Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied
Mathematics:
An International Journal
(AAM)

Vol. 14, Issue 2 (December 2019), pp. 890 – 909

Numerical Solution of Fractional Partial Differential Equations with Normalized Bernstein Wavelet Method

[1]**Mahsa Entezari**, [2*]**Saeid Abbasbandy and** [3] **Esmail Babolian**

[1]Department of Mathematics
Science and Research Branch
Islamic Azad University
Tehran, Iran
mahsa.entezari8017@gmail.com

[2]Department of Applied Mathematics
Faculty of Science
Imam Khomeini International University
Qazvin, Iran
abbasbandy@ikiu.ac.ir, abbasbandy@yahoo.com

[3]Department of Mathematics
Kharazmi University
Tehran, Iran
babolian@khu.ac.ir

*Corresponding Author

Received: July 23, 2019; Accepted: October 23, 2019

Abstract

In this paper, normalized Bernstein wavelets are presented. Next, the fractional order integration and Bernstein wavelets operational matrices of integration are derived and finally are used for solving fractional partial differential equations. The operational matrices merged with the collocation method are used in order to convert fractional problems to a number of algebraic equations. In the suggested method the boundary conditions are automatically taken into consideration. An assessment of the error of function approximation based on the normalized Bernstein wavelet is also presented. Some numerical instances are given to manifest the versatility and applicability of the suggested method. Founded numerical results are correlated with the best reported results in the literature and the analytical solutions in order to prove the accuracy and applicability of the suggested method.

Keywords: Fractional partial differential equations; Normalized Bernstein wavelet; Operational matrices

MSC 2010 No.: 65M99, 35R11

1. Introduction

Recently many natural physical phenomena, engineering models and dynamical systems like fluid mechanics, electromagnetics, electrochemistry, viscoelasticity, electrochemistry and biological population models have been simulated using fractional differential equations (Gómez-Aguilar et al. (2016); Kumar et al. (2015); Martynyuk and Ortigueira (2015); Sin et al. (2017)). The description of them using fractional calculus have many advantages versus of classical one. The fractional calculus including the derivatives and fractional integrals has many applications in different type of sciences and engineering and become more popular in last decades. Hence, the investigation of effective techniques for the fractional problems has attracted much attention and its importance and necessity have become more and more evident. Wavelets theory has served as a basis for different kinds of sciences. For example, system analysis, signal analysis for waveform representation, segmentation, numerical analysis, and optimal control. During recent decades, extensive researches have been devoted to the presentation and investigation of the numerical techniques based on wavelet methods (Heydari et al. (2014); Keshavarz et al. (2014); Kumar and Vijesh (2017); Mohammadi and Cattani (2018); Rahimkhani et al. (2016); Xu and Da (2018)). The numerical methods based on the orthogonal family of functions such as Bernstein polynomials, Block pulse functions, Chebyshev, Hermite, Laguerre, Legendre, sine and cosine functions and Wavelets had a great application in the approximation theory (Baseri et al. (2017); Bhrawy and Zaky (2016); Bhrawy et al. (2014); Chen et al. (2010); Heydari et al. (2014); Kazemi Nasab et al. (2013); Keshavarz et al. (2014); Kumar and Vijesh (2017); Li and Sun (2011); Mohammadi and Cattani (2018); Mokhtary et al. (2016); Rad et al. (2016); Parand et al. (2018); Rahimkhani et al. (2016); Saadatmandi et al. (2012); Xu and Da (2018); Zhou and Xu (2016)). The problem is shortened to a system of algebraic equations by truncation of a number of orthogonal basis functions and use of operational matrices (Abbasbandy and Taati (2009); Biazar et al. (2012); Heydari et al. (2014)).

In this study, we apply fractional order and Bernstein wavelets operational matrices of integration of integer order to find a solution for the following general fractional partial differential equation:

$$\frac{\partial^\alpha u}{\partial x^\alpha} = F\left(x, t, u, \frac{\partial^\beta u}{\partial x^\beta}, \frac{\partial^\mu u}{\partial t^\mu}, \frac{\partial^\gamma u}{\partial t^\gamma}\right), \quad 0 < \beta, \gamma \leq 1,\ 1 < \alpha, \mu \leq 2, \tag{1}$$

with Dirichlet boundary conditions:

$$\begin{aligned} u(x,0) &= f_0(x), \qquad u(0,t) = g_0(t), \\ u(x,1) &= f_1(x), \qquad u(1,t) = g_1(t), \end{aligned} \tag{2}$$

where the parameters $\alpha, \beta, \gamma, \mu$ refer to the fractional order of derivatives and f_i and g_i, for $i = 0, 1$ are two times continuously differentiable functions on $[0, 1]$.

Recently, a couple of numerical methods have been suggested to deal with fractional partial differential equations. Saadamandi et al had used Sinc-Legendre collocation method for solving the problem (1)-(2) (Saadatmandi et al. (2012)). Uddin and Haq used radial basis functions via constant coefficients for this problem (Uddin and Haq (2011)). Chen et al. had applied the Haar wavelet for solving the above problem (Chen et al. (2010)). In Izadkhah and Saberi-Nadjafi (2015), the spectral method based on the Gegenbauer collocation was applied for approximation of the solution of problem (1)-(2). In this study, a numerical method based on the normalized Bernstein wavelets is suggested in order to solve fractional partial differential equations. Here, the normalized Bernstein wavelets are introduced and an estimation of the error of function approximation based on them is given. In this suggested method, both integer orders and operational matrices of integration of fractional order were used at the same time in order to approximate the solutions of the fractional problems. Because the boundary conditions are going to be taken into the consideration automatically in the approximation process, the suggested method is very easy for solving these boundary value problems.

Several test samples are presented to show the strength of the suggested method and computational efficiency. Numerical solutions are the outcomes of the present method that have been compared with the best results reported in the literature and analytical solutions. The numerical results manifested applicability and high accuracy of the suggested normalized Bernstein wavelets method to solve the partial differential equations.

The other parts of this paper is formed as follows: In Section 2, mathematical preliminaries of fractional calculus and some necessary definitions will be introduced. Section 3 represents the normalized Bernstein wavelets and their effects. In Section 4, the Normalized Bernstein Wavelets operational matrices of integration which includes fractional order and integer are derived. Section 5 introduces the suggested method of this paper. In Section 6, several numerical examples are presented. And finally, Section 7 ends with a short conclusion.

2. Fractional calculus

In this section, we briefly recall some necessary definitions and properties of the fractional calculus.

Definition 2.1.

A real function h on $[0, \infty)$, is said to be in the space $C_\sigma, \sigma \in R$, if there is a real number ρ with

$\rho > \sigma$ such that $h(t) = t^\rho h_0(t)$, where $h_0 \in C[0,\infty)$, and $h \in C_\sigma^n$ if $h^{(n)} \in C_\sigma, n \in N$.

Definition 2.2. (See Podlubny (1998))

The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ is defined as:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, \quad x > 0,$$
$$J^0 f(x) = f(x),$$

which has the following property:

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \quad \gamma > -1.$$

The Riemann-Liouville derivative and integral had a very important role in advancement of theoretical and pure fractional calculus, but since solving equations with it fractional derivatives depends on initial and boundary conditions of fractional derivatives, it could not be used effectively in practice, especially for solving fractional equations.

With respect to new scientific advancements and finding some new applications of fractional derivatives in modeling various phenomena, there was a need to develop a new definition of fractional derivatives that can model various phenomena more effectively in a way that the model would be more consistent to initial and boundary conditions of problems. Therefore, Caputo introduced a new fractional derivative. For this reason, Caputo's definition of fractional derivative is used in this article.

Definition 2.3. (See Podlubny (1998))

The Caputo's definition of fractional derivative operator is given as follows:

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt,$$

where $m-1 < \alpha \leq m,\ m \in N,\ x > 0$. It has the following two basic properties:

$$D^\alpha J^\alpha f(x) = f(x),$$
$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0.$$

Definition 2.4. (See Nemati and Yousefi (2017))

The Caputo fractional partial derivatives of order α of the function u are defined as follows

$$\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}}=\frac{1}{\Gamma(m-\alpha)}\int_{0}^{x}(x-s)^{m-\alpha-1}\frac{\partial^{m}u(s,t)}{\partial s^{m}}ds,\qquad m-1<\alpha\leq m,$$

and

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\frac{1}{\Gamma(m-\alpha)}\int_{0}^{t}(t-s)^{m-\alpha-1}\frac{\partial^{m}u(x,s)}{\partial s^{m}}ds,\qquad m-1<\alpha\leq m.$$

3. The normalized Bernstein wavelets and their properties

In the present section, the orthonormal Bernstein wavelet is constructed by using the orthonormal Bernstein polynomial and some properties of this wavelet are given. The normalized Bernstein wavelets act much better than the orthogonal Bernstein polynomials for approximation because wavelets can be assigned to the smaller part of intervals, but the orthogonal polynomials are defined only in one interval also in order to estimate the small details of physical problems wavelets are more effective.

A family of functions constructed from dilation and translation of a single function ψ are constituted wavelets. If the dilation parameter a and the translation parameter b are continuous, the family of continuous wavelets is as follows

$$\psi_{ab}(t)=|a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right),\quad a,b\in R,a\neq 0.$$

By restriction of the parameters a and b to discrete values as $a=a_0^{-k}, b=nb_0a_0^{-k}, a_0>1, b_0>0$, the following family of discrete wavelets is obtained

$$\psi_{kn}(t)=|a_0|^{\frac{k}{2}}\psi(a_0^{k}t-nb_0),\quad k,n\in Z.$$

The sequence ψ_{kn} form a wavelet basis for $L^2(R)$ and if $a_0=2$ and $b_0=1$, then we have an orthonormal basis.

The orthonormal Bernstein wavelet $\psi_{nm}(t)=\psi(k,n,m,t)$ has four arguments, defined over $[0,1)$ by

$$\psi_{nm}(t)=\begin{cases}2^{\frac{k}{2}}\tilde{B}_{mM}(2^{k}t-n), & \text{if } \frac{n}{2^k}\leq t<\frac{n+1}{2^k},\\ 0, & \text{otherwise,}\end{cases}$$

where $n=0,\dots,2^k-1, m=0,\dots,M$, m is the degree of the orthonormal Bernstein polynomial and t is the normalized time. Moreover,

$$\tilde{B}_{mM}(t) = (\sqrt{2(M-m)+1})(1-t)^{M-m}\sum_{k=0}^{m}(-1)^k \binom{2M+1-k}{m-k}\binom{m}{k} t^{m-k}.$$

Or

$$\tilde{B}_{mM}(t) = (\sqrt{2(M-m)+1})\sum_{k=0}^{m}(-1)^k \frac{\binom{2M+1-k}{m-k}\binom{m}{k}}{\binom{M-k}{m-k}} B_{m-kM-k}(t).$$

Here, B_{mM} are Bernstein polynomials of degree m defined on the interval $[0,1]$ as

$$B_{mM}(t) = \binom{M}{m} t^m (1-t)^{M-m}, \quad m = 0, 1, 2, ..., M,$$

or

$$B_{mM}(t) = \sum_{j=m}^{M}(-1)^{j-m}\binom{M}{m}\binom{M-m}{j-m} t^j, \quad m = 0, 1, 2, ..., M,$$

where

$$\binom{M}{m} = \frac{M!}{m!(M-m)!}.$$

A function f defined over $[0,1)$ may be expanded as

$$f(t) = \sum_{n=0}^{\infty}\sum_{m\in Z} c_{nm}\psi_{nm}(t), \tag{3}$$

where

$$c_{nm} = (f, \psi_{nm}) = \int_0^1 \psi_{nm}(t) f(t) dt,$$

with $(.,.)$ as the inner product in $L^2[0,1]$. If the infinite series in Equation (3) is truncated, then, it can be written as

$$f(t) \simeq (\Pi_M^{2^k-1} f)(t) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c_{nm}\psi_{nm}(t) = C^T\Psi(t) = \tilde{f}(t),$$

where T indicates transposition and, C and $\Psi(t)$ are the following $\acute{m} = 2^k(M+1)$ column vectors

$$C = [c_{00}, c_{01}, \cdots, c_{0M}, c_{10}, \cdots, c_{1M}, c_{(2^k-1)0}, \cdots, c_{(2^k-1)M}]^T$$

and

$$\Psi(t) = [\psi_{00}, \psi_{01}, \cdots, \psi_{0M}, \psi_{10}, \cdots, \psi_{1M}, \psi_{(2^k-1)0}, \cdots, \psi_{(2^k-1)M}]^T. \tag{4}$$

Now we want to obtain an approximate value for truncated error, $f - \Pi_M^{2^k-1} f$, in normalized Bernstein wavelets.

Theorem 3.1. (See Javadi et al. (2016))

Assume $f \in C^{M+1}[0,1)$ and $\Pi_M f = \sum_{k=0}^{M} c_k \tilde{B}_{kM}$ is the best approximation of f in $S = Span\{\tilde{B}_{0M}, \tilde{B}_{1M}, ..., \tilde{B}_{MM}\}$. Then, one can obtain the following inequality,

$$\|f - \Pi_M f\|_{L^2[0,1)} \leq \kappa \frac{1}{(M+1)!2^{2M+1}}, \tag{5}$$

where $\kappa = Max_{t\in[0,1)} |f^{(M+1)}(t)|$.

Lemma 3.2.

For $n = 0, 1, ..., 2^k - 1$, suppose $f_n : [\frac{n}{2^k}, \frac{n+1}{2^k}) \to \mathbb{R}$ is a function in $L^2[\frac{n}{2^k}, \frac{n+1}{2^k})$. Consider the function $F_n f_n : [0,1) \to \mathbb{R}$ such that $(F_n f_n)(t) = f_n(\frac{t+n}{2^k})$ for all $t \in [0,1)$, then, we have

$$\|F_n f_n\|^2_{L^2[0,1)} = 2^k \|f_n\|^2_{L^2[\frac{n}{2^k}, \frac{n+1}{2^k})}.$$

Proof:

From the definition of inner product of $L^2[0,1)$ we have

$$\begin{aligned}\|F_n f_n\|^2_{L^2[0,1)} &= \int_0^1 |(F_n f_n)(t)|^2 dt = \int_0^1 |f_n(\tfrac{t+n}{2^k})|^2 dt \\ &= 2^k \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} |f_n(x)|^2 dx = 2^k \|f_n\|^2_{L^2[\frac{n}{2^k}, \frac{n+1}{2^k})},\end{aligned}$$

where the change of variable $x = \frac{t+n}{2^k}$ is used. ■

Now an error estimation for normalized Bernstein wavelets expansion is obtained as follows.

Theorem 3.3.

Suppose that $f \in C^{M+1}[0,1)$ and $|f^i| \leqslant \kappa$, for $i = 0, ..., M$, therefore,

$$\|f - \Pi_M^{2^k-1} f\|_{L^2[0,1]} \leq \frac{\kappa}{(M+1)!2^{2M+1}}.$$

Proof:

Let us consider functions $f_n : [\frac{n}{2^k}, \frac{n+1}{2^k}) \to \mathbb{R}$ for all $n = 0, 1, ..., 2^k - 1$ such that $f_n(t) = f(t)$ for all $t \in [\frac{n}{2^k}, \frac{n+1}{2^k})$ and equal to zero otherwise. This gives $f = \sum_{n=0}^{2^k-1} f_n$. If $\Pi_M f = \sum_{m=0}^{M} c_m \tilde{B}_{mM}$ and $\Pi_M^{2^k-1} f = \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} c_{nm}\psi_{nm}$ where

$$c_k = \int_0^1 f(t)\tilde{B}_{kM}(t)dt,$$

and

$$c_{nm} = \int_0^1 f(t)\psi_{nm}(t)dt.$$

Then, we have

$$\begin{aligned} c_{nm} &= \int_0^1 f_n(t)\psi_{nm}(t)dt = \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f_n(t)\psi_{nm}(t)dt \\ &= \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f_n(t)2^{\frac{k}{2}}\tilde{B}_{mM}(2^k t - n)dt = \int_0^1 (F_n f_n)(x)2^{\frac{k}{2}}\tilde{B}_{mM}(x)2^{-k}dx \\ &= 2^{\frac{-k}{2}} \int_0^1 (F_n f_n)(x)\tilde{B}_{mM}(x)dx, \end{aligned} \tag{6}$$

where the change of variable $x = 2^k t - n$ is used. Also for $t \in (\frac{n}{2^k}, \frac{n+1}{2^k})$ we have

$$F_n(\psi_{nm}(t)) = (F_n\psi_{nm})(x) = \psi_{nm}(\frac{x+n}{2^k}) = 2^{\frac{k}{2}}\tilde{B}_{mM}(x), \tag{7}$$

where $x \in [0, 1]$. Then, using (6) and (7) we can see that

$$\begin{aligned} F_n(\textstyle\sum_{m=0}^{M} c_{nm}\psi_{nm}(t)) &= \textstyle\sum_{m=0}^{M} c_{nm}F_n(\psi_{nm}(t)) = \sum_{m=0}^{M} c_{nm}(F_n\psi_{nm})(x) \\ &= \textstyle\sum_{m=0}^{M}(\int_0^1 (F_n f_n)(x)\tilde{B}_{mM}(x)dx)\tilde{B}_{mM}(x) \\ &= \Pi_M F_n f_n. \end{aligned} \tag{8}$$

Now, by using Lemma 3.2, (5) and (8), we get

$$\begin{aligned} \|f - \Pi_M^{2^k-1} f\|^2_{L^2[0,1)} &= \|\textstyle\sum_{n=0}^{2^k-1} f_n - \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c_{nm}\psi_{nm}\|^2_{L^2[0,1)} \\ &\le \textstyle\sum_{n=0}^{2^k-1} \|f_n - \sum_{m=0}^{M} c_{nm}\psi_{nm}\|^2_{L^2[\frac{n}{2^k}, \frac{n+1}{2^k})} \\ &= 2^{-k}\textstyle\sum_{n=0}^{2^k-1} \|F_n f_n - \Pi_M(F_n f_n)\|^2_{L^2[0,1)} \\ &= 2^{-k}\textstyle\sum_{n=0}^{2^k-1} (\frac{\kappa_n}{(M+1)!2^{2M+1}})^2 \le (\frac{\kappa}{(M+1)!2^{2M+1}})^2, \end{aligned}$$

where $\kappa_n = Max_{t\in[\frac{n}{2^k}, \frac{n+1}{2^k})}|F_n f_n^{(M+1)}(t)|$ and obviously $\kappa_n \le \kappa$. This completes the proof. ■

By this fact that the normalized Bernstein wavelets are orthonormal on $[0,1)$, we observe that $\{\psi_{nm}(x)\ \psi_{\acute{n}\acute{m}}(t) : n, \acute{n} = 1,2,3,...,2^k-1,\ m, \acute{m} = 0,1,2,...,M\}$ is an orthonormal set over $[0,1)\times[0,1)$.

A function $u \in L^2(R^2)$ defined over $[0,1)\times[0,1)$ may be expanded as

$$u(x,t) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{\infty}\sum_{\acute{n}=0}^{2^k-1}\sum_{\acute{m}=0}^{\infty} d_{nm\acute{n}\acute{m}}\ \psi_{nm}(x)\ \psi_{\acute{n}\acute{m}}(t), \tag{9}$$

where $d_{nm\acute{n}\acute{m}} = \langle u, \psi_{nm}\psi_{\acute{n}\acute{m}}\rangle_{L^2([0,1)\times[0,1))}$. Now, we truncate the infinite series (9) as follows

$$\begin{aligned} u(x,t) &\simeq \sum_{n=0}^{2^k-1}\sum_{m=0}^{M}\sum_{\acute{n}=0}^{2^k-1}\sum_{\acute{m}=0}^{M} d_{nm\acute{n}\acute{m}}\ \psi_{nm}(x)\ \psi_{\acute{n}\acute{m}}(t) \\ &= \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} \psi_{nm}(x)\ D_{nm}^T\ \psi_{nm}(t), \end{aligned}$$

where D_{nm} and $\psi(t)$ are $(2^k-1)M\times 1$ matrices given by

$$\begin{aligned} &D_{nm} = (d_{nm10}\ d_{nm11}\ ...\ d_{nm1M}\ ...\ d_{nm2^k-10}\ d_{nm2^k-11}\ ...\ d_{nm2^k-1M})^T, \\ &\Psi(t) = (\psi_{10}(t)\ \psi_{11}(t)\ ...\ \psi_{1M}(t)\ ...\ \psi_{2^k-10}(t)\ \psi_{2^k-11}(t)\ ...\ \psi_{2^k-1M}(t))^T. \end{aligned}$$

For all $n = 1,2,3,...,2^k-1$, let us consider $\Psi_n(x) = (\psi_{n0}(x)\ \psi_{n1}(x)\ ...\ \psi_{nM}(x))$ and $D_n = (D_{n0}\ D_{n1}\ ...\ D_{nM})^T$, then,

$$u(x,t) \simeq \sum_{n=1}^{2^k-1}\sum_{m=0}^{M} \psi_{nm}(x) D_{nm}^T \Psi(t) = \sum_{n=1}^{2^k-1} \Psi_n(x) D_n \Psi(t).$$

Now, if

$$\Psi^T(x) = (\Psi_1(x),\ ...\ ,\Psi_{2^k-1}(x)), \qquad D = \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_{2^k-1} \end{pmatrix},$$

we can write

$$u(x,t) \simeq \Psi^T(x)\ D\ \Psi(t).$$

4. Operational matrix of integration

In the following section, the operational matrix of integration of integer order and the operational matrix of integration of fractional order, respectively, are obtained. These operational matrices play an important role in the proposed method for dealing with the problem (1)-(2).

4.1. Operational matrix of integration of order integer order

An $\acute{m}$-set of Block Pulse Functions is defined over the interval $[0, T)$ as

$$b_j(t) = \begin{cases} 1, & \text{if } \frac{jT}{\acute{m}} \leq t < \frac{(j+1)T}{\acute{m}}, \\ 0, & \text{otherwise,} \end{cases}$$

where $j = 0, 1, ..., \acute{m} - 1$ with a positive integer value for $\acute{m}$ and when $T = 1$, then BPFs are defined over $[0, 1)$. The useful properties of BPFs are listed in Kilicman and Zhour (2007).

If f is a given function that is square integrable on the interval $[0, 1)$, then, using the orthogonality of BPFs, it f can be approximated as

$$f(t) \simeq \sum_{i=0}^{\acute{m}-1} f_i b_i(t) = \mathbf{f}^T \mathbf{B}_{\acute{m}}(t),$$

where

$$f_i = \acute{m} \int_0^1 f(s) b_i(s) ds, i = 0, 1, ..., \acute{m} - 1,$$

and

$$\mathbf{f} = (f_0, f_1, \cdots, f_{\acute{m}-1})^T, \ \ \mathbf{B}_{\acute{m}} = (b_0(t), b_1(t), \cdots, b_{\acute{m}-1}(t))^T.$$

Let $t_i = \frac{2i-1}{2\acute{m}}$ be collocation points for $i = 1, 2, \cdots, \acute{m}$. Using $(2^k - 1)M$-set of BPFs, the orthonormal Bernstein is expanded as

$$\Psi(t) = \Psi_{\acute{m}\times\acute{m}} B_{\acute{m}}(t), \tag{10}$$

where $\Psi_{\acute{m}\times\acute{m}} = \Big(\Psi(t_1) \ \Psi(t_2) \ \ldots \ \Psi(t_{\acute{m}})\Big)$.

The block pulse operational matrix $F_{\acute{m}}$ of integer order integration is given as follows (Deb et al. (1994)),

$$\int_0^t B_{\acute{m}}(x) dx \simeq F_{\acute{m}} B_{\acute{m}}(t), \tag{11}$$

where

$$F_{\acute{m}} := h \begin{pmatrix} 1/2 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1/2 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1/2 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1/2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1/2 \end{pmatrix}, \tag{12}$$

(13)

with $h = \frac{1}{\acute{m}}$. The matrix $F_{\acute{m}}$ performs as an integrator in the BPF domain and it is pivotal in any BPF domain analysis.

To obtain the Bernstein wavelets operational matrix of the integer order integration, $P_{\acute{m}}$, consider

$$\int_0^t \Psi(x)dx \simeq (P_{\acute{m}}\Psi)(t) = P_{(2^k-1)M\times(2^k-1)M}\ \Psi(t). \tag{14}$$

Using (10) and (11), we have

$$\begin{aligned}(P_{\acute{m}}\Psi)(t) &\simeq (P_{\acute{m}}\Psi_{(2^k-1)M\times(2^k-1)M}\ B_{(2^k-1)M})(t)\\ &\simeq \Psi_{(2^k-1)M\times(2^k-1)M}\ F_{(2^k-1)M}\ B_{(2^k-1)M}(t).\end{aligned} \tag{15}$$

From Equations (14) and (15), we conclude that

$$P_{(2^k-1)M\times(2^k-1)M}\ \Psi(t) \simeq \Psi_{(2^k-1)M\times(2^k-1)M}\ F_{(2^k-1)M}\ B_{(2^k-1)M}(t),$$

and by attention to Equation (10), we can write

$$P_{(2^k-1)m\times(2^k-1)M} = \Psi_{(2^k-1)M\times(2^k-1)M}\ F_{\acute{m}}\ \Psi^{-1}_{(2^k-1)M\times(2^k-1)M}.$$

4.2. Operational matrix of the fractional order integration

According to the previous section, we will use the block pulse operational matrix to construct the Bernstein wavelets operational matrix of integration of fractional order. Kilicman 2007 introduced the block pulse operational matrix F^α of integration of fractional order as follows,

$$(I^\alpha B_{\acute{m}})(t) \simeq F^\alpha B_{\acute{m}}(t),$$

where

$$F^\alpha = \frac{1}{\acute{m}^\alpha}\frac{1}{\Gamma(\alpha+2)}\begin{pmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{\acute{m}-1}\\ 0 & 1 & \xi_1 & \xi_2 & \cdots & \xi_{\acute{m}-2}\\ 0 & 0 & 1 & \xi_1 & \cdots & \xi_{\acute{m}-3}\\ \vdots & \vdots & \ddots & \ddots & & \vdots\\ 0 & 0 & \cdots & 0 & 1 & \xi_1\\ 0 & 0 & 0 & \cdots & 0 & 1\end{pmatrix}, \tag{16}$$

and $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$, for $k = 1, 2, \ldots, \acute{m}-1$.

Now, we present the Bernstein wavelets operational matrix of the fractional order integration, P^α. Assume that

$$(I^\alpha\Psi)(t) \simeq P^\alpha\ \Psi(t). \tag{17}$$

Using (10) and (16), we have

$$(I^{\alpha}\Psi)(t) \simeq (I^{\alpha}\Psi\ B(t)) = \Psi\ (I^{\alpha}\ B(t)) \approx \Psi\ F^{\alpha}\ B(t).$$

From Eqs. (17) and (18), it follows

$$P^{\alpha}\Psi(t) \simeq \Psi\ F^{\alpha}\ B(t).$$

Together with (10), we have

$$P^{\alpha} = \Psi\ F^{\alpha}\ \Psi^{-1}.$$

Using this fact that the operational matrix P^{α} contains many zero entries, the calculations are fast.

5. Description of the proposed method

In this section, we will show how the Bernstein wavelets operational matrices of integration including integer and fractional order is used for solving a class of fractional partial differential equations (1) with the boundary conditions (2). For this purpose, $\frac{\partial^{\alpha+\mu}u}{\partial x^{\alpha}\ \partial t^{\mu}}$ approximates as follows

$$\frac{\partial^{\alpha+\mu}u}{\partial x^{\alpha}\ \partial t^{\mu}} \approx \Psi(x)^{T}\ U\Psi(t), \tag{18}$$

where $U = [u_{i,j}]_{\acute{m}\times\acute{m}}$ is an unknown matrix which should be found and $\Psi(.)$ is the vector that is defined in (4). Now, by using integrating the Riemann-Liouville fractional integral of order μ, we integrate of (18) with respect to t, therefore,

$$\frac{\partial^{\alpha}u}{\partial x^{\alpha}} \simeq \Psi(x)^{T}\ U\ P^{\mu}\ \Psi(t) + \frac{\partial^{\alpha}u}{\partial x^{\alpha}}|_{t=0} + t\frac{\partial}{\partial t}(\frac{\partial^{\alpha}u}{\partial x^{\alpha}})|_{t=0}. \tag{19}$$

To obtain the value of $\frac{\partial}{\partial t}(\frac{\partial^{\alpha}u}{\partial x^{\alpha}})|_{t=0}$, we put $t = 1$ in (19), hence, by (2) we conclude that

$$\frac{\partial}{\partial t}(\frac{\partial^{\alpha}u}{\partial x^{\alpha}})|_{t=0} \simeq \frac{\partial^{\alpha}f_1}{\partial x^{\alpha}} - \frac{\partial^{\alpha}f_0}{\partial x^{\alpha}} - \Psi(x)^{T}\ U\ P^{\mu}\ \Psi(1). \tag{20}$$

Consequently by (20) and (19), we obtain

$$\frac{\partial^{\alpha}u}{\partial x^{\alpha}} \simeq \Psi(x)^{T}\ U\ P^{\mu}\ \Psi(t) - t\Psi(x)^{T}\ U\ P^{\mu}\ \Psi(1) + (1-t)\frac{\partial^{\alpha}f_0}{\partial x^{\alpha}} + t\frac{\partial^{\alpha}f_1}{\partial x^{\alpha}}. \tag{21}$$

Next, we evaluate $\frac{\partial^{\mu}u}{\partial t^{\mu}}$ by using the fractional integrating of order α of (18) with respect to x, so

$$\frac{\partial^{\mu}u}{\partial t^{\mu}} \simeq (P^{\alpha}\ \Psi(x))^{T}\ U\ \Psi(t) + \frac{\partial^{\mu}u}{\partial t^{\mu}}|_{x=0} + x\frac{\partial}{\partial x}(\frac{\partial^{\mu}u}{\partial t^{\mu}})|_{x=0}. \tag{22}$$

According to the process described above, in (22) let $x = 1$ and by attention to (2), we can rewrite (22) as:

$$\frac{\partial^\mu u}{\partial t^\mu} \simeq (P^\alpha\, \Psi(x))^T\, U\, \Psi(t) - x(P^\alpha\, \Psi(1))^T\, U\, \Psi(t) + (1-x)\frac{\partial^\mu g_0}{\partial t^\mu} + x\frac{\partial^\mu g_1}{\partial t^\mu}. \tag{23}$$

Finally, we integrate of (21) with respect to x by using the fractional integrating of order α, and using (2) we get

$$\begin{aligned} u(x,t) &= (P^\alpha\, \Psi(x))^T\, U\, \Psi(t) - t(P^\alpha\, \Psi(x))^T\, U\, P^\mu \Psi(1) - x(P^\alpha\, \Psi(1))^T\, U\, P^\mu \Psi(t) \\ &\quad + xt(P^\alpha\, \Psi(1))^T\, U\, P^\mu \Psi(1) + H(x,t), \end{aligned} \tag{24}$$

where

$$\begin{aligned} H(x,t) &\simeq g_0(t) + (1-t)(f_0(x) - f_0(0) - x\acute{f_0}(0)) + t(f_1(x) - f_1(0) - x\acute{f_1}(0)) \\ &\quad + x(g_1(t) - g_0(t)) - x(1-t)(f_0(1) - f_0(0) - \acute{f_0}(0)) - xt(f_1(1) - f_1(0) - \acute{f_1}(0)). \end{aligned}$$

The $\frac{\partial^\beta u}{\partial x^\beta}$ and $\frac{\partial^\gamma u}{\partial t^\gamma}$ are approximated by using fractional differentiation of order β and γ of equation (24) with respect to x and t, respectively.

By inserting all these approximate values in equation (1) and putting collocation points $x_i, y_i = \frac{2i-1}{2\acute{m}}, i = 1, 2, ..., \acute{m}$ into the obtained equation and replacing $\simeq$ by $=$, the following nonlinear system of algebraic equations is obtained

$$\frac{\partial^\alpha u}{\partial x^\alpha} - F(x, y, u, \frac{\partial^\beta u}{\partial x^\beta}, \frac{\partial^\mu u}{\partial t^\mu}, \frac{\partial^\gamma u}{\partial t^\gamma}) = 0, \quad i, j = 1, 2, ..., \acute{m}. \tag{25}$$

By solving this nonlinear system using any iterative method such as Newton iteration method, we can find the unknown coefficients U, and then, we have the approximate solution by substituting U into equation (24).

6. Numerical examples

In this section, we will use the above proposed method to demonstrate the efficiency of the proposed method for solving some problems. In the case of having the exact solution, the efficiency of the proposed method is shown by using the root mean square error L_2 and maximum absolute error L_∞, in order to be able to to compare the numerical result

$$\begin{aligned} L_\infty &:= \max_{1\le j\le \acute{m}} |u(x_j, t_j) - U(x_j, t_j)|, \\ L_2 &:= (\frac{1}{\acute{m}} \sum_{j=1}^{\acute{m}} |u(x_j, t_j) - U(x_j, t_j)|^2)^{1/2}. \end{aligned}$$

All the computations associated with the examples are performed by using Mathematica 9.1 on a personal computer with 4 GHz speed and 4 GB DDR3 memory operating under Windows 7 system.

Example 6.1.

Consider the following fractional partial differential equation

$$\frac{\partial^{3/2}u(x,t)}{\partial x^{3/2}}+\frac{\partial^{3/4}u(x,t)}{\partial x^{3/4}}+\frac{\partial^{4/3}u(x,t)}{\partial t^{4/3}}+u(x,t)=f(x,t),$$

with $f(x,t)=x^2+t+4\frac{\sqrt{x}}{\sqrt{\pi}}+\frac{16\sqrt{2}\Gamma(3/4)}{5\pi}$, and the boundary conditions:

$$\begin{aligned} u(x,0)&=x^2, \qquad u(0,t)=t,\\ u(x,1)&=x^2+1, \quad u(1,t)=1+t. \end{aligned}$$

The exact solution of this problem is $u(x,t)=x^2+t$.

Table 1 shows the root-mean-square error L_2 and maximum absolute error L_∞ in some nodes $(x,t)\in[0,1]$ for $\acute{m}=12$. To show the efficiency of this method, the absolute error obtained by the present method has been compared with the Legendre wavelet method (Heydari et al. (2014)) in Table 2 and Figure 1 shows the approximate solution and absolute error for Example 6.1 with $\acute{m}=12$. It is evident from Table 2 and Figure 1, that the proposed method is accurate for solving this kind of problems and the obtained approximate solution is very close to the exact one.

Table 1. The L_∞ and L_2 errors in some different values of t for Example 6.1

t	$t=0.1$	$t=0.3$	$t=0.5$	$t=0.7$	$t=0.9$	$t=1$
L_∞	1.11E-16	2.22E-16	1.99E-15	2.88E-15	5.55E-15	2.22E-16
L_2	3.78E-17	1.02E-16	6.43E-16	9.07E-16	1.73E-15	6.41E-17

Table 2. Comparison of absolute errors for Example 6.1

t	$t=0.1$	$t=0.3$	$t=0.5$	$t=0.7$	$t=0.9$	$t=1$
Legendre wavelet method (Heydari et al. (2014))	4.90E-5	1.47E-6	2.28E-6	1.17E-6	6.45E-6	0.00
Bernstein wavelet method	1.11E-16	2.22E-16	1.99E-15	5.55E-15	2.22E-16	2.22E-16

Example 6.2.

Consider the following time-fractional convection diffusion equation of order $\alpha\ (0<\alpha<1)$

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+x\frac{\partial u(x,t)}{\partial x}+\frac{\partial^2u(x,t)}{\partial x^2}=f(x,t),$$

with $f(x,t)=2x^2+2t^\alpha+2$, and the boundary conditions:

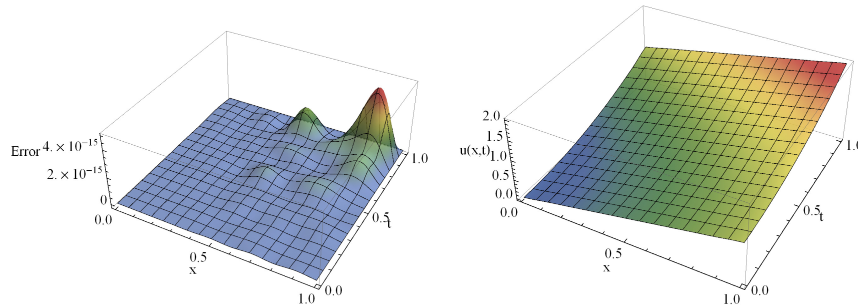

Figure 1. Plot of the approximate solution (left) and the absolute error (right) for Example 6.1

$$
\begin{aligned}
&u(x,0) = x^2,\ 0 < x < 1, \\
&u(0,t) = \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} t^{2\alpha}, \\
&u(1,t) = 1 + \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} t^{2\alpha},\ \ 0 \le t \le 1.
\end{aligned}
$$

This problem has the following exact solution

$$
u(x,t) = x^2 + \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} t^{2\alpha}.
$$

The absolute errors for different values of α at different points using presented method for $\acute{m} = 12$ are given in Table 3. We compare the error obtained by the present method with the Sinc-Legendre method (Saadatmandi et al. (2012)), Haar wavelet method (Chen et al. (2010)) and Third Chebyshev wavelet method (Zhou and Xu (2016)) for $\alpha = 0.5$ in Table 4. Figure 2 shows the absolute error for $\alpha = 0.5$ and $\acute{m} = 6$. By attention to Figure 3 you can see that the approximate solution have good agreement with the exact solution. Our numerical experience show that errors $L_\infty$and L_2 do not change drastically for non-diagonal points.

Example 6.3.

Consider the following non homogeneous fractional partial differential equation

$$
\frac{\partial^{1.5} u(x,t)}{\partial x^{1.5}} + \frac{\partial^{1.2} u(x,t)}{\partial t^{1.2}} = \frac{4\sqrt{x}}{\sqrt{\pi}} + \frac{5t^{\frac{4}{5}}}{2\Gamma\frac{4}{5}}
$$

with initial-boundary conditions as

$$
\begin{aligned}
&u(x,0) = t^2, \qquad u(0,t) = x^2, \\
&u(x,1) = x^2 + 1, \qquad u(1,t) = 1 + t^2.
\end{aligned}
$$

The exact solution of this equation is $u(x,t) = x^2 + t^2$.

Table 3. Absolute errors of Example 6.2

(x,t)	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
$(0.1, 0.1)$	0	0	0	0	3.469E-18
$(0.2, 0.2)$	0	1.110E-16	0	0	2.776E-17
$(0.3, 0.3)$	0	0	0	0	1.110E-16
$(0.4, 0.4)$	0	0	0	0	2.775E-16
$(0.5, 0.5)$	8.881E-16	6.661E-16	0	8.881E-16	4.041E-14
$(0.6, 0.6)$	0	4.441E-16	0	0	5.995E-15
$(0.7, 0.7)$	0	4.441E-16	2.220E-16	2.220E-16	1.332E-15
$(0.8, 0.8)$	0	8.882E-16	4.441E-16	0	6.661E-16
$(0.9, 0.9)$	0	4.441E-16	4.441E-16	4.440E-16	1.220E-14

Table 4. Comparison of absolute errors with $\alpha = 0.5, t = 0.5$ for Example 6.2

x	third-Chebyshev wavelet (Zhou and Xu (2016)) $\acute{m} = 6$	Haar wavelet (Chen et al. (2010)) $\acute{m} = 64$	Sinc-Legendre wavelet (Saadatmandi et al. (2012)) $\acute{m} = 25$	Present method $\acute{m} = 6$
0.1	1.110E-16	1.210E-3	6.462E-6	0
0.2	1.110E-16	1.259E-3	1.578E-5	0
0.3	2.220E-16	1.865E-3	2.272E-5	1.110E-16
0.4	2.220E-16	7.412E-3	2.674E-5	0
0.5	0	1.000E-6	2.759E-5	0
0.6	0	7.460E-3	2.534E-5	2.220E-16
0.7	2.220E-16	1.724E-3	2.035E-5	0
0.8	0	4.990E-3	1.320E-5	2.220E-16
0.9	0	1.678E-2	4.653E-6	0

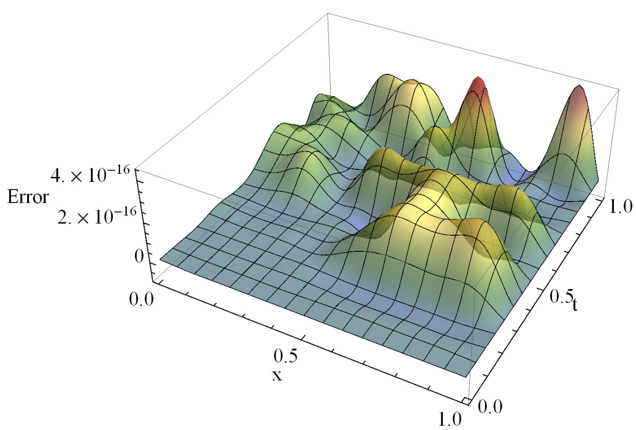

Figure 2. Graph of absolute error for Example 6.2

The computational results for various $\acute{m}$ have been presented in Table 5 and Figure 4 has been shown the error function graph for $\acute{m} = 12$. The maximum absolute errors of method (Singh and Singh (2018)) and present method are compared in Table 6.

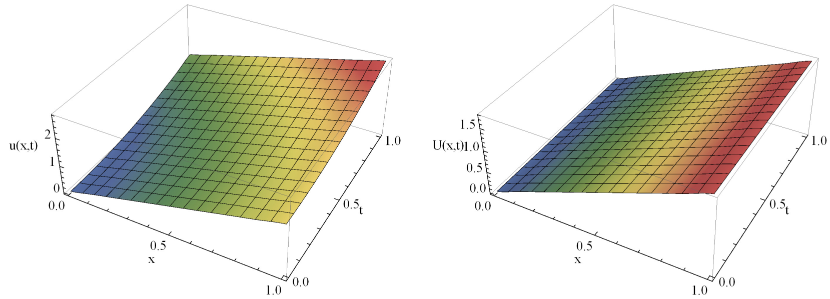

Figure 3. Graph of the approximate solution (left) and the exact solution (right) for Example 6.2

Table 5. Absolute errors for different value of $\acute{m}$ of Example 6.3

	Presented method	Presented method
(x, y)	$\acute{m} = 12$	$\acute{m} = 16$
$(0.2, 0.2)$	0	0
$(0.4, 0.4)$	0	0
$(0.6, 0.6)$	1.11×10^{-16}	0
$(0.8, 0.8)$	0	0

Table 6. Maximum absolute error (L_∞) of Example 6.3

Method in (Singh and Singh (2018))		Presented method	
$N = 10$	$N = 20$	$\acute{m} = 12$	$\acute{m} = 16$
7.1002×10^{-3}	7.1002×10^{-3}	1.11×10^{-16}	0

Example 6.4. (See Daftardar-Gejji and Bhalekar (2010))

Consider the following fractional diffusion equation with initial and boundary conditions

$$\begin{aligned} &\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < l, \quad t > 0, \quad 0 < \alpha \leq 1 \\ &u(x,0) = x^2 + 2x + 1, \\ &u(0,t) = \frac{2t^\alpha}{\Gamma(\alpha+1)} + 1, \\ &u(l,t) = l^2 + 2l + 1 + \frac{2t^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

This equation has the exact solution $u(x,t) = x^2 + 2x + \frac{2t^\alpha}{\Gamma(\alpha+1)} + 1$. Varsha Daftardar-Gejji and Bhalekar 2010, using a new iterative method, obtained this exact solution.

By using the above proposed method for $l = 1$ and $\acute{m} = 16$, we obtain the exact solution. Since the exact solution is obtained, the numerical results will not be presented.

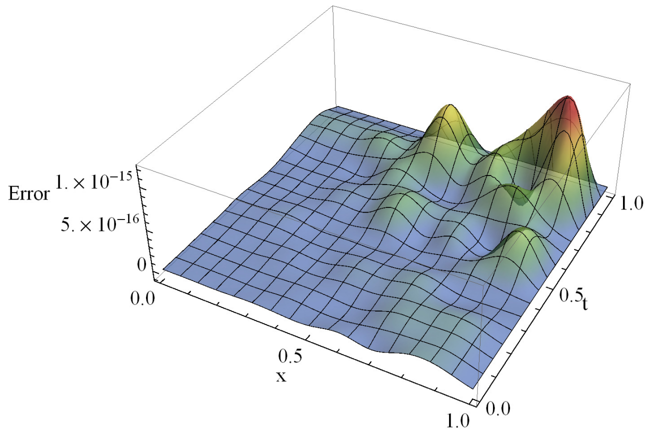

Figure 4. The error function graph for Example 6.3 for $\acute{m} = 12$

7. Conclusion

In this paper, to get the numerical solutions of the fractional partial differential equations, a pattern of the normalized Bernstein wavelets method was suggested. In addition, an error estimate of function approximation based on the introduced wavelets was given. A fractional order and the Bernstein wavelets operational matrices of integration of integer order have been derived. The operational matrices merged with the collocation method were used in order to diminish fractional problems for a number of algebraic equations. In this suggested method the boundary conditions would be considered automatically. Some numerical examples were presented to show the accuracy and applicability of the suggested method.

Acknowledgment:

We are grateful to the anonymous reviewers for their helpful comments, which undoubtedly led to the definite improvement in the manuscript.

REFERENCES

Abbasbandy, S. and Taati, E. (2009). Numerical solution of the system of nonlinear Volterra integro-differential equations with nonlinear differential part by the operational Tau method and error estimation, J. Comput. Appl. Math, Vol. 231, pp. 106–113.

Baseri, A., Babolian, E. and Abbasbandy, S. (2017). Normalized Bernstein polynomials in solving space-time fractional diffusion equation, Advances in Difference Equations, pp. 2017:346.

Biazar, J. and Ebrahimi, H. (2012). Chebyshev wavelets approach for nonlinear systems of Volterra integral equations, Comput. Math. Appl, Vol. 63, pp. 608–616.

Bhrawy, A.H., Alhamed, Y.A. and Baleanu, D. (2014). New spectral techniques for systems of fractional differential equations using fractional-order generalized Laguerre orthogonal functions, Fractional Calculus and Applied Analysis, Vol. 17, pp. 1138–1157.

Bhrawy, A.H. and Zaky, M.A. (2016). A shifted fractional-order Jacobi orthogonal functions: An application for system of fractional differential equations, Appl. Math. Modelling, Vol. 40, No. 2, pp. 832–845.

Chen, Y., Wu, y., Cui, Y., Wang, Z. and Jin, D. (2010). Wavelet method for a class of fractional convection–diffusion equation with variable coefficients, J. Comput. Sci, Vol. 1, pp. 146–149.

Daftardar-Gejji, V. and Bhalekar, S. (2010). Solving fractional boundary value problems with Dirichlet boundary conditions using a new iterative method, Computers and Mathematics with Applications, Vol. 59, pp. 1801–1809.

Deb, A., Sarkar, G. and Sen, S. (1994). Block pulse functions, the most fundamental of all piecewise constant basis functions, Int. J. Syst. Sci, Vol. 25, pp. 351–363.

Gómez-Aguilar, F.G., Miranda-Hernández, M., López-López, M.G., Alvarado-Martinez, V.M. and Baleanu, D. (2016). Modeling and simulation of the fractional space-time diffusion equation, Communications in Nonlinear Science and Numerical Simulation, Vol. 30, No. 1-3, pp. 115–127.

Heydari, M.H., Hooshmandasl, M.R. and Mohammadi, F. (2014). Legendre wavelets method for solving fractional partial differential equations with Dirichlet boundary conditions, Appl. Math. Comput, Vol. 234, pp. 267–276.

Heydari, M.H., Hooshmandasl, M.R., Mohammadi, F. and Cattani, C. (2014). Wavelets method for solving systems of nonlinear singular fractional Volterra integro-differential equations, Commun. Nonlinear Sci. Numer. Simul. Vol. 19, No. 1, pp. 37–48.

Izadkhah, M.M. and Saberi-Nadjafi, J. (2015). Gegenbauer spectral method for time-fractional convection-diffusion equations with variable coefficients, Math. Method Appl. Sci, Vol. 38, pp. 3183–3194.

Javadi, S., Babolian, E. and Taheri, Z. (2016). Solving generalized pantograph equations by shifted orthonormal Bernstein polynomials, Journal of Computational and Applied Mathematics, Vol. 303, pp. 1–14.

Kazemi Nasab, A., Kilicman, A., Pashazadeh Atabakan Z. and Abbasbandy, S. (2013). Chebyshev wavelet finite difference method: A new approach for solving initial and boundary value problems of fractional order, Abstract and Applied Analysis, Vol. 2013, Article ID 916456.

Kilicman, A. and Zhour, Z.A. (2007). Kronecker operational matrices for fractional calculus and some applications, Appl. Math. Comput, Vol. 187, No. 1, pp. 250–265.

Keshavarz, E., Ordokhani, Y. and Razzaghi, M. (2014). Bernoulli wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations, Appl. Math. Model, Vol. 38, pp. 6038–6051.

Kumar, D., Singh, J. and Kumar, S. (2013). A fractional model of NavierâĂŞStokes equation arising in unsteady flow of a viscous fluid, Journal of the Association of Arab Universities for Basic and Applied Sciences, Vol. 17, No. 1, pp. 14–19.

Kumar, K.H. and Vijesh, V.A. (2017). Chebyshev wavelet quasilinearization scheme for coupled nonlinear sine-Gordon equations, Journal of Computational and Nonlinear Dynamics, Vol. 12, No. 1, pp. 011018.

Li, Y. and Sun, N. (2011). Numerical solution of fractional differential equations using the generalized block pulse operational matrix, Computers and Mathematics with Applications, Vol. 62, pp. 1046–1054.

Martynyuk, V. and Ortigueira, M. (2015). Fractional model of an electrochemical capacitor, Signal Processing, Vol. 107, pp. 355–360.

Mohammadi, F. and Cattani, C. (2018). A generalized fractional-order Legendre wavelet Tau method for solving fractional differential equations, Journal of Computational and Applied Mathematics, Vol. 339, pp. 306–316.

Mokhtary, P., Ghoreishi, F. and Srivastava, H.M. (2016). The Müntz-Legendre Tau method for fractional differential equations, Appl. Math. Modelling, Vol. 40, No. 2, pp. 671-684.

Nemati, A. and Yousefi, S.A. (2017) A numerical scheme for solving two-dimensional fractional optimal control problems by the Ritz method combined with fractional operational matrix, IMA J. Math. Control Inf. , Vol. 34, pp. 1079–1097.

Rad, J.A., Kazem, S., Shaban, M., Parand, K. and Yildirim, A. (2016). Numerical solution of fractional differential equations with a Tau method based on Legendre and Bernstein polynomials, Mathematical Methods in the Applied Sciences, Vol. 37, No. 3, pp. 329-342

Parand, K., Latifi, S., Delkhosh, M. and Moayeri, M.M. (2018). Generalized Lagrangian Jacobi Gauss collocation method for solving unsteady isothermal gas through a micro-nano porous medium, The European Physical Journal Plus, Vol. 133, No. 1, pp. 28.

Podlubny I. (1998). *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of Their Applications*, Vol. 198, Academic Press.

Rahimkhani, P., Ordokhani, Y. and Babolian, E. (2016). Fractional-order Bernoulli wavelets and their applications, Appl. Math. Modell, Vol. 40, pp. 8087–8107.

Saadatmandi, A., Dehghan, m. and Azizi, M.R. (2012). The Sinc-Legendre collocation method for a class of fractional convection-diffusion equations with variable coefficients, Commun. Nonlinear Sci, Vol. 17, pp. 4125–4136.

Sin, C.S., Zheng, L., Sin, J.S., Liu, F. and Liu, L. (2017). Unsteady flow of viscoelastic fluid with the fractional K–BKZ model between two parallel plates, Applied Mathematical Modelling, Vol. 47, pp. 114–127.

Singh, H. and Singh, C.S. (2018). Stable numerical solutions of fractional partial differential equations using Legendre scaling functions operational matrix, Ain Shams Engineering Journal, Vol. 9, No. 4, pp. 717–725.

Uddin, M. and Haq, S. (2011). RBFs approximation method for time fractional partial differential equations, Commun. Nonlinear Sci, Vol. 16, pp. 4208–4214.

Xu, X. and Da, X. (2018). Legendre wavelets method for approximate solution of fractional-order differential equations under multi-point boundary conditions, International Journal of Computer Mathematics Vol. 95, No. 5, pp. 998–1014.

Zhou, F. and Xu, X. (2016). The third kind Chebyshev wavelets collocation method for solving the time-fractional convection diffusion equations with variable coefficients, Appl. Math. Comput, Vol. 280, pp. 11–29.