Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 14, Issue 2 (December 2019), pp. 836 – 846

# Fuzzy Semi-S-irresolute Continuous Mappings in Šostak's Fuzzy Topological Spaces

[1]B. Vijayalakshmi, [2]J. Praba, [3]M. Saraswathi and [4]A. Vadivel

[1]Department of Mathematics
Government Arts College
Chidambaram, Tamil Nadu-608 102
mathvijaya2006au@gmail.com

[2]Department of Mathematics
Kandaswami Kandar's College
P-velur, Tamil Nadu-638 182
lallu.jklap@gmail.com

[3]Department of Mathematics
Thiruvalluvar University College of Arts & Science
Tirupattur, Tamil Nadu-635 901
vimesh.sarash75@gmail.com

[4]Department of Mathematics
Government Arts College
(Autonomous)
Karur, Tamil Nadu-639 005
avmaths@gmail.com

Received: March 19, 2019; Accepted: July 8, 2019

## Abstract

In this paper, the concepts of fuzzy semi-S-irresolute open map, fuzzy semi-S-irresolute closed map and fuzzy semi-S-irresolute homeomorphism to the fuzzy topological spaces in Šostak's sense are introduced and studied. Some of their characteristic properties are considered. Also a comparison between these new types of functions are established by giving examples.

**Keywords:** Fuzzy semi-S-irresolute continuous; Fuzzy semi-S-irresolute open; Fuzzy semi-S-irresolute closed and fuzzy semi-S-irresolute homeomorphism

**MSC 2010 No.:** 54A40, 54C05, 03E72

## 1. Introduction

After the introduction of fuzzy sets by Zadeh (1965), Chang (1968) was the one who initiated the idea of a fuzzy topology in a set $X$, in which a collection, $T$,of open sets of fuzzy subsets of

$X$. He also defined new type of fuzzy topology. In Chattopadhyay (1993) and Hazra et al (1992), the authors defined the idea of gradation of openness of fuzzy subsets of $X$. They also used the concept of fuzzy topology in the sense of Chang. Höhle, in 1980, the basic idea of topology is itself fuzzy. As in 1985 by the customized work of Kubiak (1985) and Šostak (1985) in Lattice $L = I$ was initiated in 1995 by Höhle and Šostak and further by Kubiak and Šostak in 1997, in that topology was used to explain the two membership latices namely $L^X$ to $M$, where $L$ and $M$ are appropriate lattices. With all the basic ideas of topologies the basic lattices are fixed in (Kubiak (1985); Šostak (1985)) and another significant generalizations of lattices $L$ varies from space to space (Höhle and Šostak (1999)) is also represented. Using earlier developments (Höhle (1980); Höhle and Šostak (1995); Höhle and Šostak (1999); Kubiak (1985); Kubiak and Šostak (1997); Rodabaugh (1991); Šostak (1985); Šostak (1989b); Šostak (1989a); Šostak (1999)), the notion of a fuzzy topology as a fuzzy subset of the power set is independently consider with the base work of Höhle (1980), Kubiak (1985), and Šostak (1985). At present authors Chattopadhyay (1993), Ramadan (1992) and Ying (1991) also worked in this concept.

## 2. Preliminaries

$X,\ Y$, etc., denote the non empty sets, $I = [0,\ 1]$ and $I_0 = (0,\ 1]$. In 1992, Ramadan gives the definition of smooth topological spaces (in short, sts) and the operators $C_\tau : I^X \times I_0 \to I^X$ as $C_\tau(\lambda,\ r) = \bigwedge\{\mu \in I^X : \lambda \leq \mu,\ \tau(\overline{1} - \mu) \geq r\}$. and $I_\tau : I^X \times I_0 \to I^X$ as $I_\tau(\lambda,\ r) = \bigvee\{\mu \in I^X : \lambda \geq \mu,\ \tau(\mu) \geq r\}$. Kim et al (2003) discussed in a sts $(X,\ \tau)$, $\forall r \in I_0$. $\lambda$ is called $r$-fuzzy strongly semiopen (respectively, $r$-fuzzy semiopen) ($r$-fsso (respectively, $r$-fso) for short) iff $\lambda \leq I_\tau(C_\tau(I_\tau(\lambda, r), r),\ r)$ (respectively, $\lambda \leq C_\tau(I_\tau(\lambda, r), r)$). The respective compliment sets are closed set.

The operators $r$-fuzzy strongly semi-interior (respectively, $r$-fuzzy strongly semi-closure) of $\lambda$, denoted by $SSI_\tau(\lambda,\ r)$ (respectively, $SSC_\tau\ (\lambda,\ r)$) as $SSI_\tau(\lambda,\ r) = \bigvee\{\mu \in I^X : \mu \leq \lambda, \mu \text{ is } r\text{ -fsso}\}$ (respectively, $SSC_\tau\ (\lambda,\ r) = \bigwedge\{\mu \in I^X : \mu \geq \lambda, \mu \text{ is } r\text{ -fssc}\}$).

$f : (X,\ \tau) \to (Y,\ \eta)$ is fuzzy continuous (FCts for short) if $\eta(\mu) \leq \tau(f^{-1}(\mu))$ for each $\mu \in I^Y$. $f : (X,\ \tau) \to (Y,\ \eta)$ is fuzzy semi continuous, (respectively, fuzzy irresolute, fuzzy strongly semi continuous and fuzzy S-irresolute continuous) (FSCts (respectively, FI, FsSCts and FS-ICts) for short) if $f^{-1}(\mu)$ is $r$-fso (respectively, $r$-fso, $r$-fsso and $r$-fsso) set of $X$ for each $\eta(\mu) \geq r$, (respectively, $\mu$ is $r$-fso, $\eta(\mu) \geq r$ and $r$-fsso) set of $Y$ $r \in I_0$. $f : (X,\ \tau) \to (Y,\ \eta)$ is fuzzy open (respectively, fuzzy closed, fuzzy irresolute semiopen and fuzzy irresolute semiclosed) (FO (respectively, FC, FISO and FISC) for short) if $\tau(\lambda) \leq \eta(f(\lambda))$ (respectively, $\tau(\overline{1} - \lambda) \leq \eta(\overline{1} - f(\lambda))$, $r$-fso and $r$-fsc ) for each $\lambda \in I^X$ (resp. $r$-fso and $r$-fsc). $f : (X,\ \tau) \to (Y,\ \eta)$ is fuzzy S-irresolute open (respectively, fuzzy strongly semi-open and fuzzy strongly semi-closed) (FS-IO (resp. FsSO and FsSCC) for short) if $f(\mu)$ is $r$-fsso (respectively, $r$-fsso and $r$-fssc) set of $Y$ for each $r$-fsso (respectively, $\tau(\mu) \geq r$ & $\tau(\overline{1} - \mu) \geq r$,) set $r \in I_0$.

## 3. Fuzzy Semi-S-irresolute Open and Fuzzy Semi-S-irresolute Closed

In the following section, we introduce the concepts of fuzzy semi-S-irresolute open, fuzzy semi-S-irresolute closed and fuzzy semi-S-irresolute homeomorphism to the fuzzy topological spaces in Šostak's sense.

**Definition 3.1.**

Let $(X,\ \tau)$ and $(Y, \eta)$ be fts's and let $f : X \to Y$ be a mapping. Then $f$ is called

(1) fuzzy semi-S-irresolute continuous (briefly, FSS-ICts) (Abbas and Azab Abd-alla (2004)) if $f^{-1}(\mu)$ is $r$-fso set of $X$ for each $r$-fsso set $\mu \in I^Y,\ r \in I_0$.
(2) fuzzy semi-S-irresolute open (briefly, FSS-IO) if $f(\mu)$ is $r$-fso set of $Y$ for each $r$-fsso set $\mu \in I^X$, $r \in I_0$.
(3) fuzzy semi-S-irresolute closed (briefly, FSS-IC) if $f(\mu)$ is $r$-fsc set of $Y$ for each $r$-fssc set $\mu \in I^X$, $r \in I_0$.
(4) fuzzy semi-S-irresolute homeomorphism (briefly, FSS-IH) if $f$ is bijective and both of $f$ and $f^{-1}$ are FSS-ICts.

**Example 3.2.**

Let $\mu_1$ and $\mu_2$ be fuzzy subsets of $X = Y = \{a, b, c\}$ defined as follows:

$$\mu_1(a) = 0.1,\ \mu_1(b) = 0.3,\ \mu_1(c) = 0.2;$$
$$\mu_2(a) = 0.2,\ \mu_2(b) = 0.4,\ \mu_2(c) = 0.5;$$

Then $\tau, \eta : I^X \to I$ defined as

$$\tau(\lambda)=\begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2}, & \text{if } \lambda = \mu_1, \\ 0, & \text{otherwise}, \end{cases}$$
$$\eta(\mu)=\begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2}, & \text{if } \mu = \mu_2, \\ 0, & \text{otherwise}, \end{cases}$$

are smooth fuzzy topologies on $X$ and $Y$, respectively. Then the identity function $F : X \to Y$ is fuzzy semi-S-irresolute continuous.

**Example 3.3.**

Let $\mu_1$ and $\mu_2$ be fuzzy subsets of $X = Y = \{a, b, c\}$ defined as follows:

$$\mu_1(a) = 0.2,\ \mu_1(b) = 0.4,\ \mu_1(c) = 0.5;$$
$$\mu_2(a) = 0.1,\ \mu_2(b) = 0.3,\ \mu_2(c) = 0.2;$$

Then $\tau$, $\eta : I^X \to I$ defined as

$$\tau(\lambda)= \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2}, & \text{if } \lambda = \mu_1, \\ 0, & \text{otherwise}, \end{cases}$$
$$\eta(\mu)= \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2}, & \text{if } \mu = \mu_2, \\ 0, & \text{otherwise}, \end{cases}$$

are smooth fuzzy topologies on $X$ and $Y$, respectively. Then the identity function $F : X \to Y$ is fuzzy semi-S-irresolute open.

**Example 3.4.**

Let $\mu_1$ and $\mu_2$ be fuzzy subsets of $X = Y = \{a, b, c\}$ defined as follows:

$$\mu_1(a) = 0.7,\ \mu_1(b) = 0.7,\ \mu_1(c) = 0.7;$$
$$\mu_2(a) = 0.1,\ \mu_2(b) = 0.1,\ \mu_2(c) = 0.1;$$

Then $\tau$, $\eta : I^X \to I$ defined as

$$\tau(\lambda)= \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2}, & \text{if } \lambda = \mu_1, \\ 0, & \text{otherwise}, \end{cases}$$
$$\eta(\mu)= \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2}, & \text{if } \mu = \mu_2, \\ 0, & \text{otherwise}, \end{cases}$$

are smooth fuzzy topologies on $X$ and $Y$, respectively. Then the identity function $F : X \to Y$ is fuzzy semi-S-irresolute closed.

**Theorem 3.5.**

Let $(X,\ \tau_1)$ and $(Y,\ \tau_2)$ be fts's and $f : X \to Y$ be a mapping. The following statements are equivalent.

(1) A map $f$ is FSS-ICts.
(2) For each $r$-fssc $\mu \in I^Y,\ f^{-1}(\mu)$ is $r$-fsc.
(3) $f(I_{\tau_1}(C_{\tau_1}(\lambda, r), r)) \leq SSC_{\tau_2}(f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.
(4) $I_{\tau_1}(C_{\tau_1}(f^{-1}(\mu), r), r) \leq f^{-1}(SSC_{\tau_2}(\mu, r)$, for each $\mu \in I^Y$ and $r \in I_0$.
(5) $f^{-1}(SSI_{\tau_2}(\mu,\ r)) \leq C_{\tau_1}(I_{\tau_1}(f^{-1}(\mu), r), r)$, for each $r$-fsso set $\mu \in I^Y$ and $r \in I_0$.

***Proof:***

(1) $\Rightarrow$ (2): Let $\mu$ be $r$-fssc set in $Y$. Then $\overline{1} - \mu$ is $r$-fsso in $Y$. By (1), $f^{-1}(\overline{1} - \mu)$ is $r$-fso in $X$. Since $f^{-1}(\overline{1} - \mu) = \overline{1} - f^{-1}(\mu),\ f^{-1}(\overline{1} - \mu) = \overline{1} - f^{-1}(\mu)$ is $r$-fso set in $X$. This implies that $f^{-1}(\mu)$ is $r$-fsc set in $X$.

(2) $\Rightarrow$ (3): Suppose there exists $\lambda \in I^Y$ and $r \in I_0$ such that $f(I_{\tau_1}(C_{\tau_1}(\lambda, r), r)) \nleq SSC_{\tau_2}(f(\lambda), r)$. There exists $y \in Y$ and $t \in (0, 1)$ such that $f(I_{\tau_1}(C_{\tau_1}(\lambda, r), r))(y) > t > SSC_{\tau_2}(f(\lambda), r)(y)$. If $f^{-1}(\{y\}) = \phi$, it is a contradiction because $f(I_{\tau_1}(C_{\tau_1}(\lambda,\ r),\ r))(y) = 0$. If $f^{-1}(\{y\}) \neq \phi$, there exists $x \in f^{-1}(\{y\})$ such that

$$f(I_{\tau_1}(C_{\tau_1}(\lambda,\ r),\ r))(y) \geq I_{\tau_1}(C_{\tau_1}(\lambda,\ r),\ r)(x) > t > SSC_{\tau_2}(f(\lambda), r)(f(x)) \qquad (1)$$

Since $SSC_{\tau_2}(f(\lambda),\ r)(f(x)) < t$, there exists $r$-fssc $\mu \in I^Y$ with $f(\lambda) \leq \mu$ such that $SSC_{\tau_2}(f(\lambda),\ r)(f(x)) \leq \mu(f(x)) < t$. Moreover, $f(\lambda) \leq \mu$ implies $\lambda \leq f^{-1}(\mu)$. From (2), $f^{-1}(\mu)$ is $r$-fsc. Thus $I_{\tau_1}(C_{\tau_1}(\lambda,\ r),\ r)(x) \leq f^{-1}(\mu)(x) = \mu(f(x)) < t$. It is a contradiction for (1). Hence $f(I_{\tau_1}(C_{\tau_1}(\lambda,\ r),\ r) \leq SSC_{\tau_2}(f(\lambda),\ r)$.

(3) $\Rightarrow$ (4): For all $\mu \in I^Y$, $r \in I_0$, put $\lambda = f^{-1}(\mu)$. From (3), we have $f(I_{\tau_1}(C_{\tau_1}(f^{-1}(\mu),\ r),\ r)) \leq SSC_{\tau_2}(f(f^{-1}(\mu)),\ r) \leq SSC_{\tau_2}(\mu, r)$. It implies $I_{\tau_1}(C_{\tau_1}(f^{-1}(\mu),\ r),\ r) \leq f^{-1}(I_{\tau_1}(C_{\tau_1}(f^{-1}(\mu),\ r),\ r)) \leq f^{-1}(SSC_{\tau_2}(\mu, r))$.

(4) $\Rightarrow$ (3): Let $\lambda \in I^X$ and $r \in I_0$. By (4), we obtain $I_{\tau_1}(C_{\tau_1}(\lambda, r), r) \leq I_{\tau_1}(C_{\tau_1}(f^{-1}(f(\lambda)), r), r) \leq f^{-1}(SSC_{\tau_2}(f(\lambda), r))$. Thus $f(I_{\tau_1}(C_{\tau_1}(\lambda,\ r),\ r) \leq SSC_{\tau_2}(f(\lambda),\ r)$.

(3) $\Rightarrow$ (1): Let $\mu$ be any $r$-fsso set of $Y$ and $r \in I_0$. Since $f^{-1}(\overline{1} - \mu) = \overline{1} - f^{-1}(\mu)$, by (3), we have $f(I_{\tau_1}(C_{\tau_1}(f^{-1}(\overline{1} - \mu),\ r),\ r)) \leq SSC_{\tau_2}(ff^{-1}(\overline{1} - \mu),\ r) \leq SSC_{\tau_2}(\overline{1} - \mu,\ r) = \overline{1} - SSI_{\tau_2}(\mu,\ r) \leq \overline{1} - \mu$ and hence,

$$\begin{aligned}
\overline{1} - C_{\tau_1}(I_{\tau_1}(f^{-1}(\mu), r), r) &= I_{\tau_1}(C_{\tau_1}(\overline{1} - f^{-1}(\mu), r), r) \\
&= I_{\tau_1}(C_{\tau_1}(f^{-1}(\overline{1} - \mu), r), r) \\
&\leq f^{-1}f(I_{\tau_1}(C_{\tau_1}(f^{-1}(\overline{1} - \mu), r), r)) \\
&\leq f^{-1}(\overline{1} - \mu) \\
&\leq \overline{1} - f^{-1}(\mu).
\end{aligned}$$

Therefore, we obtain $f^{-1}(\mu) \leq C_{\tau_1}(I_{\tau_1}(f^{-1}(\mu), r), r) \Rightarrow f^{-1}(\mu)$ is $r$-fso set in $X$. Hence the mapping $f$ is fuzzy semi-S-irresolute continuous. ∎

**Theorem 3.6.**

Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's and $f : X \to Y$ be a bijective mapping. Then the map $f$ is FSS-ICts iff $SSI_\eta(f(\lambda),\ r) \leq f(C_\tau(I_\tau(\lambda, r), r))$, for each $\lambda \in I^X$ and $r \in I_0$.

***Proof:***

(1) ⇒ (2): Let $f$ be a FSS-ICts mapping and $\lambda \in I^X$ and $r \in I_0$. Then $f^{-1}(SSI_\eta(f(\lambda), r))$ is $r$-fso set in $X$. By Theorem 3.5 and the fact that $f$ is 1-1, we have,

$$f^{-1}(SSI_\eta(\mu,\ r)) \leq C_\tau(I_\tau(f^{-1}(\mu), r), r).$$

Let $\mu = f(\lambda)$. Then, $f^{-1}(SSI_\eta(f(\lambda),\ r)) \leq C_\tau(I_\tau(f^{-1}(f(\lambda),\ r), r) = C_\tau(I_\tau(\lambda,\ r), r)$. Again, since $f$ is onto, we have $SSI_\eta(f(\lambda),\ r) = ff^{-1}(SSI_\eta(f(\lambda),\ r) \leq f(C_\tau(I_\tau(\lambda,\ r), r))$.

Conversely, Let $\mu$ be $r$-fsso set of $Y$. Then we have $\mu = SSI_\eta(\mu, r)$. By (2), $f(C_\tau(I_\tau(f^{-1}(\mu), r), r)) \geq SSI_\eta(f(f^{-1}(\mu),\ r)) = SSI_\eta(\mu,\ r) = \mu$ and $C_\tau(I_\tau(f^{-1}(\mu), r), r) = f^{-1}f(C_\tau(I_\tau(f^{-1}(\mu),\ r), r)) \geq f^{-1}(\mu)$. This implies $C_\tau(I_\tau(f^{-1}(\mu), r), r) \geq f^{-1}(\mu)$. Thus, $f^{-1}(\mu)$ is $r$-fso. Thus (1). ■

**Theorem 3.7.**

Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's and let $f : X \to Y$ be a mapping. Then the following statements are equivalent.

(1) $f$ is called FSS-IO.
(2) $f(SSI_\tau(\lambda,\ r)) \leq C_\eta(I_\eta(f(\lambda),\ r), r)$, for each $\lambda \in I^X$ and $r \in I_0$.
(3) $SSI_\tau(f^{-1}(\mu),\ r) \leq f^{-1}(C_\eta(I_\eta(\mu,\ r), r))$, for each $\mu \in I^Y$ and $r \in I_0$.
(4) For any $\mu \in I^Y$ and any $r$-fssc with $f^{-1}(\mu) \leq \lambda$, there exists a $r$-fsc $\rho \in I^Y$ with $\mu \leq \rho$ such that $f^{-1}(\rho) \leq \lambda$.

***Proof:***

(1) ⇒ (2): Let $f$ be FSS-IO. For each $\lambda \in I^X$, since $SSI_\tau(\lambda,\ r) \leq \lambda$, we have $f(SSI_\tau(\lambda,\ r)) \leq f(\lambda)$. From (1), $f(SSI_\tau(\lambda,\ r))$ is $r$-fso. Hence $f(SSI_\tau(\lambda,\ r)) \leq C_\eta(I_\eta(f(\lambda),\ r), r)$. Thus proved (2).

(2) ⇒ (3): For all $\mu \in I^Y,\ r \in I_0$, put $\lambda = f^{-1}(\mu)$ from (2). Then, $f(SSI_\tau(f^{-1}(\mu),\ r)) \leq C_\eta(I_\eta(f(f^{-1}(\mu)),\ r),\ r) \leq C_\eta(I_\eta(\mu,\ r),\ r)$.
It implies $SSI_\tau(f^{-1}(\mu),\ r) \leq f^{-1}(C_\eta(I_\eta(\mu,\ r),\ r))$.

(3) ⇒ (4): Let $\lambda$ be $r$-fssc set of $X$ such that $f^{-1}(\mu) \leq \lambda$. Since $\overline{1} - \lambda \leq f^{-1}(\overline{1} - \mu)$ and $SSI_\tau(\overline{1} - \lambda,\ r) = \overline{1} - \lambda$. $SSI_\tau(\overline{1} - \lambda,\ r) = \overline{1} - \lambda \leq SSI_\tau(f^{-1}(\overline{1} - \mu),\ r)$. From (3),

$$\overline{1} - \lambda \leq SSI_\tau(f^{-1}(\overline{1} - \mu),\ r) \leq f^{-1}(C_\eta(I_\eta(\overline{1} - \mu,\ r), r)).$$

It implies $\lambda \geq \overline{1} - f^{-1}(C_\eta(I_\eta(\overline{1} - \mu,\ r), r)) = f^{-1}(\overline{1} - C_\eta(I_\eta(\overline{1} - \mu,\ r), r))$ $\lambda \geq f^{-1}(I_\eta(C_\eta(\mu,\ r), r))$. Hence there exists a $r$-fsc $\mu \in I^Y$ with $\mu \leq I_\eta(C_\eta(\mu,\ r), r)$ such that $f^{-1}(I_\eta(C_\eta(\mu,\ r), r)) \leq \lambda$. Thus (4) is proved.

(4) ⇒ (1): Let $\omega$ be $r$-fsso set of $X$. Now, we have to prove that $f(\omega)$ is $r$-fso set of $Y$. Put $\mu = \overline{1} - f(\omega)$ and $\lambda = \overline{1} - \omega$ such that $\lambda$ is $r$-fssc. We obtain $f^{-1}(\mu) = f^{-1}(\overline{1} - f(\omega)) = \overline{1} - f^{-1}(f(\omega)) \leq \overline{1} - \omega = \lambda \Rightarrow\ f^{-1}(\mu) \leq \lambda$. From (4), there exists a $r$-fsc set $\rho$ with $\mu \leq \rho$ such

that $f^{-1}(\rho) \le \lambda = \overline{1} - \omega$. It implies $\omega \le \overline{1} - f^{-1}(\rho) = f^{-1}(\overline{1} - \rho)$. Thus,

$$f(\omega) \le f(f^{-1}(\overline{1} - \rho)) \le \overline{1} - \rho \tag{2}$$

On the other hand, since $\mu \le \rho$,

$$f(\omega) = \overline{1} - \mu \ge \overline{1} - \rho, \tag{3}$$

Hence from equations (2) and (3), we get, $f(\omega) = \overline{1} - \rho$. (i.e)$f(\omega)$ is $r$-fso. ∎

**Theorem 3.8.**

Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's and $f : X \to Y$ be a mapping. The following statements are equivalent.

(1) $f$ is called FSS-IC.
(2) $f(SSC_\tau(\lambda,\ r)) \ge I_\eta(C_\eta(f(\lambda),\ r), r)$, for each $\lambda \in I^X$ and $r \in I_0$.
(3) $SSC_\tau(f^{-1}(\mu),\ r) \ge f^{-1}(I_\eta(C_\eta(\mu,\ r), r))$, for each $\mu \in I^Y$ and $r \in I_0$.
(4) For any $\mu \in I^Y$ and $r$-fsso $\lambda \in I^X$ with $f^{-1}(\mu) \le \lambda$, there exists a $r$-fso, $\rho \in I^Y$ with $\mu \ge \rho$ such that $f^{-1}(\rho) \ge \lambda$.

***Proof:***

(1) $\Rightarrow$ (2): Let $f$ be fuzzy semi-S-irresolute closed. For each $\lambda \in I^X$ and $r \in I_0$, since $SSC_\tau(\lambda,\ r) \ge \lambda$, we have $f(SSC_\tau(\lambda,\ r)) \ge f(\lambda)$. From (1), $f(SSC_\tau(\lambda,\ r))$ is $r$-fsc. Hence $f(SSC_\tau(\lambda,\ r)) \ge I_\eta(C_\eta(f(\lambda),\ r),\ r)$.

(2) $\Rightarrow$ (3): For all $\mu \in I^Y,\ r \in I_0$ put $\lambda = f^{-1}(\mu)$ from (2). Then,

$$f(SSC_\tau(f^{-1}(\mu),\ r)) \ge I_\eta(C_\eta(f(f^{-1}(\mu)), r), r) \ge I_\eta(C_\eta(\mu,\ r),\ r).$$

It implies

$$SSC_\tau(f^{-1}(\mu),\ r) \ge f^{-1}(I_\eta(C_\eta(\mu,\ r),\ r)).$$

(3) $\Rightarrow$ (4): Let $\lambda$ be $r$-fsso set of $X$ such that $f^{-1}(\mu) \ge \lambda$. Since $\overline{1} - \lambda \ge f^{-1}(\overline{1} - \mu)$ and $SSC_\tau(\overline{1} - \lambda,\ r) = \overline{1} - \lambda\ SSC_\tau(\overline{1} - \lambda,\ r) = \overline{1} - \lambda \ge SSC_\tau(f^{-1}(\overline{1} - \mu),\ r)$. From (3), $\overline{1} - \lambda \ge SSC_\tau(f^{-1}(\overline{1} - \mu),\ r) \ge f^{-1}(I_\eta(C_\eta(\overline{1} - \mu,\ r),\ r))$ It implies $\lambda \le \overline{1} - f^{-1}(I_\eta(C_\eta(\overline{1} - \mu,\ r),\ r))$ $= f^{-1}(\overline{1} - I_\eta(C_\eta(\overline{1} - \mu,\ r),\ r))$ $\lambda \le f^{-1}(C_\eta(I_\eta(\mu,\ r),\ r))$ Hence there exists a $r$-fso set $\mu \in I^Y$ with $\mu \ge C_\eta(I_\eta(\mu,\ r),\ r)$ such that $f^{-1}(C_\eta(I_\eta(\mu,\ r),\ r)) \ge \lambda$.

(4) $\Rightarrow$ (1): Let $\omega$ be $r$-fssc set of $X$. Now, we have to prove that $f(\omega)$ is $r$-fsc. Put $\mu = \overline{1} - f(\omega)$ and $\lambda = \overline{1} - \omega$ such that $\lambda$ is $r$-fsso. We obtain $f^{-1}(\mu) = f^{-1}(\overline{1} - f(\omega)) = \overline{1} - f^{-1}(f(\omega)) \le \overline{1} - \omega = \lambda$. From (4), there exists a $r$-fso set $\rho \in I^Y$ with $\mu \ge \rho$ such that $f^{-1}(\rho) \ge \lambda = \overline{1} - \omega$. It implies $\omega \ge \overline{1} - f^{-1}(\rho) = f^{-1}(\overline{1} - \rho)$. Thus,

$$f(\omega) \ge f(f^{-1}(\overline{1} - \rho)) = \overline{1} - \rho. \tag{4}$$

On the other hand, since $\mu \le \rho$,

$$f(\omega) = \overline{1} - \mu \le \overline{1} - \rho \tag{5}$$

Hence from (4) and (5), we have $f(\omega) = \overline{1} - \rho$. (i.e)$f(\omega)$ is $r$-fsc. ∎

**Theorem 3.9.**

Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's and let $f : X \to Y$ be a bijective mapping. Then the following statements hold :

(1) $f$ is FSS-IC iff $f^{-1}(I_\eta(C_\eta(\mu, r), r)) \leq SSC_\tau(f^{-1}(\mu),\ r)$, for each $\mu \in I^Y$ and $r \in I_0$.
(2) $f$ is FSS-IC iff $f$ if FSS-IO.

***Proof:***

(1) Let $f$ be FSS-IC. From Theorem 3.8(2), for each $\lambda \in I^X$ and $r \in I_0$, $f(SSC_\tau(\lambda,\ r)) \geq I_\eta(C_\eta(f(\lambda),\ r), r)$. For all $\mu \in I^Y,\ r \in I_0$. Put $\lambda = f^{-1}(\mu)$, since $f$ is onto, $f(f^{-1}(\mu)) = \mu$. Thus, $f(SSC_\tau(f^{-1}(\mu),\ r)) \geq I_\eta(C_\eta(f(f^{-1}(\mu)),\ r), r) = I_\eta(C_\eta(\mu,\ r), r)$. It implies $SSC_\tau(f^{-1}(\mu),\ r) = f^{-1}(f(SSC_\tau(f^{-1}(\mu),\ r))) \geq f^{-1}(I_\eta(C_\eta(\mu,\ r), r))$.

Conversely, put $\mu = f(\lambda)$. Since $f$ is injective, $f^{-1}(I_\eta(C_\eta(f(\lambda),\ r), r)) \leq SSC_\tau(f^{-1}(f(\lambda)),\ r) = SSC_\tau(\lambda,\ r)$. Since $f$ is onto, $I_\eta(C_\eta(f(\lambda),\ r), r) \leq f(SSC_\tau(\lambda,\ r))$. $\Rightarrow$ $f$ is FSS-IC.

(2) It is easily proved from :

$$\begin{aligned}
& f^{-1}(I_\eta(C_\eta(\mu,\ r), r)) \leq SSC_\tau(f^{-1}(\mu),\ r) \\
\Leftrightarrow\ & f^{-1}(\overline{1} - C_\eta(I_\eta(\overline{1} - \mu,\ r), r)) \leq \overline{1} - SSI_\tau(\overline{1} - f^{-1}(\mu),\ r) \\
\Leftrightarrow\ & \overline{1} - f^{-1}(C_\eta(I_\eta(\overline{1} - \mu,\ r), r)) \leq \overline{1} - SSI_\tau(f^{-1}(\overline{1} - \mu),\ r) \\
\Leftrightarrow\ & f^{-1}(C_\eta(I_\eta(\overline{1} - \mu,\ r), r)) \geq SSI_\tau(f^{-1}(\overline{1} - \mu),\ r) \\
\Leftrightarrow\ & \text{Hence } f \text{ is } FSS - IO.
\end{aligned}$$

■

**Remark 3.10.**

For a mapping $f : X \to Y$, the following statements are valid:

(1) $f$ is fuzzy S-irresolute continuous $\Rightarrow$ $f$ is fuzzy semi-S-irresolute continuous.
(2) $f$ is fuzzy S-irresolute open $\Rightarrow$ $f$ is fuzzy semi-S-irresolute open.
(3) $f$ is fuzzy S-irresolute closed $\Rightarrow$ $f$ is fuzzy semi-S-irresolute closed.
(4) $f$ is fuzzy continuous $\Rightarrow$ $f$ is fuzzy semi-S-irresolute continuous.

But the converses need not be true as shown by the following examples.

(1) In Example 3.2, $F$ is fuzzy semi-S-irresolute continuous but not fuzzy S-irresolute continuous because $\mu_2$ is $\frac{1}{2}$-fsso in $(Y,\ \eta)$ but $f^{-1}(\mu_2) = \mu_2$ is not $\frac{1}{2}$-fsso in $(X,\ \tau)$.
(2) In Example 3.3, $F$ is fuzzy semi-S-irresolute open but not fuzzy S-irresolute open because $\mu_1$ is $\frac{1}{2}$-fsso in $(X,\ \tau)$ but $f(\mu_1) = \mu_1$ is not $\frac{1}{2}$-fsso in $(Y,\ \eta)$.
(3) In Example 3.4, $F$ is fuzzy semi-S-irresolute closed but not fuzzy S-irresolute closed because $\mu_1$ is $\frac{1}{2}$-fssc in $(X,\ \tau)$ but $f(\mu_1) = \mu_1$ is not $\frac{1}{2}$-fssc in $(Y,\ \eta)$.

**Theorem 3.11.**

Let $f : X \to Y$ and $g : Y \to Z$ be mappings. If $f$ is FS-IO and $g$ is FSS-IO, then $g \circ f : X \to Z$ is FSS-IO, but not conversely.

***Proof:***

Let $\lambda$ be $r$-fsso set in $X$. As $f$ is FS-IO, $f(\lambda)$ is $r$-fsso set of $Y$. Also since $g : Y \to Z$ is FSS-IO, $g(f(\lambda))$ is $r$-fso in $Z$. (i.e) $(g \circ f)(\lambda) = g(f(\lambda))$ is $r$-fso in $Z$. Thus $g \circ f$ is FSS-IO. ∎

**Theorem 3.12.**

Let $f : X \to Y$ and $g : Y \to Z$ be mappings. If $f$ is FS-IC and $g$ is FSS-IC, then $g \circ f : X \to Z$ is FSS-IC, but not conversely.

***Proof:***

Let $\lambda$ be $r$-fssc set in $X$. As $f$ is FS-IC, $f(\lambda)$ is $r$-fssc set of $Y$. Also since $g : Y \to Z$ is FSS-IC, $g(f(\lambda))$ is $r$-fsc in $Z$. (i.e) $(g \circ f)(\lambda) = g(f(\lambda))$ is $r$-fsc in $Z$. Thus $g \circ f$ is FSS-IC. ∎

**Theorem 3.13.**

Let $f : X \to Y$ and $g : Y \to Z$ be mappings. If $f$ is FSS-IC and $g$ is FISC, then $g \circ f : X \to Z$ is FSS-IC, but not conversely.

***Proof:***

Let $\lambda$ be $r$-fssc set in $X$. As $f$ is FSS-IC, $f(\lambda)$ is $r$-fsc set of $Y$. Also since $g : Y \to Z$ is FISC, $g(f(\lambda))$ is $r$-fsc in $Z$. (i.e) $(g \circ f)(\lambda) = g(f(\lambda))$ is $r$-fsc in $Z$. Thus $g \circ f$ is FSS-IC map. ∎

**Theorem 3.14.**

Let $f : X \to Y$ and $g : Y \to Z$ be mappings. If $f$ is fuzzy semi-S-irresolute open and $g$ is FISO, then $g \circ f : X \to Z$ is FSS-IO, but not conversely.

***Proof:***

Let $\lambda$ be $r$-fsso set in $X$. As $f$ is FSS-IO, $f(\lambda)$ is $r$-fso set of $Y$. Also since $g : Y \to Z$ is FISO, $g(f(\lambda))$ is $r$-fso in $Z$. (i.e) $(g \circ f)(\lambda) = g(f(\lambda))$ is $r$-fso in $Z$. Thus $g \circ f$ is FSS-IO map. ∎

**Theorem 3.15.**

Let $f : X \to Y$ and $g : Y \to Z$ be mappings. If $f$ is FSS-ICts and $g$ is FsSCts, then $g \circ f : X \to Z$ is FSCts, but not conversely.

***Proof:***

Let $\mu$ be $r$-fo set in $Z$. As $g$ is FsSCts, $g^{-1}(\mu)$ is $r$-fsso set of $Y$. Also, since $f : X \to Y$ is FSS-ICts, $f^{-1}(g^{-1}(\mu))$ is $r$-fso in $X$. (i.e) $(g \circ f)^{-1}(\mu) = f^{-1}(g^{-1}(\mu))$ is $r$-fso set in $X$. Thus $g \circ f$ is FSCts. ∎

**Theorem 3.16.**

Let $f : X \to Y$ and $g : Y \to Z$ be mappings. If $f$ is FSS-ICts and $g$ is FS-ICts, then $g \circ f : X \to Z$ is FSS-ICts, but not conversely.

***Proof:***

Let $\mu$ be $r$-fsso set in $Z$. As $g$ is FS-ICts, $g^{-1}(\mu)$ is $r$-fsso set of $Y$. Also, since $f : X \to Y$ is FSS-ICts, $f^{-1}(g^{-1}(\mu))$ is $r$-fso set in $X$. (i.e) $(g \circ f)^{-1}(\mu) = f^{-1}(g^{-1}(\mu))$ is $r$-fso set in $X$. Thus $g \circ f$ is FSS-ICts. ■

## 4. Conclusion

Šostak's fuzzy topology has been recently of major interest among fuzzy topologies. The concepts of fuzzy semi-S-irresolute open, fuzzy semi-S-irresolute closed mappings and fuzzy semi-S-irresolute homeomorphism to the fuzzy topological spaces in Šostak's sense are introduced and studied. Some of their charachteristic properties are considered. Also a comparison between these new types of functions are established and counter examples are also given. These results will help to extend the some generalized continuous mappings, compactness and hence it will help to improve smooth topological and bi-topological spaces.

***Acknowledgement:***

*The authors wish to thank the reviewers for their careful reviews which improve the presentation of this paper considerably.*

## REFERENCES

Abbas, S. E., and Azab Abd-alla, M., (2004). On fuzzy $S$-irresolute continuous mappings, The Journal of Fuzzy Mathematics, Vol. 12, No. 4, pp. 905–920.

Chang, C. L. (1968). Fuzzy topological spaces, J. Math. Anal. Appl., Vol. 24 , pp. 182–189.

Chattopadhyay, K.C. and Samanta, S.K. (1993). Fuzzy topology: fuzzy closure operator, fuzzy compactness and fuzzy connectedness, Fuzzy Sets and Systems, Vol. 54, No. 2, pp. 207–212.

Hazra, R.N., Samanta, S.K. and Chattopadhyay, K.C. (1992). Fuzzy topology redefined, Fuzzy Sets and Systems, Vol. 4, pp. 79–82.

Höhle, U. (1980). Upper semicontinuous fuzzy sets and applications, J. Math. Anall. Apll., Vol. 78, pp. 659–673.

Höhle, U. and Šostak, A.P. (1995). A general theory of fuzzy topological spaces, Fuzzy sets and systems, Vol. 73, pp. 131–149.

Höhle, U. and Šostak, A.P. (1999) Axiomatic Foundations of Fixed-Basis Fuzzy Topology. In

*The Handbooks of Fuzzy Sets series, Volume 3*, pp. 123–272, Kluwer Academic Publishers, Dordrecht.

Kim, Y.C., Ramadam, A.A. and Abbas, S.E. (2003). Weaker forms of continuity in Šostak fuzzy topology, Indian J. Pure Appl. Math., Vol. 34, No. 2, pp. 311–333.

Kubiak, T. (1985). *On Fuzzy Topologies*, Ph.D. Thesis, A. Mickiewicz, Poznan.

Kubiak, T. and Šostak, A.P. (1997). Lower set-valued fuzzy topologies, Questions Math., Vol. 20, No. 3, pp. 423–429.

Ramadan, A.A. (1992). Smooth topological spaces, Fuzzy Sets and Systems, Vol. 48, pp. 371–375.

Rodabaugh, S.E. (1991). Categorical foundations of variable basis topology. In *The Handbooks of Fuzzy Sets series, Vol 3, Fuzzy sets: Logic, topology and Measure Theory* (U.Höhle and S. E. Rodabaugh, Editors), Kluwer Academic Publishers, Dordrecht.

Šostak, A.P. (1985). On a fuzzy topological structure, Rend. Circ. Matem. Palermo Ser. II, Vol. 11, pp. 89–103.

Šostak, A.P. (1989a). On some modifications of fuzzy topologies, Mathematick Vesnik, Vol. 41, pp. 20–37.

Šostak, A.P. (1989b) Two decades of fuzzy topology: Basic ideas, notion and results, Russian Math. Surveys, Vol. 44, No. 6, pp. 125–186.

Šostak, A.P. (1999). On the neighbourhood structure of fuzzy topologies, Zb. Radova Univ. Nisu, Vol. 4, pp. 7–14.

Ying, M.S. (1991). A new approach for fuzzy topology (I), Fuzzy sets and systems, Vol. 39, pp. 303–321.

Zadeh, L.A. (1965). Fuzzy sets, Inform. Control, Vol. 8, pp. 338–353.