given by

$$J(0,0,\mu) = \begin{pmatrix} -a & \frac{-\alpha(1-\mu-B)^2\left(\frac{a}{b}\right)^2}{\beta^2+(1-\mu-B)^2\left(\frac{a}{b}\right)^2} \\ 0 & -c + \frac{-k\alpha(1-\mu-B)^2\left(\frac{a}{b}\right)^2}{\beta^2+(1-\mu-B)^2\left(\frac{a}{b}\right)^2} \end{pmatrix}. \tag{5}$$

For $\mu = 0$, we get the result

$$J(0,0,0) = \begin{pmatrix} -a & -\frac{c}{k} \\ 0 & 0 \end{pmatrix}. \tag{6}$$

Noticing that $det(J(0, 0, 0)) = 0$, it implies that there is a bifurcation at the equilibrium P_1 when $\mu = 0$. Besides, we can easily establish the eigenvalues of (6) which are $\lambda_1 = -a < 0$ and $\lambda_2 = 0$. Then, by the Center Manifold theorem (Carr (1982); Dang-Vu and Delcarte (2000)), we have the existence of a center manifold W^c, a stable manifold W^s and an empty unstable manifold W^u, for which

$$dimW^c = 1, \;\; dimW^s = 1, \;\; dimW^u = 0 \Leftrightarrow W^u = \emptyset.$$

The next step is focused on changing the coordinate basis of the system into the basis containing the vectors from the span set of eigenspaces of $J(0, 0, 0)$.

By a simple calculation, we establish the eigenspaces of J in $(0, 0, 0)$:

$$E^s(0) = \left\{(x,y) \in \mathbb{R}^2 | y = 0\right\},$$
$$E^c(0) = \left\{(x,y) \in \mathbb{R}^2 | x = -\frac{c}{ak}y\right\}.$$

Thereby, we can notice that $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{c}{ak} \\ 1 \end{pmatrix} \right\}$ is a base of $\mathbb{R}^2$.

In these coordinates, the differential equations (4) becomes:

$$\dot{U} = \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = T^{-1}J(0,0,0)TU + \hat{F}(U,\mu), \tag{7}$$

where $T = \begin{pmatrix} -\frac{c}{ak} & 1 \\ 1 & 0 \end{pmatrix}$, $T^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & \frac{c}{ak} \end{pmatrix}$, $U = T^{-1}X$ and $\hat{F}(U,\mu) = T^{-1}F(TU,\mu)$ is the nonlinear term.

Then, we can notice that

$$T^{-1}J(0,0,0)T = \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix},$$

and

$$\hat{F}(U,\mu) = \begin{pmatrix} \hat{F}_1(U,\mu) \\ \hat{F}_2(U,\mu) \end{pmatrix},$$

with

$$\begin{aligned} \hat{F}_1(U,\mu) =& \frac{2bc(k\alpha - c)}{ak\alpha}\left(-\frac{c}{ak} + v\right)u - \frac{2a(k\alpha - c)^2}{\beta bk\alpha}\sqrt{\frac{c}{k\alpha - c}}\mu u \\ &+ \frac{b^2c(k\alpha - c)(k\alpha - 4c)}{a^2k^2\alpha^2}\left(-\frac{c}{ak}u + v\right)^2 u - \frac{a^2(k\alpha - c)^2(k\alpha - 4c)}{b^2\beta^2k^2\alpha^2}\mu^2 u, \end{aligned}$$

and

$$\begin{aligned} \hat{F}_2(U,\mu) =& -b\left(-\frac{c}{ak}u + v\right)^2 + \left(-1 + \frac{c}{a}\right)\Bigg[\frac{2bc(k\alpha - c)}{ak^2\alpha}\left(-\frac{c}{ak} + v\right)u \\ &- \frac{2a(k\alpha - c)^2}{\beta bk^2\alpha}\sqrt{\frac{c}{k\alpha - c}}\mu u + \frac{b^2c(k\alpha - c)(k\alpha - 4c)}{a^2k^3\alpha^2}\left(-\frac{c}{ak}u + v\right)^2 u \\ &- \frac{a^2(k\alpha - c)^2(k\alpha - 4c)}{b^2\beta^2k^3\alpha^2}\mu^2 u\Bigg]. \end{aligned}$$

We add to the system (7) the trivial equation

$$\dot{\mu} = 0. \tag{8}$$

To identify the center manifold for the system (7)-(8), we express the variable v as a function of u and μ. Considering a function h_s such that

$$v = h_s(u,\mu).$$

We know that v is a solution of the second equation of the system (7). Therefore, we obtain by substituting v with the Taylor expansion of $h_s(u,\mu)$, i.e,

$$h_s(u,\mu) = d_{12}u^2 + d_{22}u\mu + d_{21}\mu^2 + d_{13}u^3 + d_{32}u^2\mu + d_{23}u\mu^2 + d_{31}\mu^3$$

in the first equation of (7), so that

$$Dh_s(u,\mu)\hat{F}_1(u, h_s(u,\mu),\mu) = -ah_s(u, h_s(u,\mu),\mu) + \hat{F}_2(u, h_s(u,\mu),\mu),$$

where $Dh_s(u,\mu)$ is the Jacobian matrix of $h_s(u,\mu)$.

Hence, we establish the coefficients d_{ij} as next

$$d_{21} = d_{31} = 0,\ d_{22} = \frac{2(k\alpha - c)^2}{\beta b k^2\alpha}\sqrt{\frac{c}{k\alpha - c}},\ d_{12} = -\frac{bc^2}{a^3k^2}\left[1 + \frac{2(-1+\frac{c}{a})(k\alpha - c)}{k\alpha}\right],$$

$$d_{13} = \frac{bc^2(k\alpha - c)}{a^2k^2\alpha}\left[4 + \frac{bc(k\alpha - c)}{a^2k^3\alpha}\left(-1 + \frac{c}{a}\right)\right] - \frac{2b^2c^3}{a^4}\left(1 + 2\left(-1+\frac{c}{a}\right)\frac{k\alpha - c}{k\alpha}\right)^2,$$

$$d_{23} = -\frac{(k\alpha - c)}{\beta^2b^2k^3\alpha}\left[\frac{4c(k\alpha - c)}{\alpha} + (c-a)(k\alpha - 4c)\right],$$

$$d_{32} = \sqrt{\frac{c}{k\alpha - c}}\Bigg(\frac{-2c(k\alpha - c)^2}{a^2\beta k^3\alpha}\left[\frac{-2c(k\alpha - c)}{ak\alpha} + \frac{k\alpha - c}{k\alpha} - 1\right] + \frac{4c^2(k\alpha - c)^2}{a^3\beta k^3\alpha}\left(1 + 2\left(-1 + \frac{c}{a}\right)\frac{k\alpha - c}{k\alpha}\right)\Bigg).$$

Then,

$$v = h_s(u,\mu) = d_{12}u^2 + d_{22}u\mu + d_{13}u^3 + d_{32}u^2\mu + d_{23}u\mu^2. \tag{9}$$

Therefore, substituting (9) in the system (7)-(8), we get

$$\begin{cases} \dot{u} &= \hat{F}_1(u, h(u,\mu)) := G(u,\mu), \\ \dot{\mu} &= 0\ . \end{cases} \tag{10}$$

Let

$$G_p := \frac{\partial G}{\partial p} \text{ and } G_{pq} := \frac{\partial^2 G}{\partial p \partial q}.$$

The Taylor expansion of G around the origin gives, by adding the conditions $G(0,0) = G_u(0,0) = 0$, which express that the center manifold is tangent to the axis of u at the origin, that

$$\dot{u} = G(u,\mu) = G_\mu(0,0)\mu + \frac{1}{2}\left(G_{uu}(0,0)u^2 + 2G_{u\mu}(0,0)\mu u + G_{\mu\mu}(0,0)\mu^2\right) + O(3). \tag{11}$$

To provide that the bifurcation is transcritical, we need to prove the following

$$G_\mu(0,0) = 0, \quad G_{uu}(0,0) \neq 0.$$

We compute the Taylor expansion of the first equation of (10) and we identify it with the expansion (11). It results that

$$G_\mu(u,\mu) = \frac{2bc(k\alpha-c)}{ak\alpha}d_{22}\mu^2 + 2\left[\frac{bc(k\alpha-c)}{ak\alpha}d_{21} - \frac{a^2(k\alpha-c)^2(k\alpha-4c)}{b^2\beta^2k^2\alpha}\right]u\mu + \frac{2a(k\alpha-c)^2}{\beta bk\alpha}\sqrt{\frac{c}{k\alpha-c}}\mu.$$

So $G_\mu(0,0)=0$.

Moreover,

$$G_{uu}(u,\mu) = -\frac{4bc^2(k\alpha-c)}{a^2k^2\alpha} + 6\left[\frac{2bc(k\alpha-c)}{ak\alpha}d_{21} + \frac{b^2c^3(k\alpha-c)^2(k\alpha-4c)}{a^4k^4\alpha^2}\right]u + \frac{4bc(k\alpha-c)^2}{ak\alpha}d_{22}\mu,$$

which implies that

$$G_{uu}(0,0) = -\frac{4bc^2(k\alpha-c)}{a^2k^2\alpha} \neq 0.$$

Hence, the bifurcation is transcritical as required. ■

3.2. Hopf bifurcation

Previously, we notice that exactly one stable limit cycle occurred in the system with a small refuge parameter (i.e, when the positive equilibria are unstable), which is in accordance with ecological considerations, for which populations are reported to oscillate in a preferred reproducible periodic manner.

Now, we will study the Hopf bifurcation for the system (2) and determine the properties of its unique limit cycle.

Theorem 3.2.

The system (2) undergoes a supercritical Hopf bifurcation at $(P_2, m = B)$. It yields a unique asymptotically stable limit cycle of radius $r = \sqrt{\frac{-\tau'(A)(m-A)}{a_1}}$ and a period $T \approx \frac{2\pi}{\omega_0}$, where a_1 (the first Lyapunov coefficient), $\tau^{'}(m)$ and ω_0 are respectively given by (30), (19) and (20) in the proof .

Proof:

The proof is based on the algorithm of Hassard, Kazarinoff and Wan (see (Dang-Vu and Delcarte

(2000))), which leads to calculate the first coefficients of Poincare's normal form for Hopf bifurcation. Its real part a_1, called the first Lyapunov coefficient, provides to evaluate the periodic orbit's radius emerging at the bifurcation point. Moreover, the sign of this coefficient determine the supercritical or subcritical behavior of the bifurcation.

We started out by the substitution $x \to \bar{x} + x_0$ and $y \to \bar{y} + y_0$ to bring back the fixed point to the origin. Then, we get the following system

$$\begin{cases} \frac{dx}{dt} = a(x+x_0) - b(x+x_0)^2 - \frac{\alpha(1-m)^2(x+x_0)^2(y+y_0)}{\beta^2+(1-m)^2(x+x_0)^2}, \\ \frac{dy}{dt} = -c(y+y_0) + \frac{k\alpha(1-m)^2(x+x_0)^2(y+y_0)}{\beta^2+(1-m)^2(x+x_0)^2}. \end{cases} \tag{12}$$

The matrix

$$\mathfrak{J}(m) = \begin{pmatrix} a_{11}(m) & a_{12}(m) \\ a_{21}(m) & a_{22}(m) \end{pmatrix} = \begin{pmatrix} a - \frac{2\beta b}{1-m}\sqrt{\frac{c}{k\alpha-c}} - \frac{2(k\alpha-c)\left(a-\frac{\beta b}{1-m}\sqrt{\frac{c}{k\alpha-c}}\right)}{k\alpha} & -\frac{c}{k} \\ \frac{2(k\alpha-c)\left(a-\frac{\beta b}{1-m}\sqrt{\frac{c}{k\alpha-c}}\right)}{\alpha} & 0 \end{pmatrix} \tag{13}$$

is the Jacobian matrix for system (12) at the origin.

Furthermore, our system can be expressed by the equation

$$\dot{X} = \mathfrak{J}(m)X + \tilde{F}(X,m), \tag{14}$$

where $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\tilde{F}$ is a nonlinear term to determinate after.

The characteristic equation of (13) is

$$det(\mathfrak{J}(m) - \lambda I) \equiv \lambda^2 + P_2(m)\lambda + Q(m) = 0, \tag{15}$$

$$P_2(m) = -tr(\mathfrak{J}(m)) \ and \ Q(m) = det(\mathfrak{J}(m)).$$

By simple computation, we get

$$P_2(m) = -a + \frac{2\beta b}{1-m}\sqrt{\frac{c}{k\alpha - c}} + \frac{2(k\alpha - c)\left(a - \frac{\beta b}{1-m}\sqrt{\frac{c}{k\alpha-c}}\right)}{k\alpha}, \tag{16}$$

and

$$Q(m) = \frac{2c(k\alpha - c)\left(a - \frac{\beta b}{1-m}\sqrt{\frac{c}{k\alpha-c}}\right)}{k\alpha}. \tag{17}$$

This lead us to prove respectively that $P_2(A) = 0$ and $Q(A) = a(k\alpha - c) > 0$.

Now, we conclude that the characteristic equation has a pure imaginary roots for the parameter value $m = A$. Therefore, by the theorem of Poincare-Andronov-Hopf, $(P_2(A), A)$ is Hopf bifurcation point.

Carrying $\lambda(m) = \tau(m) + i\omega(m)$ into the derivative of (15), will provide us to have for $m = A$ the following system

$$\begin{cases} -2\omega(A)\omega^{'}(A) + Q^{'}(A) = 0, \\ 2\omega(A)\tau^{'}(A) + \omega(A)P_2^{'}(A) = 0. \end{cases} \tag{18}$$

Hence, we can notice that $\tau(A) = 0$ and

$$\tau^{'}(A) = -\frac{P_2^{'}(A)}{2} = -\frac{\beta bc}{k\alpha(1-A)^2}\sqrt{\frac{c}{k\alpha - c}} < 0. \tag{19}$$

Thus, we have a center manifold of dimension 2. Moreover, we get an empty stable and unstable manifolds.

To obtain the center manifold equation, we need at first to establish the passage matrix T from the system's base to the base $\{e_1, e_2\}$ where $e_1 = Real(v_1) = \Re v_1$ and $e_2 = -Imaginary(v_1) = \Im v_1$, v_1 is the eigenvector associated to $\lambda(A) = i\omega_0$ with

$$w_0 = \sqrt{Q(A)} = \sqrt{a(k\alpha - c)}. \tag{20}$$

At first, let $v_1 = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$, we can provide

$$v_1 = \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{a_{12}(A)}\begin{pmatrix} a_{12}(A) \\ -a_{11}(A) + i\omega_0 \end{pmatrix}. \tag{21}$$

So,

$$\begin{aligned} T = \begin{bmatrix} \Re v_1 & -\Im v_1 \end{bmatrix} &= \frac{1}{a_{12}(A)}\begin{pmatrix} a_{12}(A) & 0 \\ -a_{11}(A) & -\omega_0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{k}{c}\sqrt{a(k\alpha - c)} \end{pmatrix}. \end{aligned} \tag{22}$$

Then,

$$T^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{c}{k\sqrt{a(k\alpha - c)}} \end{pmatrix}. \tag{23}$$

Carrying the transformation $U = T^{-1}X$, where $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, into (14) for $m = A$ we obtain

$$\begin{cases} \dot{u}_1 = -\omega_0 u_2 + F_1(u_1, u_2), \\ \dot{u}_2 = \omega_0 u_1 + F_2(u_1, u_2), \end{cases} \tag{24}$$

with $F = (F_1, F_2)$ is defined by $F = T^{-1}\tilde{F}(TX, A)$.

Now, we need to determine the function F.

If we rewrite (12) as

$$\dot{X} = f(X) = \begin{pmatrix} f_1(X) \\ f_2(X) \end{pmatrix},$$

the function $\tilde{F}$ can be formulated as follow

$$\begin{pmatrix} \tilde{F}_1(X,m) \\ \tilde{F}_2(X,m) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\frac{\partial^2 f_1}{\partial x^2}x^2 + \frac{\partial^2 f_1}{\partial x \partial y}xy + \frac{1}{2}\frac{\partial^2 f_1}{\partial y^2}y^2 + \frac{1}{6}\frac{\partial^3 f_1}{\partial x^3}x^3 + \frac{1}{2}\frac{\partial^3 f_1}{\partial x^2 \partial y}x^2 y + \frac{1}{2}\frac{\partial^3 f_1}{\partial x \partial y^2}xy^2 + \frac{1}{6}\frac{\partial^3 f_1}{\partial y^3}y^3 \\ \frac{1}{2}\frac{\partial^2 f_2}{\partial x^2}x^2 + \frac{\partial^2 f_2}{\partial x \partial y}xy + \frac{1}{2}\frac{\partial^2 f_2}{\partial y^2}y^2 + \frac{1}{6}\frac{\partial^3 f_2}{\partial x^3}x^3 + \frac{1}{2}\frac{\partial^3 f_2}{\partial x^2 \partial y}x^2 y + \frac{1}{2}\frac{\partial^3 f_2}{\partial x \partial y^2}xy^2 + \frac{1}{6}\frac{\partial^3 f_2}{\partial y^3}y^3 \end{pmatrix}, \tag{25}$$

where $\dfrac{\partial^i f_p}{\partial x^{j_1} y^{j_2}} = \dfrac{\partial^i f_p}{\partial x^{j_1} y^{j_2}}(X, m)$.

To simplify the previous expression, we denote the derivatives as

$$\begin{pmatrix} \tilde{F}_1(X,m) \\ \tilde{F}_2(X,m) \end{pmatrix} = \begin{pmatrix} \epsilon_{20}x^2 + \epsilon_{11}xy + \epsilon_{02}y^2 + \epsilon_{30}x^3 + \epsilon_{21}x^2y + \epsilon_{12}xy^2 + \epsilon_{03}y^3 \\ \delta_{20}x^2 + \delta_{11}xy + \delta_{02}y^2 + \delta_{30}x^3 + \delta_{21}x^2y + \delta_{12}xy^2 + \delta_{03}y^3 \end{pmatrix}, \tag{26}$$

where $\epsilon_{i,j} = \epsilon_{i,j}(X, m)$ and $\delta_{i,j} = \delta_{i,j}(X, m)$.

After calculating those derivatives at the origin, we get

$$\begin{aligned} &\epsilon_{20} = -2b + \frac{2b(k\alpha - c)(k\alpha - 4c)}{k\alpha(k\alpha - 2c)}, \quad \epsilon_{11} = \frac{4bc^2(k\alpha - c)}{ak^2\alpha(k\alpha - 2c)}, \quad \epsilon_{30} = \frac{48b^2c^2(k\alpha - c)}{(k\alpha - 2c)(k\alpha)^2}, \\ &\epsilon_{21} = \frac{-8b^2c^3(k\alpha - 4c)(k\alpha - c)}{a^2k^3\alpha^2(k\alpha - 2c)^2}, \quad \epsilon_{02} = \epsilon_{12} = \epsilon_{03} = 0. \\ &\delta_{20} = \frac{2b(k\alpha - c)(k\alpha - 4c)}{\alpha(k\alpha - 2c)}, \quad \delta_{11} = \frac{4bc^2(k\alpha - c)}{ak\alpha(k\alpha - 2c)}, \quad \delta_{30} = \frac{-48b^2c^2(k\alpha - c)}{(k\alpha^- 2c)k\alpha^2}, \\ &\delta_{21} = \frac{-8b^2c^3(k\alpha - 4c)(k\alpha - c)}{a^2k^3\alpha^2(k\alpha - 2c)^2}, \quad \delta_{02} = \delta_{12} = \delta_{03} = 0. \end{aligned} \tag{27}$$

Knowing that $TU = \begin{pmatrix} u_1 \\ \frac{k}{c}\sqrt{a(k\alpha - c)}u_2 \end{pmatrix}$, we provide

$$\begin{aligned} F(U) &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{c}{k\sqrt{a(k\alpha-c)}} \end{pmatrix} \tilde{F}\left(\begin{pmatrix} u_1 \\ \frac{k}{c}\sqrt{a(k\alpha - c)}u_2 \end{pmatrix}, A \right) \\ &= \begin{pmatrix} \alpha_{20}x^2 + \alpha_{11}xy + \alpha_{02}y^2 + \alpha_{30}x^3 + \alpha_{21}x^2y + \alpha_{12}xy^2 + \alpha_{03}y^3 \\ \beta_{20}x^2 + \beta_{11}xy + \beta_{02}y^2 + \beta_{30}x^3 + \beta_{21}x^2y + \beta_{12}xy^2 + \beta_{03}y^3 \end{pmatrix}, \end{aligned} \tag{28}$$

where,

$$\begin{aligned} \alpha_{02} &= \alpha_{12} = \alpha_{03} = 0, \\ \alpha_{11} &= \frac{4bc(k\alpha - c)\sqrt{a(k\alpha - c)}}{ak\alpha(k\alpha - 2c)}, \\ \alpha_{30} &= \frac{8b^2c^2(k\alpha - c)}{k^2\alpha^2(k\alpha - 2c)}, \\ \alpha_{21} &= \frac{-4b^2c^2(k\alpha - 4c)(k\alpha - c)}{a^2k^2\alpha^2(k\alpha - 2c)^2}\frac{k}{c}\sqrt{a(k\alpha - c)}, \\ \alpha_{20} &= -b + \frac{b(k\alpha - c)(k\alpha - 4c)}{k\alpha(k\alpha - 2c)}. \\ \beta_{02} &= \beta_{12} = \beta_{03} = 0, \\ \beta_{11} &= \frac{-4bc^2(k\alpha - c)}{ak\alpha(k\alpha - 2c)}, \\ \beta_{30} &= \frac{-8b^2c^3(k\alpha - c)}{k^2\alpha^2(k\alpha - 2c)\sqrt{a(k\alpha - c)}}, \\ \beta_{21} &= \frac{4b^2c^2(k\alpha - 4c)(k\alpha - c)}{a^2k^2\alpha^2(k\alpha - 2c)^2}, \\ \beta_{20} &= \frac{bc(k\alpha - c)(k\alpha - 4c)}{k\alpha(k\alpha - 2c)\sqrt{a(k\alpha - c)}}. \end{aligned} \tag{29}$$

Now, we can calculate the coefficient a_1, which is defined by the expression

$$\begin{aligned} 8a_1 &= \frac{1}{\sqrt{a(k\alpha - c)}}[\alpha_{20}\alpha_{11} - \beta_{20}\beta_{11} - 2\alpha_{20}\beta_{20}] + 3\alpha_{30} + \beta_{21} \\ &= \frac{6b^2c}{ak\alpha}\left(-1 + \frac{(k\alpha - c)(k\alpha - 4c)}{k\alpha(k\alpha - 2c)}\right) - \frac{4b^2c^3(k\alpha - 4c)(k\alpha - c)}{a^2k^2\alpha^2(k\alpha - 2c)^2} \\ &+ \frac{24b^2c^2(k\alpha - c)}{ak^2\alpha^2(k\alpha - 2c)} + \frac{4b^2c^2(k\alpha - c)^2(k\alpha - 4c)}{a^2k^2\alpha^2(k\alpha - 2c)^2}. \end{aligned} \tag{30}$$

Set

$$\zeta = \frac{6b^2c}{ak\alpha}\left(-1 + \frac{(k\alpha - c)(k\alpha - 4c)}{k\alpha(k\alpha - 2c)}\right) - \frac{4b^2c^3(k\alpha - 4c)(k\alpha - c)}{a^2k^2\alpha^2(k\alpha - 2c)^2}. \tag{31}$$

Then, the terms α_{30} and β_{21} are evidently negative. Next, we need to determine the sign of ζ.

Simplifying the expression of ζ, we can prove that

$$\zeta = \frac{2b^2c}{ak\alpha}\left[-3 + \frac{(k\alpha - c)(k\alpha - 4c)}{k\alpha(k\alpha - 2c)}\left[3 - \frac{2c^2}{a(k\alpha - 2c)}\right]\right], \tag{32}$$

which is negative; this proves that a_1 is negative. Then, the Hopf bifurcation point is supercritical.

To end up, we have $\tau^{'}(A) < 0$ and $a_1 < 0$ so the equilibrium point P_2 is asymptotically stable if $m > A$. Moreover, it is unstable if $m < A$ added to the existence of a unique asymptotically stable limit cycle of radius $r = \sqrt{\frac{-\tau^{'}(A)(m - A)}{a_1}}$ and a period $T \approx \frac{2\pi}{\sqrt{a(k\alpha - c)}}$, with $\tau^{'}(A)$ expression is given in (19). ■

4. Numerical Simulation

Keeping the same parameter values as subsection (2.2.2), we will simulate the bifurcation diagram for both the transcritical and Hopf bifurcation, in addition to the radius of the limit cycle relatively to the parameter refuge.

$$\begin{cases} x = \text{the preys density.} \\ y = \text{the predators density.} \end{cases}$$

4.1. Transcritical bifurcation diagram

In Figures (6) and (7), the stability of the fixed points is illustrated for the initial conditions $(u_0, v_0) = (5, 3)$. Thus, those simulations undergo a transcritical bifurcation at $m = 0.76 = B$. We model an ecological system and then look forward to observe the system in a stable equilibrium state, in a neighborhood of $m = B$, which is the extinction of predators. For $m \leq B$, the fixed points, except P_0, converge to collide at the parameter value B and become locally asymptotically stable equilibrium for $m > B$.

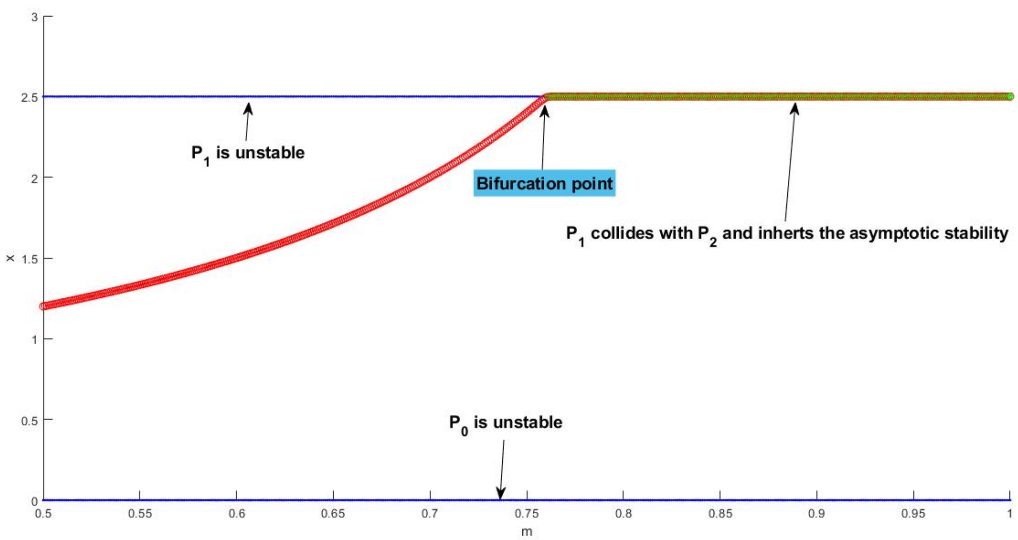

Figure 6. Bifurcation diagram (preys)

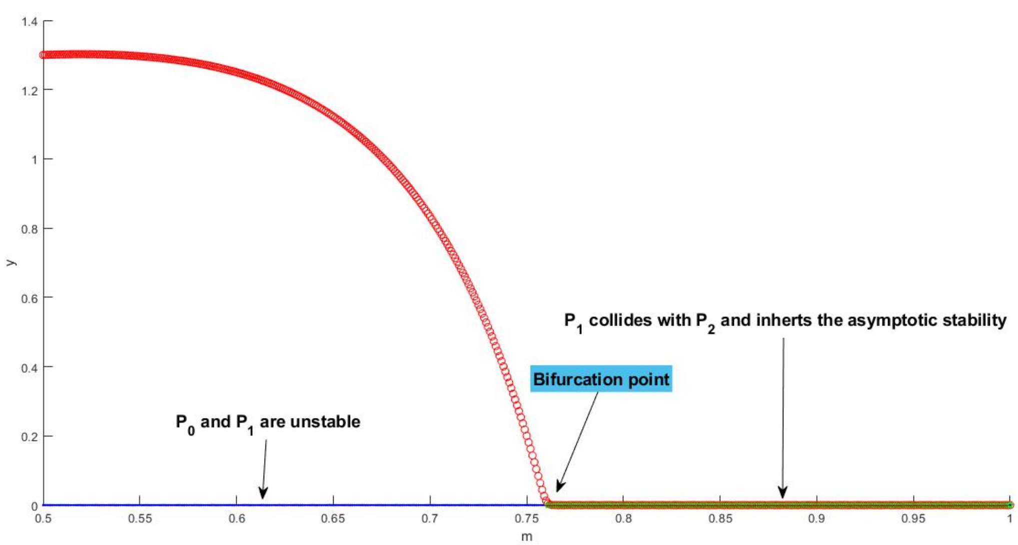

Figure 7. Bifurcation diagram (predators)

4.2. Hopf bifurcation diagram and limit cycle radius simulation

In an ecological system, we are interested in the coexistence. Keeping the same initial conditions as the previous simulation, in Figures (8) and (9), we show the occurrence of a periodic orbit that is pointed asymptotically stable, when the fixed point P_2 is unstable (i.e, $m \leq 0.36$), while the system moves forward to the locally asymptotically stable fixed point P_2 representing the coexistence.

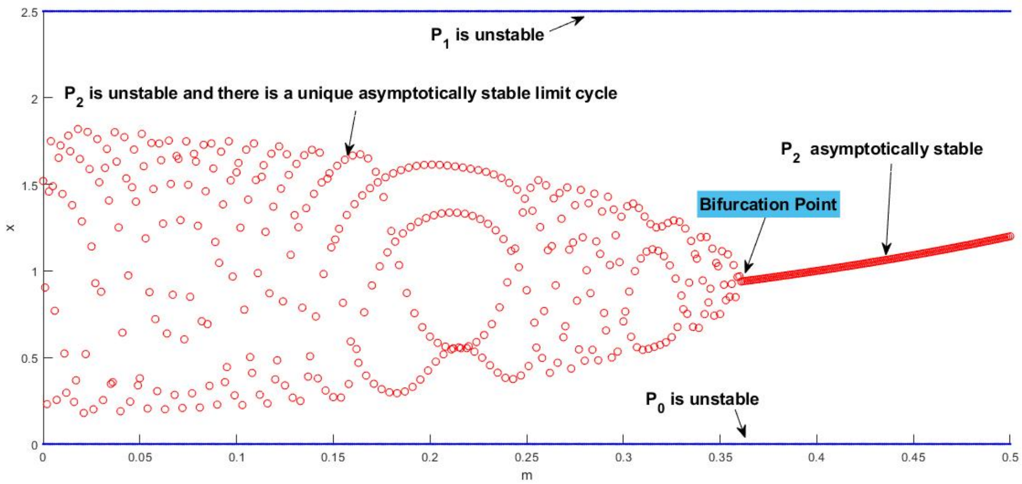

Figure 8. Bifurcation diagram (preys)

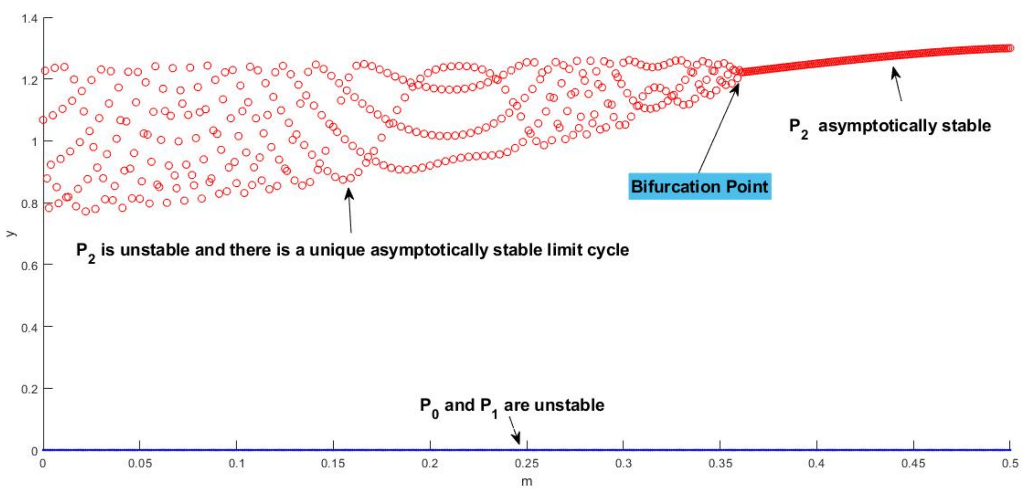

Figure 9. Bifurcation diagram (predators)

In Figure (10), we show how the radius of periodic orbit varies relatively to the refuge parameter. We notice that the radius value decline until vanishing at the bifurcation point, where the limit cycle switches to an equilibrium that is locally asymptotically stable.

5. Conclusion

In this worksheet, we studied the impact of the prey refuge on the dynamic behavior of the model based on a Lotka-Volterra model following the Holling type III functional response. We have noticed the major effect of "refugia" on the stability of system (2). As an ecological model, we search for the coexistence of species in our system which is realizable for small parameter values of refuge where the Hopf bifurcation phenomena appears. Furthermore, as expected for large values

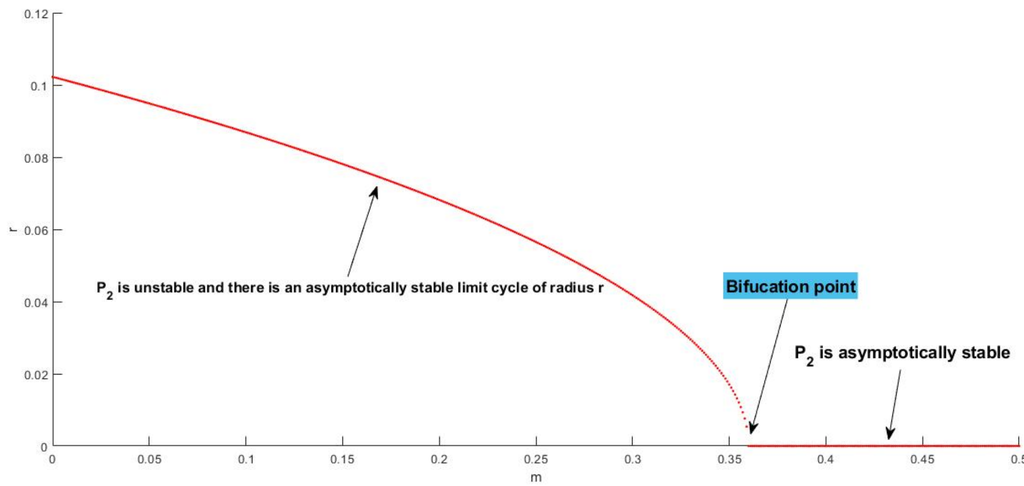

Figure 10. The radius of the cycle limit r with respect to m

of refuge, the system shift from the stability of coexistence to the extinction of predators. This was clearly explained by the occurrence of a transcritical bifurcation. Then, we can notice the interest of studying this model taking into consideration the prey's mobility away from predation and understand the changes of behavior for our dynamical system by bifurcation analysis.

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