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Slow Flow Past Porous Shell of Variable Permeability with Cavity at the Centre

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Abstract

In this paper slow flow of a viscous, incompressible fluid past a heterogeneous porous spherical shell with cavity is discussed. The permeability of porous sphere is varying with radial distance. Flow outside the porous spherical shell and inside the central cavity region is governed by the Stoke's equation. Brinkman equation is used to analyze the flow inside the porous region. The boundary conditions used at the interface of porous and clear region are the continuity of velocity and stress. Exact solution of the problem is obtained and relevant quantities such as stream lines, velocity, pressure and drag on surface of the spherical shell are evaluated and exhibited graphically. The effect of various parameters on the flow has been discussed. Obtained results are useful for the flow past porous particles of variable permeability.

Keywords: Porous media; Spherical shell; Variable permeability flow; Brinkman model; Stoke's flow; Drag force

MSC 2010 No.: 76D07, 76S05

1. Introduction

A porous medium usually consists of a large number of interconnected pores. The exact form of the structure is usually complicated and can differ from medium to medium and it may be of homogeneous permeability or of heterogeneous permeability. Flow through porous medium may be governed by either Darcy's law or Brinkman's equation, depending on the nature of the porous material. Darcy law is a first order equation which balances the volume flow rate in the medium with pressure gradient and it does not take in to account the viscous forces in the medium. Darcy proposed semi empirical law which describes the flow through porous mediums. For homogeneous media this law can be expressed as

$$V = -\frac{k}{\mu}\nabla p,$$

where V is the filter velocity, k is permeability of the medium, μ is the viscosity of the fluid, p is the pressure. Brinkman (1947) introduced an alternative equation commonly known as Brinkman equation. Omitting the inertial terms this equation takes the form

$$\nabla p = -\frac{\mu}{k}\, V + \mu_e \nabla^2 V,$$

where μ_e is the effective viscosity in porous medium.

The porous particles available in nature are of various shapes and sizes. These particles may be only porous or a rigid core covered with a porous layer and sometimes these particles are referred to as composite particles. Most of the available literature on flow through porous medium is on homogeneous porous medium but in real problems porous medium may be of variable permeability. A few examples are that, in colloidal motion when colloids coalesce to form aggregate structures with spatially varying porosities, surface covered with variable porosity porous medium, the catalyzer grains produced by calcination consist of layers with different porosities, tissues in human body, spherical particles having sticky or hairy surfaces.

Masliyah et al. (1987) studied slow flow past a solid sphere with porous shell of homogeneous permeability using the Stoke's equation in clear fluid region and Brinkman equation in porous region. They used continuity of normal and tangential velocity and stress at fluid-porous interface and no slip condition on impermeable core as boundary condition. Qin and Kaloni (1993) investigated the creeping flow past a porous spherical shell by using Stoke's and Brinkman equation and shown that the solution obtained by Darcy's law can be derived with the appropriate choice of boundary conditions.

Keh and Chou (2004) have been investigated analytically the quasi steady translation and steady rotation of a spherically symmetric composite particle composed of a solid core and a surrounding porous shell located at the center of a spherical cavity filled with an incompressible Newtonian fluid. Srinivasacharya (2007) studied the flow of an incompressible viscous liquid past a porous approximate spherical shell. He used Navier-Stoke's equation for the flow in free fluid region and Darcy's law for porous region and obtained exact solution of the problem. The boundary conditions used at the interface are continuity of the normal velocity, continuity of the pressure and Beavers and Joseph slip condition.

Seth et al. (2010) investigated unsteady hydromagnetic convective flow of a viscous incompressible electrically conducting heat generating/absorbing fluid within a parallel plate rotating channel in a uniform porous medium under slip boundary conditions and obtained exact solution of the governing equations for fully developed flow. They analyzed asymptotic behavior of the solution for the fluid velocity for large values of frequency parameter to gain some physical insight into the flow pattern. Tamamneh and Bataineh (2011) investigated axisymmetric viscous, two dimensional steady and incompressible fluid flow past a solid sphere with porous shell at moderate Reynolds numbers. They found that drag coefficient as well as separation angle.

Verma and Datta (2012) found analytical solution of slow flow past a porous sphere of variable permeability and used Stoke's equation for clear fluid flow and Brinkman equation for flow in porous sphere. Saad and Faltas (2012) presented combined analytical numerical method to study the quasi-steady axisymmetrical flow of an incompressible viscous fluid past an assemblage of porous eccentric spherical particle-in-cell models. The flow inside the porous particle is governed by the Brinkman model and the flow in the fictitious envelope region is governed by Stoke's equations. Boundary conditions on the particles surface and fictitious spherical envelope that correspond to the Happel, Kuwabara, Kvashnin and Cunningham/Mehta-Morse models are satisfied by a collocation technique.

Verma and Singh (2014) studied laminar flow of a viscous incompressible fluid in an annular region between two infinitely long coaxial circular cylinders filled by a porous medium of variable permeability. Authors presented permeability of the porous medium varies with the radial distance and flow within the porous annular region is governed by the Brinkman law and found analytical solutions for relevant quantities. Seth et al. (2015) investigated unsteady hydromagnetic natural convection flow of a viscous, incompressible, electrically conducting and temperature dependent heat absorbing fluid confined within a parallel plate rotating vertical channel in porous medium. Fluid flow within the channel is induced due to impulsive movement of one of the plates of the channel. The authors obtained exact solution for the governing equations for fluid velocity and fluid temperature by Laplace transform technique.

Seth et al. (2016) studied unsteady MHD convective Couette flow of a viscous, incompressible, electrically conducting, and temperature dependent heat absorbing fluid within a rotating vertical channel embedded in a fluid saturated porous medium taking Hall current into account. Fluid flow within the channel was induced due to accelerated movement of one of the plates of the channel. They obtained exact solution for fluid velocity and fluid temperature was obtained in closed form by the Laplace transform technique and found that Hall current, thermal buoyancy force and permeability of the medium tended to accelerate fluid flow in both the primary and secondary flow directions whereas heat absorption had a reverse effect on it.

Seth et al. (2017) studied the effects of viscous and Ohmic heating and heat generation/absorption on magnetohydrodynamic flow of an electrically conducting Casson thin film fluid over an unsteady horizontal stretching sheet in a non-Darcy porous medium. The fluid is assumed to slip along the boundary of the sheet. Authors used shooting technique in conjunction with the 4th order Runge-Kutta method to solve the problem and obtained velocity and temperature of the fluid thin film along with local skin friction coefficient and local Nusselt number for a range of values of pertinent flow parameters. Prakash and Raja Sekhar (2017) studied viscous flow past a porous spherical particle composed of a rigid core inside. This particle is located inside a spherical fluid cavity filled with incompressible Newtonian fluid, under the creeping flow conditions. The authors concluded that the presence of cavity wall retards the particle movement and boundary effect is more pronounced when the separation distance between the particle surface and the cavity wall is less.

In this article we study the slow flow of a viscous incompressible fluid through a porous spherical shell of radially varying permeability with cavity at centre. Flow outside the porous spherical shell and inside the central cavity region is governed by the Stoke's equation and Brinkman equation is used to analyze the flow inside the porous region. The boundary conditions used at the interface of porous and clear region are the continuity of velocity and stress.

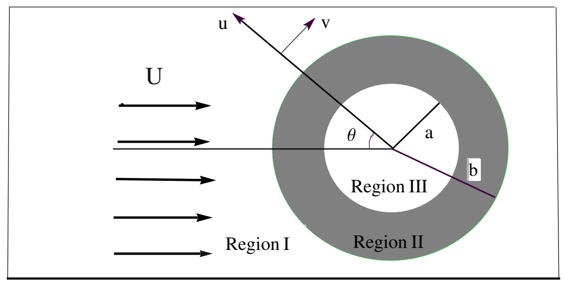

Figure 1. Flow past porous spherical shell of variable permeability with cavity at the center

2. Mathematical Formulation

A steady slow flow of viscous, incompressible fluid past a heterogeneous porous spherical shell with cavity at the centre is considered as shown in Figure 1. Radius of spherical cavity is $r^* = a$ and radius of porous spherical shell is $r^* = b$. Thickness of the porous shell is $(b - a)$. We assume that fluid have uniform velocity U far away from the spherical shell and flow is axis symmetric. Flow field is divided into three regions. Region I is the clear fluid region outside the spherical shell $(r^* \geq b)$, region II is the porous region within the spherical shell, i.e., $a \leq r^* \leq b$, $(b > a)$ and region III is the central cavity region $(0 \leq r^* \leq a)$ within the spherical shell. The flow in clear region I and III is governed by the Stoke's equation and equation of continuity that are given by

$$\nabla p_j^* = \mu \nabla^2 V_j^*, \tag{1}$$

$$\nabla . V_j^* = 0, \tag{2}$$

where V_j^*, p_j^* and μ are the velocity, pressure and viscosity of fluid, respectively. The superscript $j = 1$ correspond to region I $(r^* \geq b)$ and $j = 3$ correspond to region III $(0 \leq r^* \leq a)$, respectively. The flow in porous region II $(a \leq r^* \leq b)$ is governed by the Brinkman equation (1947) together with equation of continuity, that are given by

$$\nabla p_2^* = -\frac{\mu}{k} V_2^* + \mu_e \nabla^2 V_2^*, \tag{3}$$

$$\nabla . V_2^* = 0, \tag{4}$$

where k, μ_e, V_2^* and p_2^* are the permeability, effective viscosity, fluid velocity and pressure in region II, respectively. According to Liu and Masliyah (2005) depending upon the type of porous media, μ_e may be either greater or smaller than μ. We follow Brinkman (1947) and Chikh et al. (1995)

and take $\mu_e = \mu$ for high porosity porous media. With this consideration Brinkman equation (3) becomes

$$\nabla p_2^* = -\frac{\mu}{k} V_2^* + \mu \, \nabla^2 V_2^*. \tag{5}$$

We choose spherical polar coordinate system (r^*, θ, ϕ) with center of spherical shell as the origin and the line $\theta = 0$ as axis of symmetry which is along the direction of uniform flow U, as it is shown in figure 1. Due to axis symmetry of the problem we have $\partial/\partial\phi = 0$. Now we introduce dimensionless variables as follows

$$r = \frac{r^*}{a}, \quad u_j = \frac{u_j^*}{U}, \quad v_j = \frac{v_j^*}{U}, \quad p_j = \frac{a p_j^*}{\mu U}, \tag{6}$$

where u_j^* and v_j^* are the radial and azimuthal component of velocity V_j^* in the increasing direction of r^* and θ, respectively. Here, $j = 1, 2, 3$ corresponds to region I, II and III respectively. Using dimensionless variables equation of continuity (2) and (4) in spherical polar coordinates can be written as

$$\frac{\partial}{\partial r}\left(r^2 u_j\right) + \frac{r}{\sin\theta}\frac{\partial}{\partial\theta}\left(v_j \sin\theta\right) = 0. \tag{7}$$

Equation (1) in dimensionless variables, defined in Equation (6), provides two component equations as

$$\frac{\partial p_j}{\partial r} = \frac{\partial^2 u_j}{\partial r^2} + \frac{2}{r}\frac{\partial u_j}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_j}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u_j}{\partial\theta} - \frac{2u_j}{r^2} - \frac{2}{r^2}\frac{\partial v_j}{\partial\theta} - \frac{2v_j\cot\theta}{r^2}, \tag{8}$$

$$\frac{1}{r}\frac{\partial p_j}{\partial \theta} = \frac{\partial^2 v_j}{\partial r^2} + \frac{2}{r}\frac{\partial v_j}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_j}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial v_j}{\partial\theta} + \frac{2}{r^2}\frac{\partial u_j}{\partial\theta} - \frac{2v_j \text{cosec}^2\theta}{r^2}, \tag{9}$$

Here, $j = 1, 3$ corresponds to region I $(q \leq r \leq \infty)$ and region III $(0 \leq r \leq 1)$, respectively, and $q = b/a$ is the thickness parameter. Brinkman equation (5) for porous region $(1 \leq r \leq q)$ in dimensionless variables provides two component equations as

$$-\frac{\partial p_2}{\partial r} = \frac{a^2 u_2}{kr} - \frac{\partial^2 u_2}{\partial r^2} - \frac{2}{r}\frac{\partial u_2}{\partial r} - \frac{2}{r^2}\frac{\partial^2 u_2}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial u_2}{\partial\theta} + \frac{2u_2}{r^2} + \frac{2}{r^2}\frac{\partial v_2}{\partial\theta} + \frac{2v_2\cot\theta}{r^2}, \tag{10}$$

$$-\frac{1}{r}\frac{\partial p_2}{\partial\theta} = \frac{a^2 v_2}{kr} - \frac{\partial^2 v_2}{\partial r^2} - \frac{2}{r}\frac{\partial v_2}{\partial r} - \frac{1}{r^2}\frac{\partial^2 v_2}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial v_2}{\partial\theta} - \frac{2}{r^2}\frac{\partial u_2}{\partial\theta} + \frac{2v_2\text{cosec}^2\theta}{r^2}. \tag{11}$$

Here, $k(r)$ is the permeability of the porous shell which is varying as a radial distance according to the law $k(r) = k_0 r^2$. k_0 is the permeability on the surface of the shell at $r = 1$. The boundary conditions on the surface of spherical shell are continuity of tangential and normal velocities and stresses that can be expressed in non dimensional variables as

$$\begin{aligned}
u_1 &= u_2 && \text{at} \quad r = q,\\
v_1 &= v_2 && \text{at} \quad r = q,\\
\tau_{r\theta(1)} &= \tau_{r\theta(2)} && \text{at} \quad r = q,\\
\tau_{rr(1)} &= \tau_{rr(2)} && \text{at} \quad r = q,\\
u_2 &= u_3 && \text{at} \quad r = 1,\\
v_2 &= v_3 && \text{at} \quad r = 1,\\
\tau_{r\theta(2)} &= \tau_{r\theta(3)} && \text{at} \quad r = 1,\\
\tau_{rr(2)} &= \tau_{rr(3)} && \text{at} \quad r = 1,
\end{aligned} \tag{12}$$

where, $\tau_{r\theta(j)}$ and $\tau_{rr(j)}$ are dimensionless shear and normal stress, respectively, and are given by

$$\tau_{r\theta(j)} = \frac{1}{r}\frac{\partial u_j}{\partial \theta} + \frac{\partial v_j}{\partial r} - \frac{v_j}{r}, \tag{13}$$

$$\tau_{rr(j)} = -p_j + 2\frac{\partial u_j}{\partial r}. \tag{14}$$

At large distances from origin of spherical shell flow is uniform. Thus, we have another boundary condition

$$u_1 \to \cos\theta \;\text{ and }\; v_1 \to -\sin\theta \;\text{ as }\; r \to \infty, \tag{15}$$

3. Solution of the problem

We introduce the Stoke's stream function ψ_j such that

$$u_j = \frac{1}{r^2 \sin\theta}\frac{\partial \psi_j}{\partial \theta},\; v_j = -\frac{1}{r\sin\theta}\frac{\partial \psi_j}{\partial r};\quad j \;=\; 1, 2, 3, \tag{16}$$

where ψ_1, ψ_2 and ψ_3 are stream functions corresponding to regions I, II and III, respectively. Eliminating pressure p_j from Equations (8) and (9) and then using Equation (16) we get

$$E^4\psi_j = 0. \tag{17}$$

Here, $j = i$ correspond to region I and $j = 3$ correspond to region III, respectively, where E^2 is Stoke's stream function operator defined as,

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right),$$

and

$$E^4 = \frac{\partial^4}{\partial r^4} + \frac{6\sin\theta}{r^4}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\times\left[\frac{\partial^2}{\partial r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^3}{\partial\theta\partial r^2}\right.$$
$$\left. + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\times\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right], \tag{18}$$

Again, eliminating pressure p_2 from Equations (10) and (11), then using Equation (16) we get

$$E^4\psi_2 - \frac{a^2}{k(r)}\left(E^2 - \frac{1}{k}\frac{\partial k}{\partial r}\frac{\partial}{\partial r}\right)\psi_2 = 0, \qquad 1 \le r \le q. \tag{19}$$

Now, permeability of the shell varies according to the law $k(r) = k_0 r^2$. For this $k(r)$, Equation (19) becomes

$$E^4\psi_2 - \frac{\sigma^2}{r^2}\left(E^2 - \frac{2}{r}\frac{\partial}{\partial r}\right)\psi_2 = 0, \qquad 1 \le r \le q, \tag{20}$$

where $\sigma^2 = a^2/k_0$ is permeability variation parameter.

Boundary condition (15) in terms of Stoke's stream function can be expressed as

$$\psi_1 \to \frac{r^2}{2}\sin^2\theta \qquad \text{as} \qquad r \to \infty. \tag{21}$$

Boundary condition (21) leads to consideration of solution of Equations (17) and (20) in the form

$$\psi_1(r,\theta) = f_1(r)\sin^2\theta, \qquad q \le r \le \infty, \tag{22}$$

$$\psi_2(r,\theta) = f_2(r)\sin^2\theta, \qquad 1 \le r \le q, \tag{23}$$

$$\psi_3(r,\theta) = f_3(r)\sin^2\theta, \qquad 0 \le r \le 1. \tag{24}$$

Substituting ψ_1 and ψ_3 from the above equations in Equation (17) and ψ_2 in Equation (20), we get the following ordinary differential equations,

$$r^4 f_1^{''''} - 4r^2 f_1^{''} + 8r f_1^{'} - 8f_1 = 0, \qquad q \le r \le \infty, \tag{25}$$
$$r^4 f_2^{''''} - 4r^2 f_2^{''} + 8r f_2^{'} - 8f_2 - \sigma^2(r^2 f_2^{''} - 2r f_2^{'} - 2f_2) = 0, \qquad 1 \le r \le q, \tag{26}$$
$$r^4 f_3^{''''} - 4r^2 f_3^{''} + 8r f_3^{'} - 8f_3 = 0, \qquad 0 \le r \le 1. \tag{27}$$

Equations (25) and (27) are homogeneous ordinary differential equation of order four. The solutions of these equations are

$$f_1(r) = \frac{A_1}{r} + B_1 r + C_1 r^2 + D_1 r^4, \tag{28}$$

and

$$f_3(r) = \frac{A_3}{r} + B_3 r + C_3 r^2 + D_3 r^4. \tag{29}$$

The solution of Equation (26) is

$$\begin{aligned} f_2(r) = {} & A_2\, r^{\frac{1}{2}\left(3-\sqrt{13+2\sigma^2-2\sqrt{36-4\sigma^2+\sigma^4}}\right)} + B_2\, r^{\frac{1}{2}\left(3+\sqrt{13+2\sigma^2-2\sqrt{36-4\sigma^2+\sigma^4}}\right)} \\ & + C_2\, r^{\frac{1}{2}\left(3-\sqrt{13+2\sigma^2+2\sqrt{36-4\sigma^2+\sigma^4}}\right)} + D_2\, r^{\frac{1}{2}\left(3+\sqrt{13+2\sigma^2+2\sqrt{36-4\sigma^2+\sigma^4}}\right)}, \end{aligned} \tag{30}$$

where A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, A_3, B_3, C_3 and D_3 are constants of integration which can be determined by using boundary conditions. With the above expressions for $f_1(r)$, $f_2(r)$ and $f_3(r)$ stream function in the region I, II and III are given by equations (22), (23) and (24), respectively. From boundary conditions (21) we get $C_1 = 1/2$, $D_1 = 0$ and $A_3 = B_3 = 0$. By substituting values of C_1, $D_{1,}$ $A_{3,}$ B_3, equations (22) and (24) becomes

$$f_1(r) = \frac{A_1}{r} + B_1 r + \frac{r^2}{2}, \tag{31}$$

$$f_3(r) = C_3 r^2 + D_3 r^4. \tag{32}$$

Boundary conditions (12) in terms of $f_1(r)$, $f_2(r)$ and $f_3(r)$ can be written as

$$\begin{aligned} f_1(q) &= f_2(q), \\ f_1^{'}(q) &= f_2^{'}(q), \\ f_1^{''}(q) &= f_2^{''}(q), \\ f_1^{'''}(q) &= f_2^{'''}(q) - \sigma^2 f_2^{'}(q), \\ f_2(1) &= f_3(1), \\ f_2^{'}(1) &= f_3^{'}(1), \\ f_2^{''}(1) &= f_3^{''}(1), \\ f_2^{'''}(1) &= f_3^{'''}(1) - \sigma^2 f_3^{'}(1). \end{aligned} \tag{33}$$

Using the above boundary conditions (33), we get the values of arbitrary constants A_1, B_1, A_2, B_2, C_2, D_2, C_3 and D_3, which are given in the Appendix. Thus the stream functions in region I, II and III are given by

$$\psi_1(r,\theta) = \left(\frac{A_1}{r} + B_1 r + \frac{r^2}{2}\right)\sin^2\theta, \qquad (q \le r \le \infty), \tag{34}$$

$$\psi_2(r,\theta) = \left(A_2 r^{\alpha_1} + B_2 r^{\beta_1} + C_2 r^{\alpha_2} + D_2 r^{\beta_2}\right)\sin^2\theta, \quad (1 \le r \le q), \tag{35}$$

$$\psi_3(r,\theta) = \left(C_3 r^2 + D_3 r^4\right)\sin^2\theta, \qquad (0 \le r \le 1). \tag{36}$$

Using Equation (16), the radial and azimuthal component of the fluid velocity in region I, II and III are given by

$$u_1 = \frac{2\cos\theta}{r^2}\left(\frac{A_1}{r} + B_1 r + \frac{r^2}{2}\right), \qquad (q \le r \le \infty), \tag{37}$$

$$v_1 = -\frac{\sin\theta}{r}\left(-\frac{A_1}{r^2} + B_1 + r\right), \qquad (q \le r \le \infty), \tag{38}$$

$$u_2 = \frac{2\cos\theta}{r^2}\left(A_2 r^{\alpha_1} + B_2 r^{\beta_1} + C_2 r^{\alpha_2} + D_2 r^{\beta_2}\right), \qquad (1 \le r \le q), \tag{39}$$

$$v_2 = -\frac{\sin\theta}{r}\left(A_2\alpha_1 r^{\alpha_1-1} + B_2\beta_1 r^{\beta_1-1} + C_2\alpha_2 r^{\alpha_2-1} + D_2\beta_2 r^{\beta_2-1}\right), \quad (1 \le r \le q), \tag{40}$$

$$u_3 = \frac{2\cos\theta}{r^2}\left(C_3 r^2 + D_3 r^4\right), \qquad (0 \le r \le 1), \tag{41}$$

$$v_3 = -\frac{\sin\theta}{r}\left(2C_3 r + 4D_3 r^3\right), \quad (0 \le r \le 1), \tag{42}$$

where constants A_1, B_1, A_2, B_2, C_2, D_2, C_3 and D_3 are given in the Appendix. Substituting velocity from Equations (37) and (38) in Equation (8) and integrating the resulting equation, we get the pressure p_1 outside the spherical shell as

$$p_1 = \frac{2\cos\theta B_1}{r^2} \qquad (q \le r \le \infty). \tag{43}$$

Similarly, the pressure p_2 inside the spherical shell (porous region) ($1 \le r \le q$) is obtained by using Equations (39) and (40) in Equation (10) as

$$\begin{aligned} p_2 = {} & A_2\, r^{\alpha_1-3}(4 - \alpha_1\sigma^2 - 3\alpha_1^2 + \alpha_1^3) + B_2\, r^{\beta_1-3}(4 - \beta_1\sigma^2 - 3\beta_1^2 + \beta_1^3) \\ & + C_2\, r^{\alpha_2-3}(4 - \alpha_2\sigma^2 - 3\alpha_2^2 + \alpha_2^3) + D_2\, r^{\beta_2-3}(4 - \beta_2\sigma^2 - 3\beta_2^2 + \beta_2^3), \end{aligned} \tag{44}$$

and the pressure p_3 inside the spherical shell (cavity region) ($0 \le r \le 1$) is obtained by using Equations (41) and (42) in Equation (8) as

$$p_3 = 20 D_3 r\cos\theta \qquad (0 \le r \le 1). \tag{45}$$

3.1. Drag Force on the Spherical Shell

The non-dimensional drag force acting on the surface of porous spherical shell is given by

$$D = 2\pi \int_0^\pi \left(\tau_{rr(1)} \cos\theta - \tau_{r\theta(1)} \sin\theta\right)_{r=q} \sin\theta \, d\theta. \tag{46}$$

Substituting $\tau_{rr(2)}$ and $\tau_{r\theta(2)}$ from Equations (13) and (14) in the above equation, we get

$$D = 12\pi \int_0^\pi \left[-\left(\frac{2A_1}{q^4} + \frac{B_1}{q^2}\right)\cos^2\theta \sin\theta + \frac{A_1}{q^4}\sin^3\theta\right] d\theta. \tag{47}$$

After integration we get

$$D = -\frac{8\pi B_1}{q^2}, \tag{48}$$

where B_1 is given in the Appendix. If permeability is zero then drag on spherical shell is

$$D_\infty = \lim_{\sigma\to\infty} D = \frac{6\pi}{q}.S \tag{49}$$

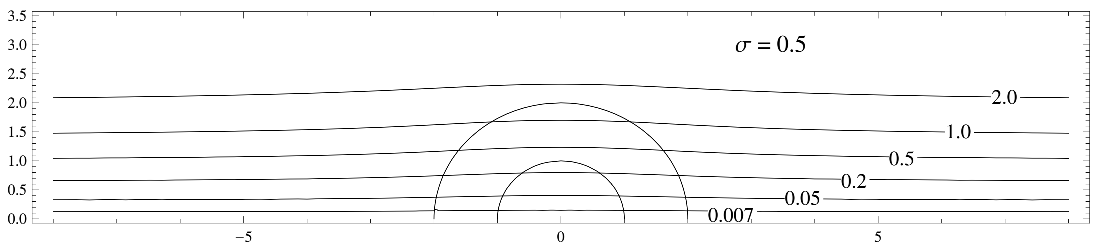

Figure 2. Stream lines $\psi = c$ for $c = 0.007, 0.05, 0.2, 0.5, 1.0, 2.0$ when $q = 2$ and $\sigma = 0.5$

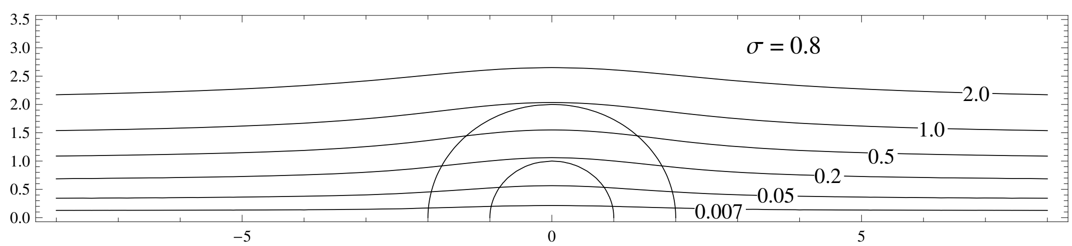

Figure 3. Stream lines $\psi = c$ for $c = 0.007, 0.05, 0.2, 0.5, 1.0, 2.0$ when $q = 2$ and $\sigma = 0.8$

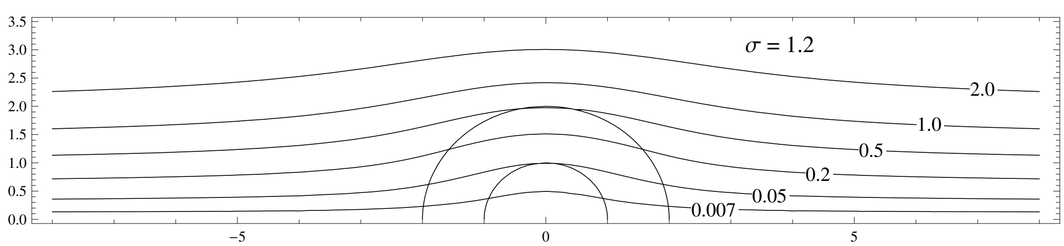

Figure 4. Stream lines $\psi = c$ for $c = 0.007, 0.05, 0.2, 0.5, 1.0, 2.0$ when $q = 2$ and $\sigma = 1.2$

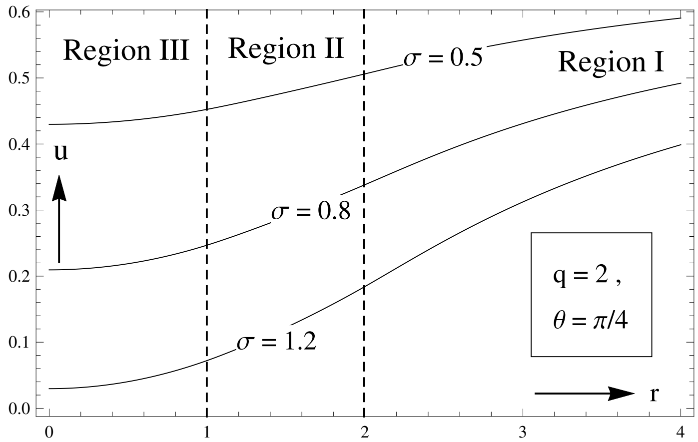

Figure 5. Variation of radial velocity u with radial distance r

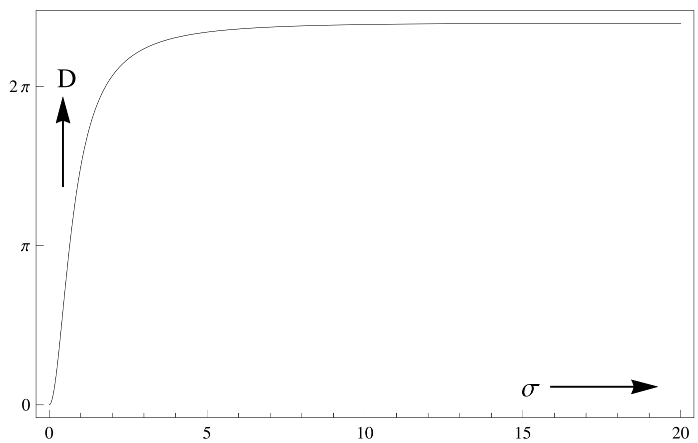

Figure 6. Variation of Drag force D with σ for fixed $q = 2$

4. Discussion

Figures 2, 3 and 4 represents the stream lines of the flow for permeability variation parameter $\sigma = 0.5, 0.8$ and 1.2, respectively. These lines are sketched by using equations (34), (35) and (36). From these figures we can see the effect of permeability variation parameter σ on the stream lines of the flow. These figures also reveal that-

• As permeability parameter σ increases, the stream lines shift away from the centre sphere. This indicate that flow through sphere reduces as σ increases. This is because increase in $\sigma^2 = a^2/k_0$ caused by decrease in the permeability of porous spherical shell.

• In region I at a large distance from the spherical shell stream lines are almost parallel to the direction of the main stream velocity U and is independent of σ.

• In region II, the stream lines have a curved shape because of permeability variation according to the law $k(r) = k_0 r^2$ and curvature of the stream lines increases with σ.

Figure 5 represents the variation of the radial velocity u with radial distance r for $q = 2$ and $\theta = \pi/4$

for $\sigma = 0.5, 0.8$ and 1.2. These graphs are sketched by using equations (37), (38), (39), (40), (41) and (42). We observe that velocity is increasing as σ is decreasing (i.e., when permeability of the porous region is increasing). We also observe that velocity is constant at large distance from the spherical shell. Velocity decreases sharply in the porous region with r whereas inside in the cavity region it is approaches to fixed value. Therefore, the size of the cavity region effects the flow very much.

In Figure 6 variation of the drag force on the spherical shell with σ for fixed $q = 2$ is shown. We observe that drag increases with increase in σ initially and then drag becomes almost constant, i.e., independent of σ. This is because increase in σ caused decrease in permeability of the porous shell. Equation (48) shows that drag on the shell is very much dependent on the size of the cavity region.

5. Conclusion

The steady slow flow of viscous incompressible fluid through a porous spherical shell of radially varying permeability with central cavity has been investigated using the Brinkman and Stoke's equation. An exact solution of the problem in closed form is obtained. We find expressions for the stream lines, fluid velocity and drag force on the spherical shell. The effect of various parameters on the flow are discussed and obtained results are exhibited graphically. We found that variation of permeability has significant effect on the flow. Also the size of cavity region effect the flow characteristics very much. In the limiting case when $\sigma \to \infty$ the obtained results reduces to the classical results of the flow past solid sphere. The obtained results are useful for the flow through porous particles of variable permeability.

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Appendix

$$A_1 = \left(\frac{K_2K_7}{\Delta} + \frac{K_1L_{10}}{T_1} + K_9\right)$$
$$B_1 = \left(\frac{K_2K_8}{\Delta} + \frac{K_1L_{11}}{T_1} + K_{10}\right)$$
$$A_2 = \left(-\frac{K_2}{\Delta}\right)$$
$$B_2 = \left(\frac{K_2T_4 - \Delta K_1}{\Delta T_1}\right)$$
$$C_2 = \left(\frac{\Delta\left(K_{16}L_1T_1 - K_1L_2\right) + K_2\left(L_5T_1 + L_2T_4\right)}{\Delta L_1T_1}\right)$$
$$D_2 = \left(\frac{\Delta\left(K_{13}T_1T_6 - K_1T_5\right) + K_2\left(T_4T_5 - T_1T_7\right)}{\Delta T_1T_6}\right)$$
$$C_3 = \left(\frac{K_2\left(-K_3L_5 - K_5T_4 + K_{11}\right)}{\Delta} + K_1K_5 + K_{14}\right)$$
$$D_3 = \left(\frac{K_2\left(-K_4L_5 - K_6T_4 + K_{12}\right)}{\Delta} + K_1K_6 + K_{15}\right)$$
$$\Delta = \left(T_1T_3 + T_2T_4\right)$$
$$K_1 = \left(L_1L_7 + L_8\right)$$
$$K_2 = \left(K_1T_2 + L_9T_1\right)$$
$$K_3 = \left(\frac{f_{11} - \frac{f_6h_{10}}{h_6}}{L_1}\right)$$
$$K_4 = \left(\frac{f_{12} - \frac{f_6h_{11}}{h_6}}{L_1}\right)$$
$$K_5 = \left(\frac{-\frac{g_6h_{10}}{h_6} + g_{10} + K_3L_2}{T_1}\right)$$
$$K_6 = \left(\frac{-\frac{g_6h_{11}}{h_6} + g_{11} + K_4L_2}{T_1}\right)$$
$$K_7 = \left(\frac{e_2y_3q^{e_1}}{2h_1} - \frac{f_2L_5y_2q^{f_1}}{2h_1L_1} - \frac{L_{10}T_4}{T_1}\right)$$
$$K_8 = \left(-\frac{e_1y_3q^{e_2}}{2h_2} + \frac{f_1L_5y_2q^{f_2}}{2h_2L_1} - \frac{L_{11}T_4}{T_1}\right)$$
$$K_9 = \left(\frac{h_3q^3}{4h_1} - \frac{3f_2h_5y_2q^{\alpha_2}}{2h_1L_1}\right)$$
$$K_{10} = \left(\frac{3f_1h_5y_2q^{f_3}}{2h_2L_1} - \frac{3h_3q}{4h_2}\right)$$
$$K_{11} = \left(e_{10} - \frac{e_6h_{10}}{h_6}\right)$$

$$
\begin{aligned}
K_{12} &= e_{11} - \frac{e_6 h_{11}}{h_6} \\
K_{13} &= -\frac{3 f_{10} q^2}{2T_6} \\
K_{14} &= -\frac{3(\alpha_2 - 4) q^2 T_8}{4T_6} \\
K_{15} &= \frac{3 f_3 q^2 T_9}{4T_6} \\
K_{16} &= \left(\frac{3h_5}{L_1 q}\right) \\
T_1 &= (L_1 R_1 + L_2 R_2) \\
T_2 &= (L_1 L_3 - L_2 L_4) \\
T_3 &= (L_1 L_6 - L_4 L_5) \\
T_4 &= (L_1 R_3 - L_5 R_2) \\
T_5 &= -\alpha_2^2 \beta_1^2 q^{\alpha_2} + \beta_1^2 q^{\alpha_2} + 6\alpha_2^2 \beta_1 q^{\alpha_2} - 6\beta_1 q^{\alpha_2} - \alpha_2^2 q^{\beta_1} + \alpha_2^2 \beta_1^2 q^{\beta_1} - 6\alpha_2 \beta_1^2 q^{\beta_1} \\
&\quad + 6\alpha_2 q^{\beta_1} - 8\alpha_2^2 q^{\alpha_2} + 8q^{\alpha_2} + 8\beta_1^2 q^{\beta_1} - 8q^{\beta_1} \\
T_6 &= \alpha_2^2 \beta_2^2 q^{\alpha_2} - \beta_2^2 q^{\alpha_2} - 6\alpha_2^2 \beta_2 q^{\alpha_2} + 6\beta_2 q^{\alpha_2} + \alpha_2^2 q^{\beta_2} - \alpha_2^2 \beta_2^2 q^{\beta_2} + 6\alpha_2 \beta_2^2 q^{\beta_2} \\
&\quad - 6\alpha_2 q^{\beta_2} + 8\alpha_2^2 q^{\alpha_2} - 8q^{\alpha_2} - 8\beta_2^2 q^{\beta_2} + 8q^{\beta_2} \\
T_7 &= 8\alpha_1^2 q^{\alpha_1} + \alpha_1^2 \alpha_2^2 q^{\alpha_1} - \alpha_2^2 q^{\alpha_1} - 6\alpha_1^2 \alpha_2 q^{\alpha_1} + 6\alpha_2 q^{\alpha_1} - 8q^{\alpha_1} + \alpha_1^2 q^{\alpha_2} - \alpha_1^2 \alpha_2^2 q^{\alpha_2} \\
&\quad + 6\alpha_1 \alpha_2^2 q^{\alpha_2} - 8\alpha_2^2 q^{\alpha_2} - 6\alpha_1 q^{\alpha_2} + 8q^{\alpha_2} \\
T_8 &= \left(-\alpha_2 \beta_2 + 4\alpha_2 + \beta_2^2 - 4\beta_2\right) \\
T_9 &= \left(-\alpha_2 \beta_2 + 2\alpha_2 + \beta_2^2 - 2\beta_2\right) \\
L_1 &= (f_5 h_6 - f_6 h_5) \\
L_2 &= (g_6 h_5 - g_5 h_6) \\
L_3 &= (g_6 h_9 - g_8 h_6) \\
L_4 &= (f_9 h_6 - f_6 h_9) \\
L_5 &= (e_5 h_6 - e_6 h_5) \\
L_6 &= (e_9 h_6 - e_6 h_9) \\
L_7 &= \left(-\frac{3h_6}{q^2} - \frac{3H_1}{q}\right) \\
L_8 &= \left(\frac{3h_5 R_2}{q}\right) \\
L_9 &= \left(\frac{3h_5 L_4}{q} - \frac{3h_9 L_1}{q}\right) \\
L_{10} &= \left(\frac{f_2 L_2 y_2 q^{f_1}}{2h_1 L_1} + \frac{g_2 y_1 q^{g_1}}{2h_1}\right) \\
L_{11} &= \left(-\frac{f_1 L_2 y_2 q^{f_2}}{2h_2 L_1} - \frac{g_1 y_1 q^{g_2}}{2h_2}\right) \\
H_1 &= (h_7 + h_8) \\
F_1 &= (f_7 + f_8) \\
G_1 &= (g_7 + g_9) \\
E_1 &= (e_7 + e_8)
\end{aligned}
$$

$$R_1 = (G_1 h_6 - g_6 H_1)$$
$$R_2 = (F_1 h_6 - f_6 H_1)$$
$$R_3 = (h_6 E_1 - e_6 H_1)$$
$$y_1 = (\beta_2 - \beta_1)$$
$$y_2 = (\beta_2 - \alpha_2)$$
$$y_3 = (\beta_2 - \alpha_1)$$
$$e_1 = (\alpha_1 + 1)$$
$$e_2 = (\alpha_1 - 1)$$
$$e_3 = (\alpha_1 - 2)$$
$$e_4 = (\alpha_1 - 3)$$
$$e_5 = 16\left(\alpha_1 - 4\right) + 2\left(12\alpha_1 - 4\alpha_1 e_2\right)$$
$$e_6 = -2\alpha_1 e_2 q^{e_4} - \frac{2\left(\alpha_1 q^{\alpha_1} - q^{\alpha_1}\right)}{q^3}$$
$$e_7 = \frac{6\left(\alpha_1 q^{\alpha_1} - q^{\alpha_1}\right)}{q^4}$$
$$e_8 = \frac{2\left(\alpha_1 \sigma^2 q^{e_2} - \alpha_1 e_3 e_2 q^{e_4}\right)}{q}$$
$$e_9 = 2\{-\alpha_1(4\sigma^2 - 24) - 4\alpha_1 e_3 e_2\} + 48(\alpha_1 - 4)$$
$$e_{10} = \left(\frac{\alpha_1}{2} - 2\right)$$
$$e_{11} = \left(1 - \frac{\alpha_1}{2}\right)$$
$$f_1 = (\alpha_2 + 1)$$
$$f_2 = (\alpha_2 - 1)$$
$$f_3 = (\alpha_2 - 2)$$
$$f_4 = (\alpha_2 - 3)$$
$$f_5 = 16\left(\alpha_2 - 4\right) + 2\left(12\alpha_2 - 4\alpha_2 f_2\right)$$
$$f_6 = \left(-2\alpha_2 f_2 q^{f_4} - \frac{2\left(\alpha_2 q^{\alpha_2} - q^{\alpha_2}\right)}{q^3}\right)$$
$$f_7 = \frac{6\left(\alpha_2 q^{\alpha_2} - q^{\alpha_2}\right)}{q^4}$$
$$f_8 = \frac{2\left(\alpha_2 \sigma^2 q^{f_2} - \alpha_2 f_3 f_2 q^{f_4}\right)}{q}$$
$$f_9 = 2\{-\alpha_2(4\sigma^2 - 24) - 4\alpha_2 f_3 f_2\} + 48(\alpha_2 - 4)$$
$$f_{10} = \left(\alpha_2^2 - 6\alpha_2 + 8\right)$$
$$f_{11} = \left(\frac{\alpha_2}{2} - 2\right)$$
$$f_{12} = \left(1 - \frac{\alpha_2}{2}\right)$$
$$g_1 = (\beta_1 + 1)$$
$$g_2 = (\beta_1 - 1)$$
$$g_3 = (\beta_1 - 2)$$
$$g_4 = (\beta_1 - 3)$$

$$g_5 = 16\left(\beta_1 - 4\right) + 2\left(12\beta_1 - 4\beta_1 g_2\right)$$
$$g_6 = -2\beta_1 g_2 q^{g_4} - \frac{2\left(\beta_1 q^{\beta_1} - q^{\beta_1}\right)}{q^3}$$
$$g_7 = \frac{2\left(\beta_1 \sigma^2 q^{g_2} - \beta_1 g_3 g_2 q^{g_4}\right)}{q}$$
$$g_8 = 2\{-\beta_1(4\sigma^2 - 24) - 4\beta_1 g_3 g_2)\} + 48(\beta_1 - 4)$$
$$g_9 = \frac{6\left(\beta_1 q^{\beta_1} - q^{\beta_1}\right)}{q^4}$$
$$g_{10} = \left(\frac{\beta_1}{2} - 2\right)$$
$$g_{11} = \left(1 - \frac{\beta_1}{2}\right)$$
$$h_1 = (\beta_2 + 1)$$
$$h_2 = (\beta_2 - 1)$$
$$h_3 = (\beta_2 - 2)$$
$$h_4 = (\beta_2 - 3)$$
$$h_5 = 16\left(\beta_2 - 4\right) + 2\left(12\beta_2 - 4\beta_2 h_2\right)$$
$$h_6 = \left(2q^{h_4} - 2\beta_2^2 q^{h_4}\right)$$
$$h_7 = \frac{6\left(\beta_2 q^{\beta_2} - q^{\beta_2}\right)}{q^4}$$
$$h_8 = \frac{2\left(\beta_2 \sigma^2 q^{h_2} - \beta_2 h_3 h_2 q^{h_4}\right)}{q}$$
$$h_9 = 2\{-\beta_2(4\sigma^2 - 24) - 4\beta_2 h_3 h_2\} + 48\left(\beta_2 - 4\right)$$
$$h_{10} = \left(\frac{\beta_2}{2} - 2\right)$$
$$h_{11} = \left(1 - \frac{\beta_2}{2}\right)$$
$$\alpha_1 = \frac{1}{2}\left(3 - \sqrt{2\sigma^2 - 2\sqrt{\sigma^4 - 4\sigma^2 + 36} + 13}\right)$$
$$\alpha_2 = \frac{1}{2}\left(3 - \sqrt{2\sigma^2 + 2\sqrt{\sigma^4 - 4\sigma^2 + 36} + 13}\right)$$
$$\beta_1 = \frac{1}{2}\left(3 + \sqrt{2\sigma^2 - 2\sqrt{\sigma^4 - 4\sigma^2 + 36} + 13}\right)$$
$$\beta_2 = \frac{1}{2}\left(3 + \sqrt{2\sigma^2 + 2\sqrt{\sigma^4 - 4\sigma^2 + 36} + 13}\right)$$