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On a Hybrid Technique to Handle Analytical and Approximate Solutions of Linear and Nonlinear Fractional Order Partial Differential Equations

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Abstract

This manuscript is devoted to consider Natural transform (NT) coupled with homotopy perturbation method (HPM) for obtaining series solutions to some linear and nonlinear fractional partial differential equations (FPDEs). By means of NT, we obtain the transformed problem which is then solved by using HPM. By means of Stehfest's numerical algorithm and using the dual relationship of NT and Laplace transform, we calculate inverse NT for approximate solutions. The series solutions we obtain using the proposed method are in close agreement with the exact solutions. We apply the proposed method to some interesting problems to illustrate our main results.

Keywords: FPDEs; Natural Transform; Homotopy Perturbation method

MSC 2010 No.: 26A33, 34B37, 35B38

1. Introduction

In last few decades FDEs have been given much attention because they have a lot of applications in different branches of science and technology. FDEs have been considered in different scientific fields for modeling different processes and phenomenon. One can see the applications of FDEs in physical, biological, chemical and social sciences as well as in engineering sides, control theory, biophysics (for detail see Hilfer (2000), Kilbas et al. (2006), Anastassiou (2009), Goufo and Nieto (2018), Goufo and Kubeka (2018), Goufo et al. (2018)).

It has found that in many situations classical differential equations cannot explain memory description and hereditary properties in mathematical models of real world problems in the best way as compared to FDEs. It has been found that FDEs are the best tools to describe certain biological, physical, chemical and psychological as well hereditary processes. To find exact solutions of FDEs, sometimes it is not an easy job to solve these differential equations due to the complexities of fractional calculus involving in these equations. However, the solution of these equations may be easily approximated with a large variety of methods, for example, finite difference method, Fourier series method, Adomian decomposition method, Homotopy analysis method, Method of radial base function, Wavelet techniques, spectral methods, etc. The mentioned methods have their own features, as few of them yield very good answers and some are less accurate. For instance, the solutions of linear and non–linear differential equations have been received by using HAM by Liao (2003), which was further extended to homotopy perturbation method (see articles of Momani and Odibat (2007), He (2000), Biazar and Ghazvini (2007), and Barari et al. (2008)). The solutions of time-fractional diffusion equations by HPM was received by Sheng and Chen (2011) and Goufo and Atangana (2016). Kexue and Jigen (2011) have applied Laplace transform methods for the solutions of FDEs with constant coefficients. Khan and Khan (2008) introduced the Natural NT for solving some fluid problems.

Silambarasn and Belgacem (2012, 2013) used NT for the solution of Maxwell's equations. Loonker and Banerji (2013) and Shah and Khan (2015) obtained the analytical solutions of some non-homogenous FDEs by NT. In the last few years some authors have used power series to compute approximate solutions to various classes of FDEs and FPDEs (see for detail Zhang et al. (2016), Kumar et al. (2016), Kumar et al. (2017), and Morales-Delgado et al. (2018)).

Motivated by the aforementioned work, we considered a new iterative technique by combining NT with HPM for the approximate solutions of FPDEs. We call this method NHPM. We solve some interesting problems by using NHPM for the approximate solutions. From the dual relationship of Laplace and NT and the numerical Stehfest's algorithm, one can calculate the inverse of the aforementioned transform. We also compare our obtained results to that of exact solutions of the corresponding problems. For piloting the results, we have used Maple 16 and Tech-plot.

This work is designed in six sections. In the first section of the paper, we have cited some basic work related to the HPM and NT and give the applications of these methods in different scientific fields. The necessary definitions and results are given in the second section from fractional calculus. In the third section, we discuss HPM. Section four contains our main work of the paper. In the fifth section, we give the applications of NHPM by solving different problems in FPDEs. Section six describes the conclusion of the paper.

2. Basic Definitions and Results

Some fundamental results that we need are taken from fractional calculus for detail (see Kilbas et al. (2006), Javidi and Ahmad (2013), Shah et al. (2018), Loonker and Banerji (2013), Belgacem and Silambarasan (2012)) are given below.

Definition 2.1.

Fractional integral of Riemann-Liouville type with order $\alpha \in \mathbb{R}_+$ of a function $h(t) \in L([0,1],\mathbb{R})$ is given by

$$I_t^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds.$$

It is to be noted that the integral on the right side converges.

Definition 2.2.

The function $f \in (\mathbb{R},\mathbb{R}^+)$ is said to be in the space C_μ for $\mu \in \mathbb{R}$ if it can be expressed as $f(x) = x^q f_1(x)$ with $q > \mu,\ f_1(x) \in C[0,\infty)$ and it is in space $f(x) \in C_\mu^n$ if $f^{(n)} \in C_\mu$ for $n \in NU\{0\}$.

Definition 2.3.

The Riemann-Liouville fractional order derivative of a function $h \in C_{-1}^n$ with $n \in NU\{0\}$ is expressed as

$$D_t^{\alpha} h(t) = \frac{d^n}{dt^n} I^{n-\alpha} h(t),\ \alpha \in (n-1,n],\ n \in N.$$

Definition 2.4.

Fractional order derivative in Caputo sense to a function $h \in C_{-1}^n$ for $n \in NU\{0\}$ is given by

$$D_t^{\alpha} h(t) = \begin{cases} I^{n-\alpha} f^{(n)}, & \alpha \in (n-1,n],\ n \in N, \\ \dfrac{d^n}{dt^n} h(t), & \alpha = n,\ n \in N. \end{cases}$$

Definition 2.5.

The Mittag-Leffler function in two parameters is given by

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha+\beta)},$$

for $\alpha = \beta = 1,\ E_{1,1}(t) = e^t,\ E_{1,1}(-t) = e^{-t}$.

Definition 2.6.

The NT of a function $v(x,t)$ for $t \geq 0$, is defined by

$$N[v(x,t)] = R(x,s,u) = \int_0^\infty e^{-st}\, v(x,\, ut)\, dt,\ s,u \in \mathbb{R}^+.$$

Definition 2.7.

Taking NT of $E_{\alpha,\beta}$, we get

$$N[E_{\alpha,\beta}] = \sum_{k=0}^{\infty} \frac{u^{k+1}\Gamma(k+1)}{s^{k+1}\Gamma(k\alpha+\beta)}.$$

Definition 2.8.

The NT of $D^\alpha f(t)$ is computed as

$$\begin{aligned} N\left(D^\alpha f(t)\right) &= N\left(I^{n-\alpha} f^{(n)}(t)\right) \\ &= N\left[\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds\right] \\ &= \frac{u^{n-\alpha}}{s^{n-\alpha}} N\{f^{(n)}(t)\} \\ &= \frac{u^{n-\alpha}}{s^{n-\alpha}} \left[\frac{s^n}{u^n} R(s,u) - \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k}} f^{(k)}(0)\right]. \end{aligned} \tag{1}$$

For the following results see Belgacem and Silambarasan (2012), Loonker and Banerji (2013), and Stehfest (2005).

Lemma 2.9.

The NT of $\frac{\partial^\alpha f(x,t)}{\partial t^\alpha}$ is calculated as

$$N\left[\frac{\partial^\alpha f(x,t)}{\partial t^\alpha}\right] = \frac{s^\alpha}{u^\alpha} R(x,s,u) - \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k}} \left[\lim_{t\to 0} \frac{\partial^\alpha f(x,t)}{\partial t^\alpha}\right]. \tag{2}$$

Lemma 2.10.

The NT of $\frac{\partial^\alpha f}{\partial x^\alpha}(x,t)$ is computed as

$$N\left[\frac{\partial^\alpha f(x,t)}{\partial x^\alpha}\right] = \frac{d^\alpha}{dx^\alpha} R(x,s,u).$$

Lemma 2.11.

The dual relationship between NT and Laplace transform is given by

$$N\left[f(x,t)\right] = R(x,s,u) = \frac{1}{u}\int_0^\infty e^{\frac{-st}{u}} f(x,t)dt = \frac{1}{u} F\left(x, \frac{s}{u}\right).$$

Definition 2.12.

The Gaver-Stehfest's algorithm for inverse NT of function $f(t)$ is defined by using the duality of NT and Laplace transform only changing the parameter in Laplace by $\frac{s}{u}$ as

$$f_n(x,t) = N^{-1}\left[\frac{1}{u}F\left(x,\frac{s}{u}\right)\right] = \frac{\ln(t)}{t}\sum_{k=1}^{2m} a_k(m)R\left(x,k\frac{\ln(2)}{t}\right), \tag{3}$$

where $a_k(m) = (-1)^{m+k}\sum_{j=[\frac{k+1}{2}]}^{\min\{k,m\}} j^{m+1}\frac{\Gamma(2j+1)}{\Gamma(m-j+1\Gamma^2(j+1)\Gamma(2j-k+1)\Gamma(k-j+1)}$, where $[c]$ denotes the integer part of the real number c.

3. Homotopy Perturbation Method

In this section, we recall the HPM as given by He (2000), Liao (2003) and Madani et al. (2011). For the general class of FDE with boundary conditions of the form

$$\mathbf{T}(v) = f(r),\ r \in \Omega, \tag{4}$$

$$B\left(v,\frac{\partial v}{\partial x}\right) = 0,\ r \in \Gamma, \tag{5}$$

such a general differential operator is represented by $\mathbf{T}$, boundary operator by B and $f(r)$ stands for an analytical function. Also Γ is the boundary of the domain Ω. Further, the operator $\mathbf{T}$ is split into two parts, the linear operator L and non-linear operator N. Hence, (4) can be written as

$$L(v) + N(v) - f(r) = 0. \tag{6}$$

Now, we define a homotopy for $r \in \Omega$ and $H(r,\epsilon) : \Omega \times [0,1] \to \mathbb{R}$ as:

$$H(v,\epsilon) = (1-\epsilon)[L(v) - L(v_0)] + \epsilon[\mathbf{T}(v) - f(r)] = 0,\ \epsilon \in [0,1],$$

or

$$H(v,\epsilon) = L(v) - L(v_0) + \epsilon L(v_0) + \epsilon[N(v) - f(r)] = 0, \epsilon \in [0,1], \tag{7}$$

such that an embedding parameter is denoted by $0 \le \epsilon \le 1$ while an initial approximation is v_0 for (4) with

$$H(v,0) = L(v) - L(v_0) = 0,\ H(v,1) = \mathbf{T}(v) - f(r) = 0. \tag{8}$$

Assume the solution of (4) in the form of power series in ϵ, i.e,

$$v = \sum_{i=0}^{\infty} \epsilon^i v_i. \tag{9}$$

Putting the value of v from (9) in (7) and comparing the coefficients of ϵ gives a successive procedure to calculate v_i. Finally, by putting $\epsilon = 1$ in (9), we obtain the required solution of (4).

4. Main Work

In this section, we use both the NT and HPM for the development of NHPM. For this purpose, we consider the following general FPDE (Javidi and Ahmad (2012)):

$$\frac{\partial^\alpha v}{\partial t^\alpha} + A(x)\frac{\partial v}{\partial x} + B(x)\frac{\partial^2 v}{\partial t^2} + C(x) = g(x,t),\ (x,t) \in [0,1] \times [0,T], \tag{10}$$

with initial conditions

$$\frac{\partial^k v(x,0)}{\partial t^k} = f_k(x),\ k = 0,1,2...,n-1, \tag{11}$$

and boundary conditions

$$v(0,t) = h_0(t),\ v(1,t) = h_1(t),\ t \geq 0, \tag{12}$$

where $f_k(k = 0,1,2,...,n-1)$, g, h_0, h_1, A, B, C, are given functions/constants and $T \in \mathbb{R}^+,\ \alpha \in (n-1,n]$.

To proceed further taking the NT of (10)-(12) and using (2), we deduce that

$$\begin{aligned} &\frac{u^{n-\alpha}}{s^{n-\alpha}}\left[\frac{s^n}{u^n}R(x,s,u) - \sum_{k=0}^{n-1}\frac{s^{n-k-1}}{u^{n-k}}f^{(k)}(0)\right] \\ &+ \left[A(x)\frac{\partial}{\partial x} + B(x)\frac{\partial^2}{\partial x^2} + C(x)\right]R(x,s,u) = G(x,s,u), \end{aligned} \tag{13}$$

where $G(x,s,u)$ is NT of $g(x,t)$ and

$$R(0,s,u) = N\{h_0(t)\},\ R(1,s,u) = N\{h_1(t)\}. \tag{14}$$

Now rewrite (13) as

$$\begin{aligned} \frac{s^\alpha}{u^\alpha}R(x,s,u) = &\frac{u^{n-\alpha}}{s^{n-\alpha}}\sum_{k=0}^{n-1}\frac{s^{n-k-1}}{u^{n-k}}f^{(k)}(0) - \left[A(x)\frac{\partial}{\partial x} + B(x)\frac{\partial^2}{\partial x^2} + C(x)\right]R(x,s,u) \\ &+ G(x,s,u). \end{aligned} \tag{15}$$

Now, we construct a homotopy for (15) as under

$$\begin{aligned} R(x,s,u) =& \epsilon\left(-\frac{u^\alpha}{s^\alpha}\left[A(x)\frac{\partial}{\partial x} + B(x)\frac{\partial^2}{\partial x^2} + C(x)\right]\right)R(x,s,u) \\ &+ \frac{u^n}{s^n}\sum_{k=0}^{n-1}\frac{s^{n-k-1}}{u^{n-k}}f^{(k)}(0) + \frac{u^\alpha}{s^\alpha}G(x,s,u). \end{aligned} \tag{16}$$

We assume the series solution of (16) as

$$R(x,s,u) = \sum_{i=0}^{\infty}\epsilon^i R_i(x,s,u), \tag{17}$$

where $R_i(x,s,u),\ i=0,1,2,\dots,$ are known functions. By the help of (16) and (17), we have

$$\sum_{i=0}^{\infty} R_i(x,s,u) = -\epsilon\frac{u^\alpha}{s^\alpha}\left[A(x)\frac{\partial}{\partial x}+B(x)\frac{\partial^2}{\partial x^2}+C(x)\right]\sum_{i=0}^{\infty}\epsilon^i R_i(x,s,u) \\ +\frac{u^n}{s^n}\sum_{k=0}^{n-1}\frac{s^{n-k-1}}{u^{n-k}}f^{(k)}(0)+\frac{u^\alpha}{s^\alpha}G(x,s,u). \tag{18}$$

Now comparing the coefficients of ϵ^i for $i=0,1,2,\dots,n+1$, we have

$$\begin{aligned}
\epsilon^0 &: R_0(x,s,u) = \frac{u^n}{s^n}\sum_{k=0}^{n-1}\frac{s^{n-k-1}}{u^{n-k}}f^{(k)}(0)+\frac{u^\alpha}{s^\alpha}G(x,s,u),\\
\epsilon^1 &: R_1(x,s,u) = -\frac{u^\alpha}{s^\alpha}\left[A(x)\frac{\partial}{\partial x}+B(x)\frac{\partial^2}{\partial x^2}+C(x)\right]R_0(x,s,u),\\
\epsilon^2 &: R_2(x,s,u) = -\frac{u^\alpha}{s^\alpha}\left[A(x)\frac{\partial}{\partial x}+B(x)\frac{\partial^2}{\partial x^2}+C(x)\right]R_1(x,s,u),\\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
\epsilon^{n+1} &: R_{n+1}(x,s,u) = -\frac{u^\alpha}{s^\alpha}\left[A(x)\frac{\partial}{\partial x}+B(x)\frac{\partial^2}{\partial x^2}+C(x)\right]R_n(x,s,u).
\end{aligned} \tag{19}$$

Assume $\epsilon\to 1$ in (17), then the approximate solution of the problem (10) – (12), is given by

$$F_n(x,s)=\sum_{j=0}^{n}R_j(x,s,u). \tag{20}$$

Taking the inverse Natural transform of (20) and using Definition 2.12, we get

$$v(x,t)\approx v_n(x,t)=N^{-1}\{F_n(x,s,u)\}. \tag{21}$$

5. Applications of NHPM

In this section, we apply the NHPM to some problems of FPDEs to demonstrate the adopted technique.

Example 5.1.

In this example, we are solving the following FPDE (Javidi and Ahmad (2012))

$$\frac{\partial^\alpha v}{\partial t^\alpha}-\frac{x^2}{2}\frac{\partial^2 v}{\partial t^2}=0,\ (t,x)\in[0,1]\times[0,1],\ \alpha\in(1,2], \tag{22}$$

subject to the initial/boundary conditions

$$v(x,0)=x,\ v_t(x,0)=x^2,\ v(0,t)=0,\ v(1,t)=1+\sum_{k=0}^{\infty}\frac{t^{k\alpha+1}}{\Gamma(k\alpha+2)}. \tag{23}$$

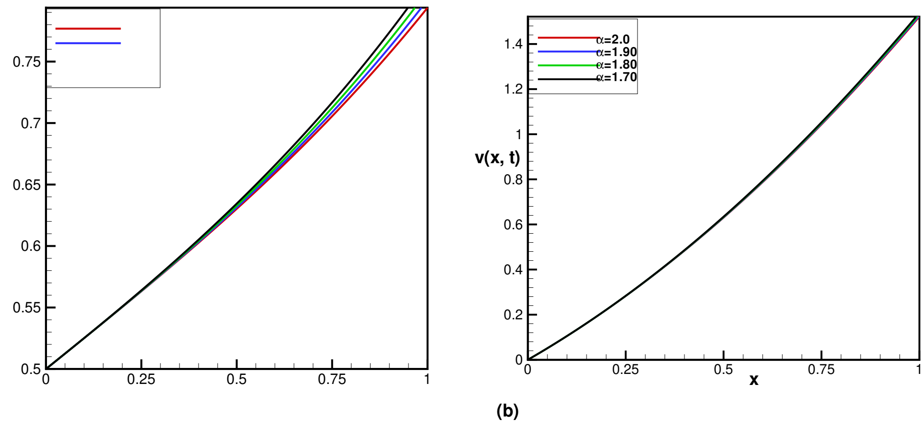

Figure 1. **(a)** Represents physical behavior of Example 5.1 at different fractional order at $x = 0.5$
(b) Represents physical behavior of Example 5.1 at different fractional order at $t = 0.5$

The particular solution of the problem (22)-(23) is

$$v(x,t) = x + x^2 \sum_{k=0}^{\infty} \frac{t^{k\alpha+1}}{\Gamma(k\alpha+2)}.$$

By the help of our proposed method (19), we get

$$\begin{aligned}
R_0(x,s,u) &= \frac{u^2}{s^2}\left(\frac{sx}{u^2} + \frac{x^2}{u}\right) = \frac{x}{s} + \frac{u}{s^2}x^2, \\
R_1(x,s,u) &= -\frac{u^\alpha}{s^\alpha}\left[-\frac{x^2}{2}\frac{\partial^2}{\partial x^2}R_0(x,s,u)\right] = \frac{u^{\alpha+1}}{s^{\alpha+2}}x^2, \\
R_2(x,s,u) &= \frac{u^\alpha}{s^\alpha}\left[\frac{x^2}{2}\frac{\partial^2}{\partial x^2}R_1(x,s,u)\right] = \frac{u^{2\alpha+1}}{s^{2\alpha+2}}x^2, \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
R_{n+1}(x,s,u) &= \frac{u^\alpha}{s^\alpha}\left[\frac{x^2}{2}\frac{\partial^2}{\partial x^2}R_n(x,s,u)\right] = \frac{u^{(n+1)\alpha+1}}{s^{(n+1)\alpha+2}}x^2.
\end{aligned} \tag{24}$$

Now according to (20), we have

$$F_n(x,s,u) = \frac{x}{s} + \frac{u}{s^2}x^2\left(1 + \frac{u^\alpha}{s^\alpha} + \frac{u^{2\alpha}}{s^{2\alpha}} + \ldots + \frac{u^{n\alpha}}{s^{n\alpha}}\right) = \frac{x}{s} + x^2\sum_{k=0}^{n}\frac{u^{k\alpha+1}}{s^{k\alpha+2}}. \tag{25}$$

Further, taking the inverse N-transform we have

$$\begin{aligned} v_n(x,t) &= N^{-1}\{F_n(x,s,u)\} \\ &= N^{-1}\left[\frac{x}{s}\right] + N^{-1}\left[x^2\sum_{k=0}^{n}\frac{u^{k\alpha+1}}{s^{k\alpha+2}}\right] \\ &= x + x^2\sum_{k=0}^{n}\frac{t^{k\alpha+1}}{\Gamma(k\alpha+2)}. \end{aligned} \tag{26}$$

On using $n \to \infty$, we get analytical solution as

$$v(x,t) = \lim_{n\to\infty} v_n(x,t) = x + x^2\sum_{k=0}^{\infty}\frac{t^{k\alpha+1}}{\Gamma(k\alpha+2)}. \tag{27}$$

We plot the analytical solutions for different values of x, t in subplots a and b respectively, of Figure 1 against various values of fractional order. At $\alpha = 2$ the exact solutions have very close agreement with the solutions of fractional order $1.90, 1.80, 1.70$ in both cases either keeping x fixed or t.

Example 5.2.

Consider the following problem (Moaddy et al. (2011)) as

$$\frac{\partial^\alpha v}{\partial t^\alpha} + x\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial t^2} = 2(t^\alpha + x^2 + 1),\ (x,t) \in [0,1]\times[0,1],\ \alpha \in (0,1], \tag{28}$$

subject to the initial conditions

$$v(x,0) = x^2, \tag{29}$$

and the boundary conditions are

$$v(0,t) = 2t^{2\alpha}\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)},\ v(1,t) = 1 + 2t^2\alpha\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}. \tag{30}$$

Now, the particular solution of the problem is

$$v(x,t) = x^2 + 2t^{2\alpha}\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}. \tag{31}$$

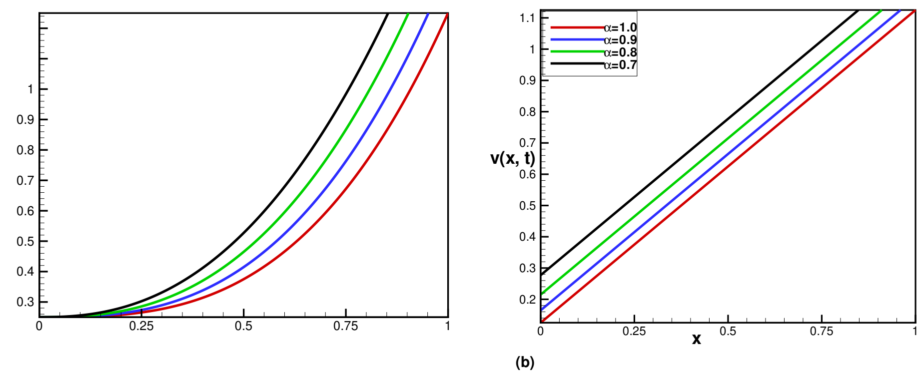

Figure 2. (a) Represents physical behavior of Example 5.2 at different fractional order at $x = 0.5$
(b) Represents physical behavior of Example 5.2 at different fractional order at $t = 0.5$

Now by the application of NHPM, we get

$$
\begin{aligned}
R_0(x,s,u) &= \frac{x^2}{s^2} + \frac{2u^\alpha}{s^\alpha}\left[\Gamma(\alpha+1)\frac{u^\alpha}{s^{(\alpha+1)}} + \frac{x^2+1}{s}\right] \\
&= \frac{x^2}{s} + \frac{2u^\alpha}{s^\alpha}\left[\Gamma(\alpha+1)\frac{u^\alpha}{s^{(\alpha+1)}} + \frac{x^2+1}{s}\right], \\
R_1(x,s,u) &= -\frac{u^\alpha}{s^\alpha}\left[x\frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2}\right] R_0(x,s,u) = -(x^2+1)\left[2\frac{u^\alpha}{s^{\alpha+1}} + 2^2\frac{u^\alpha}{s^{2\alpha+1}}\right], \\
R_2(x,s,u) &= -\frac{u^\alpha}{s^\alpha}\left[x\frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2}\right] R_1(x,s,u) \\
&= (-1)^2(x^2+1)\left[2^2\frac{u^{2\alpha}}{s^{2\alpha+1}} + 2^3\frac{u^{2\alpha}}{s^{3\alpha+1}}\right], \\
&\vdots \\
R_{n+1}(x,s,u) &= -\frac{u^\alpha}{s^\alpha}\left[x\frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2}\right] R_n(x,s,u) \\
&= (-1)^{n+1}(x^2+1)\left[2^{n+1}\frac{u^{(n+1)\alpha}}{s^{(n+1)\alpha+1}} + 2^{n+2}\frac{u^{(n+2)\alpha}}{s^{(n+2)\alpha+1}}\right].
\end{aligned} \tag{32}
$$

Further, using (20),

$$
F_n(x,s,u) = \frac{x^2}{s} + 2.\Gamma(\alpha+1)\frac{u^\alpha}{s^{\alpha+1}} + (-1)^n(x^2+1)2^{n+1}\frac{u^{(n+1)\alpha}}{s^{(n+1)\alpha+1}}, \tag{33}
$$

taking inverse NT and using $n \to \infty$, we have

$$
v(x,t) = \lim_{n\to\infty} v_n(x,t) = x^2 + 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}. \tag{34}
$$

We plot in Figure 2 the analytical solutions for various values of fractional order by taking specific values of x and t respectively.

Example 5.3.

Consider the following problem (Javidi and Ahmad (2012)) as

$$\frac{\partial^\alpha v}{\partial t^\alpha} + x\frac{\partial v}{\partial x} = \frac{\sin(x)t^{1-\alpha}}{\Gamma(2-\alpha)} + t\cos(x),\ (x,t) \in [0,1]\times[0,1],\ \alpha \in (0,1]. \tag{35}$$

The initial/boundary conditions are given as

$$v(x,0) = 0, \tag{36}$$

and

$$v(0,t) = 0,\ v(1,t) = t\sin(1). \tag{37}$$

The equation

$$v(x,t) = t\sin(x) \tag{38}$$

represents the exact solution of Example 5.3.

Thanks to the considered technique, we get

$$\begin{aligned}
R_0(x,s,u) &= \frac{u^\alpha}{s^\alpha}\left[\frac{u^{1-\alpha}}{s^{2-\alpha}}\sin(x) + \cos(x)\frac{u}{s^2}\right],\\
R_1(x,s,u) &= -\frac{u^\alpha}{s^\alpha}\left[\frac{\partial}{\partial x}R_0(x,s,u)\right]\\
&= -\frac{u^{2\alpha}}{s^{2\alpha}}\left[\frac{u^{1-\alpha}}{s^{2-\alpha}}\sin\left(x+\frac{\pi}{2}\right) + \cos\left(x+\frac{\pi}{2}\right)\frac{u}{s^2}\right],\\
R_2(x,s,u) &= (-1)^2\frac{u^{2\alpha}}{s^{2\alpha}}\left[\frac{\partial}{\partial x}R_1(x,s,u)\right]\\
&= (-1)^2\frac{u^{3\alpha}}{s^{3\alpha}}\left[\frac{u^{1-\alpha}}{s^{2-\alpha}}\sin\left(x+\frac{2\pi}{2}\right) + \cos\left(x+\frac{2\pi}{2}\right)\frac{u}{s^2}\right],\\
&\ \vdots\\
R_{n+1}(x,s,u) &= (-1)^n\frac{u^{(n+1)\alpha}}{s^{(n+1)\alpha}}\left[\frac{u^{1-\alpha}}{s^{2-\alpha}}\sin\left(x+\frac{n}{2}\pi\right) + \cos\left(x+\frac{n}{2}\pi\right)\frac{u}{s^2}\right].
\end{aligned} \tag{39}$$

Thus, by NHPM suggested in (20), we get

$$F_n(x,s,u) = \frac{u}{s^2}\sin x + (-1)^n\cos\left(x+n\frac{\pi}{2}\right)\frac{u^{(n+1)\alpha+1}}{s^{(n+1)\alpha+2}}. \tag{40}$$

Taking the inverse NT of above, we get

$$v_n(x,t) = N^{-1}\left[F_n(x,s,u)\right] = \begin{cases} t\sin(x) + t^{(n+1)\alpha+1}\cos(x), & n = 2m,\\ \sin(x)\left(t - t^{(n+1)\alpha+1}\right), & n = 2m+1. \end{cases} \tag{41}$$

Upon using $n \to \infty$ together with Definition 2.12, we get the exact solution, is given by

$$v(x,t) = \lim_{n\to\infty} v_n(x,t) = t\sin(x). \tag{42}$$

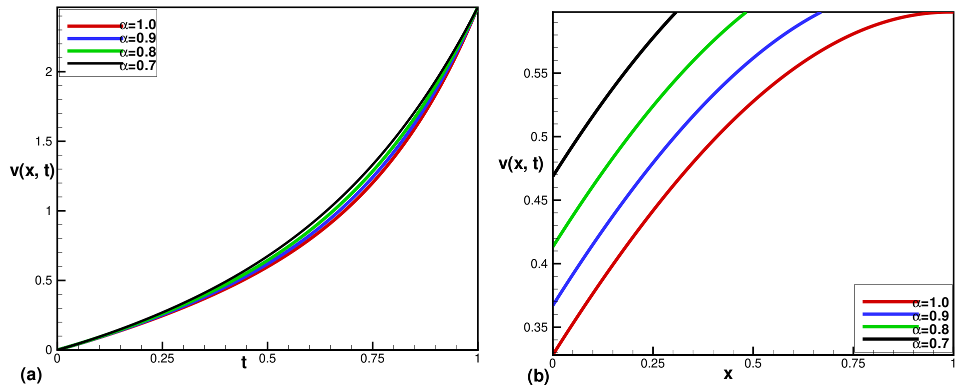

Figure 3. **(a)** Represents plots of approximate solutions at different fractional order and taking $x = 0.5$ taking $n = 4$
(b) Represents plots of approximate solutions at different fractional order and taking $t = 0.5$ taking $n = 4$

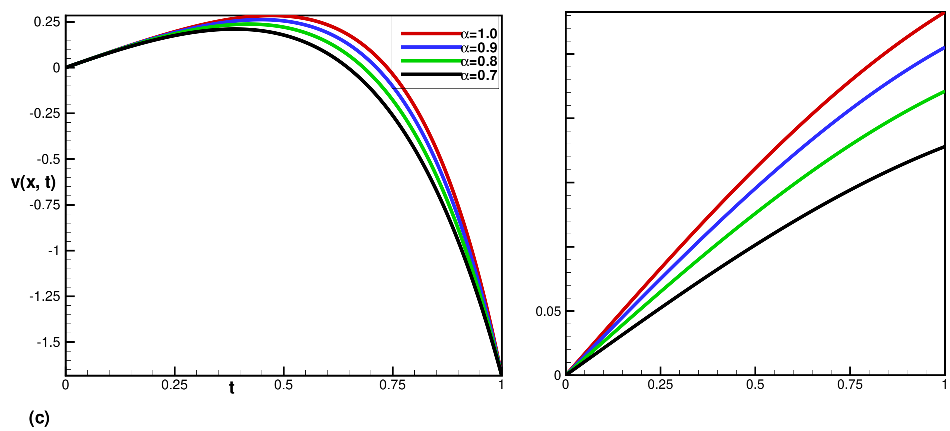

Figure 4. **(a)** Represents plots of approximate solutions at different fractional order and taking $x = 0.5$ taking $n = 3$
(b) Represents plots of approximate solutions at different fractional order and taking $t = 0.5$ taking $n = 3$

In Figures 3 and 4, we have plotted the approximate solutions obtained in (41) corresponding to even and odd values of n for different fractional order. Further enlarging the value of n, the approximate solutions goes to exact value of the solutions. This behavior demonstrates that the adapted method is quiet efficient and accurate to find analytical solutions to FPDEs with boundary and initial conditions.

Example 5.4.

Consider the FPDEs (Moaddy et al. (2011)) as

$$\frac{\partial^\alpha v}{\partial t^\alpha} + v^2(x,t) = \frac{2xt^{2-\alpha}}{\Gamma(3-\alpha)} + x^2t^4,\ (t,x) \in (0,\infty) \times [0,1],\ \alpha \in (0,1], \tag{43}$$

subject to the initial/boundary conditions as

$$v(x,0) = 0,\ v(0,t) = 0,\ v(1,t) = t^2. \tag{44}$$

At $\alpha = 1$, the classical solution is

$$v(x,t) = xt^2. \tag{45}$$

Taking the NT of (43) and (44) yield that

$$\frac{u^{1-\alpha}}{s^{1-\alpha}}\left[\frac{s}{u}R(x,s,u)\right] + H(x,s,u) = 2x\frac{u^{2-\alpha}}{s^{3-\alpha}} + 24\frac{u^4}{s^5}x^2, \tag{46}$$

where $R(x,s,u)$, $\mathfrak{H}(x,s,u)$ are the NT of $v(x,t)$ and $v^2(x,t)$ in same order.

Also

$$v(0,s,u) = 0,\ v(1,s,u) = \frac{2u^2}{s^3}.$$

Now according to our proposed method, we are constructing the homotopy for (46) such that

$$R(x,s,u) = \frac{u^\alpha}{s^\alpha}\left[\frac{u^{2-\alpha}}{s^{3-\alpha}}2x - \epsilon\mathfrak{H}(x,s,u) + 24x^2\frac{u^4}{s^5}\right]. \tag{47}$$

Going on the same technique adopted by us, we get

$$\begin{aligned}
R_0(x,s,u) &= 2x\frac{u^2}{s^3} + x^2\Gamma(5)\frac{u^{\alpha+4}}{s^{\alpha+4}},\\
R_1(x,s,u) &= \frac{u^\alpha}{s^\alpha}\left[x^2\Gamma(5)\frac{u^4}{s^5} + x^2\Gamma^2(5)\frac{\Gamma(2\alpha+9)u^{2\alpha+9}}{\Gamma^2(\alpha+5)s^{2\alpha+9}} + \frac{48\Gamma(\alpha+7)x^3u^{\alpha+7}}{\Gamma(\alpha+5)s^{\alpha+7}}\right],\\
R_2(x,s,u) &= \frac{2\Gamma(5)x^3u^{2\alpha+7}}{s^{2\alpha+7}\Gamma(\alpha+5)}\left[\Gamma(\alpha+7) + \frac{\Gamma(5)\Gamma(2\alpha+9)\Gamma(3\alpha+11)u^{2\alpha+4}}{\Gamma(\alpha+5)\Gamma(3\alpha+9)s^{2\alpha+4}}\right.\\
&\quad\left. + \frac{2x\Gamma(\alpha+7)\Gamma(2\alpha+8)u^{\alpha+1}}{\Gamma(2\alpha+7)s^{\alpha+1}} + \frac{\Gamma^2(5)x^3\Gamma(2\alpha+9)\Gamma(4\alpha+13)u^{3\alpha+6}}{\Gamma^2(\alpha+5)\Gamma(3\alpha+9)s^{3\alpha+6}}\right],
\end{aligned} \tag{48}$$

and so on.

Thanks to the proposed procedure, we have

$$\begin{aligned}
F_2(x,s,u) &= \frac{2xu^2}{s^3} + \frac{2\Gamma(5)x^3u^{2\alpha+7}}{\Gamma(\alpha+5)s^{2\alpha+7}}\left[\frac{\Gamma(5)\Gamma(2\alpha+9)\Gamma(3\alpha+11)u^{2\alpha+4}}{\Gamma(\alpha+5)\Gamma(3\alpha+9)s^{2\alpha+4}}\right.\\
&\left.\frac{2x\Gamma(\alpha+7)\Gamma(2\alpha+8)u^{\alpha+1}}{\Gamma(2\alpha+7)s^{\alpha+1}} + \frac{\Gamma^2(5)x^3\Gamma(2\alpha+9)\Gamma(4\alpha+13)u^{3\alpha+6}}{\Gamma^2(\alpha+5)\Gamma(3\alpha+9)s^{3\alpha+6}}\right].
\end{aligned} \tag{49}$$

By taking the inverse NT of (49), we obtain the following relation approximation

$$\begin{aligned}
v_2(x,t) &= xt^2 + \frac{2\Gamma(5)x^3\Gamma(2\alpha+9)\Gamma(3\alpha+11)t^{3\alpha+10}}{\Gamma^2(\alpha+5)\Gamma(3\alpha+9)\Gamma(3\alpha+11)}\\
&+ \frac{2^2\Gamma(5)x^4\Gamma(\alpha+7)\Gamma(2\alpha+8)t^{2\alpha+7}}{\Gamma(\alpha+5)\Gamma(2\alpha+7)\Gamma(2\alpha+8)} + \frac{2\Gamma^2(5)x^6\Gamma(2\alpha+7)\Gamma(4\alpha+13)t^{4\alpha+13}}{\Gamma^3(\alpha+5)\Gamma(3\alpha+9)\Gamma 4\alpha+13)}.
\end{aligned} \tag{50}$$

We plot the obtain approximate solutions in Figure 5 at different fractional order by taking various values of x and t, respectively.

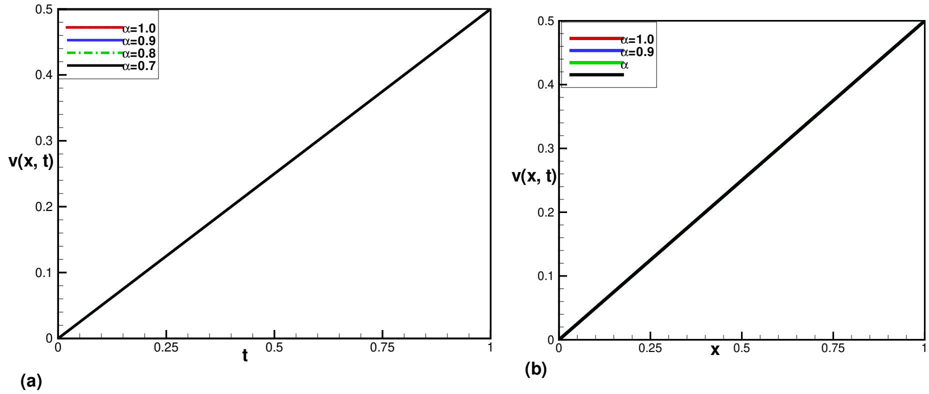

Figure 5. **(a)** Represents plot of approximate solutions at different fractional order with $x = 0.5$
(b) Represents image of approximate solutions at different fractional order with $t = 0.5$

6. Conclusion

In this article, we have derived an iterative technique for linear and nonlinear FPDEs. This method is based on the coupling of Natural transform and the homotopy perturbation method. We utilize duality of Natural and Laplace transform together with the Stehfest's numerical algorithm to calculate inverse Natural transform. To show efficiency of the proposed method, we have solved some interesting problems. It is remarkable that the series solutions computed by the proposed technique and the corresponding exact solutions have close agreement. Further our developed method is applicable to both linear and nonlinear FODEs and FPDEs. A wide class of IVPs/BVPs occurring in applied sciences involving both PDEs and ordinary FDEs can easily be solved by our proposed method.

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