Fuzzy Semi-S-irresolute Continuous Mappings in Šostak’s Fuzzy Topological Spaces

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Abstract

In this paper, the concepts of fuzzy semi-S-irresolute open map, fuzzy semi-S-irresolute closed map and fuzzy semi-S-irresolute homeomorphism to the fuzzy topological spaces in Šostak’s sense are introduced and studied. Some of their characteristic properties are considered. Also a comparison between these new types of functions are established by giving examples.

Keywords: Fuzzy semi-S-irresolute continuous; Fuzzy semi-S-irresolute open; Fuzzy semi-S-irresolute closed and fuzzy semi-S-irresolute homeomorphism

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1. Introduction

After the introduction of fuzzy sets by Zadeh (1965), Chang (1968) was the one who initiated the idea of a fuzzy topology in a set $X$, in which a collection, $T$, of open sets of fuzzy subsets of
2. Preliminaries

\(X, Y, \text{ etc.}, \) denote the non empty sets, \(I = [0, 1] \) and \(I_0 = (0, 1].\) In 1992, Ramadan gives the definition of smooth topological spaces (in short, sts) and the operators \(C_\tau : I^X \times I_0 \to I^X\) as \(C_\tau(\lambda, r) = \{\mu \in I^X : \lambda \leq \mu, \tau(1-\mu) \geq r\},\) and \(I_\tau : I^X \times I_0 \to I^X\) as \(I_\tau(\lambda, r) = \{\mu \in I^X : \lambda \geq \mu, \tau(\mu) \geq r\}.\) Kim et al (2003) discussed in a sts \((X, \tau), \forall r \in I_0, \lambda\) is called \(r\)-fuzzy strongly semiopen (respectively, \(r\)-fuzzy semiopen) \((r\text{-fso} \text{ respectively, } r\text{-fso})\) for short) iff \(\lambda \leq I_\tau(C_\tau(I_\tau(\lambda, r), r), r)\) (respectively, \(\lambda \leq C_\tau(I_\tau(\lambda, r), r))\). The respective compliment sets are closed set.

The operators \(r\text{-fuzzy strongly semi-interior} \text{ (respectively, } r\text{-fuzzy strongly semi-closure})\) of \(\lambda\), denoted by \(SSI_\tau(\lambda, r)\) (respectively, \(SSC_\tau(\lambda, r)\)) as \(SSI_\tau(\lambda, r) = \{\mu \in I^X : \mu \leq \lambda, \mu \text{ is } r\text{-fsso}\}\) (respectively, \(SSC_\tau(\lambda, r) = \{\mu \in I^X : \mu \geq \lambda, \mu \text{ is } r\text{-fssc}\}\)).

\[f : (X, \tau) \to (Y, \eta)\] is fuzzy continuous (FCts for short) if \(\eta(\mu) \leq \tau(f^{-1}(\mu))\) for each \(\mu \in I^Y.\)

\[f : (X, \tau) \to (Y, \eta)\] is fuzzy semi continuous, (respectively, fuzzy irresolute, fuzzy strongly semi continuous and fuzzy S- irresolute continuous) (FSCts (respectively, FI, FssCts and FS-ICTs) for short) if \(f^{-1}(\mu)\) is \(r\text{-fsso} \text{ (respectively, } r\text{-fso, } r\text{-fso and } r\text{-fso})\) set of \(X\) for each \(\eta(\mu) \geq r, \text{ (respectively, } \mu \text{ is } r\text{-fso, } \eta(\mu) \geq r \text{ and } r\text{-fso})\) set of \(Y r \in I_0, f : (X, \tau) \to (Y, \eta)\) is fuzzy open (respectively, fuzzy closed, fuzzy irresolute semiopen and fuzzy irresolute semiclosed) (FO (respectively, FC, FISO and FISC) for short) if \(\tau(\lambda) \leq \eta(f(\lambda))\) (respectively, \(\tau(1-\lambda) \leq \eta(1-f(\lambda))\), \(r\text{-fso and } r\text{-fso}\)) for each \(\lambda \in I^X\) (resp. \(r\text{-fso and } r\text{-fsc}\)). \(f : (X, \tau) \to (Y, \eta)\) is fuzzy S- irresolute open (respectively, fuzzy strongly semi-open and fuzzy strongly semi-closed) (FS-IO (resp. FsSO and FsSCC) for short) if \(f(\mu)\) is \(r\text{-fsso} \text{ (respectively, } r\text{-fsso and } r\text{-fssc})\) set of \(Y\) for each \(r\text{-fsso} \text{ (respectively, } \tau(\mu) \geq r \text{ & } \tau(1-\mu) \geq r, )\) set \(r \in I_0.\)
3. Fuzzy Semi-S-irresolute Open and Fuzzy Semi-S-irresolute Closed

In the following section, we introduce the concepts of fuzzy semi-S-irresolute open, fuzzy semi-S-irresolute closed and fuzzy semi-S-irresolute homeomorphism to the fuzzy topological spaces in Šostak’s sense.

**Definition 3.1.**

Let \((X, \tau)\) and \((Y, \eta)\) be fts’s and let \(f : X \rightarrow Y\) be a mapping. Then \(f\) is called

1. fuzzy semi-S-irresolute continuous (briefly, FSS-ICts) (Abbas and Azab Abd-alla (2004)) if \(f^{-1}(\mu)\) is \(r\)-fsso set of \(X\) for each \(r\)-fsso set \(\mu \in I^Y, r \in I_0\).
2. fuzzy semi-S-irresolute open (briefly, FSS-IO) if \(f(\mu)\) is \(r\)-fso set of \(Y\) for each \(r\)-fsso set \(\mu \in I^X, r \in I_0\).
3. fuzzy semi-S-irresolute closed (briefly, FSS-IC) if \(f(\mu)\) is \(r\)-fsc set of \(Y\) for each \(r\)-fssc set \(\mu \in I^X, r \in I_0\).
4. fuzzy semi-S-irresolute homeomorphism (briefly, FSS-IH) if \(f\) is bijective and both of \(f\) and \(f^{-1}\) are FSS-ICts.

**Example 3.2.**

Let \(\mu_1\) and \(\mu_2\) be fuzzy subsets of \(X = Y = \{a, b, c\}\) defined as follows:

\[
\begin{align*}
\mu_1(a) &= 0.1, \quad \mu_1(b) = 0.3, \quad \mu_1(c) = 0.2; \\
\mu_2(a) &= 0.2, \quad \mu_2(b) = 0.4, \quad \mu_2(c) = 0.5;
\end{align*}
\]

Then \(\tau, \eta : I^X \rightarrow I\) defined as

\[
\tau(\lambda) = \begin{cases} 
1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\
1/2, & \text{if } \lambda = \mu_1, \\
0, & \text{otherwise}, 
\end{cases}
\]

\[
\eta(\mu) = \begin{cases} 
1, & \text{if } \mu = \overline{0} \text{ or } \overline{1}, \\
1/2, & \text{if } \mu = \mu_2, \\
0, & \text{otherwise}, 
\end{cases}
\]

are smooth fuzzy topologies on \(X\) and \(Y\), respectively. Then the identity function \(F : X \rightarrow Y\) is fuzzy semi-S-irresolute continuous.

**Example 3.3.**

Let \(\mu_1\) and \(\mu_2\) be fuzzy subsets of \(X = Y = \{a, b, c\}\) defined as follows:
Then $\tau, \eta : I^X \to I$ defined as

$$\tau(\lambda) = \begin{cases} 
1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\
\frac{1}{2}, & \text{if } \lambda = \mu_1, \\
0, & \text{otherwise},
\end{cases}$$

$$\eta(\mu) = \begin{cases} 
1, & \text{if } \mu = \overline{0} \text{ or } \overline{1}, \\
\frac{1}{2}, & \text{if } \mu = \mu_2, \\
0, & \text{otherwise},
\end{cases}$$

are smooth fuzzy topologies on $X$ and $Y$, respectively. Then the identity function $F : X \to Y$ is fuzzy semi-S-irresolute open.

**Example 3.4.**

Let $\mu_1$ and $\mu_2$ be fuzzy subsets of $X = Y = \{a, b, c\}$ defined as follows:

$$\mu_1(a) = 0.7, \; \mu_1(b) = 0.7, \; \mu_1(c) = 0.7;$$

$$\mu_2(a) = 0.1, \; \mu_2(b) = 0.1, \; \mu_2(c) = 0.1;$$

Then $\tau, \eta : I^X \to I$ defined as

$$\tau(\lambda) = \begin{cases} 
1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\
\frac{1}{2}, & \text{if } \lambda = \mu_1, \\
0, & \text{otherwise},
\end{cases}$$

$$\eta(\mu) = \begin{cases} 
1, & \text{if } \mu = \overline{0} \text{ or } \overline{1}, \\
\frac{1}{2}, & \text{if } \mu = \mu_2, \\
0, & \text{otherwise},
\end{cases}$$

are smooth fuzzy topologies on $X$ and $Y$, respectively. Then the identity function $F : X \to Y$ is fuzzy semi-S-irresolute closed.

**Theorem 3.5.**

Let $(X, \tau_1)$ and $(Y, \tau_2)$ be fts’s and $f : X \to Y$ be a mapping. The following statements are equivalent.
(1) A map $f$ is FSS-ICts.

(2) For each $r$-fssc $\mu \in I^Y$, $f^{-1}(\mu)$ is $r$-fsc.

(3) $f(I_{r_1}(C_{r_1}(\lambda, r), r)) \leq SSC_{r_2}(f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.

(4) $I_{r_1}(C_{r_1}(f^{-1}(\mu), r), r) \leq f^{-1}(SSC_{r_2}(\mu, r))$, for each $\mu \in I^Y$ and $r \in I_0$.

(5) $f^{-1}(SSI_{r_2}(\mu, r)) \leq C_{r_1}(I_{r_1}(f^{-1}(\mu), r), r)$, for each $r$-fsso set $\mu \in I^Y$ and $r \in I_0$.

**Proof:**

(1) $\Rightarrow$ (2): Let $\mu$ be $r$-fssc set in $Y$. Then $\overline{1} - \mu$ is $r$-fsso in $Y$. By (1), $f^{-1}(\overline{1} - \mu)$ is $r$-fso in $X$. Since $f^{-1}(\overline{1} - \mu) = \overline{1} - f^{-1}(\mu)$, $f^{-1}(\overline{1} - \mu) = \overline{1} - f^{-1}(\mu)$ is $r$-fsso set in $X$. This implies that $f^{-1}(\mu)$ is $r$-fsso set in $X$.

(2) $\Rightarrow$ (3): Suppose there exists $\lambda \in I^Y$ and $r \in I_0$ such that $f(I_{r_1}(C_{r_1}(\lambda, r), r)) \notin SSC_{r_2}(f(\lambda), r)$. There exists $y \in Y$ and $t \in (0, 1)$ such that $f(I_{r_1}(C_{r_1}(\lambda, r), r))(y) > t > SSC_{r_2}(f(\lambda), r)(y)$. If $f^{-1}(\{y\}) = \phi$, it is a contradiction because $f(I_{r_1}(C_{r_1}(\lambda, r), r))(y) = 0$. If $f^{-1}(\{y\}) \neq \phi$, there exists $x \in f^{-1}(\{y\})$ such that

$$f(I_{r_1}(C_{r_1}(\lambda, r), r))(y) \geq I_{r_1}(C_{r_1}(\lambda, r), r)(x) > t > SSC_{r_2}(f(\lambda), r)(f(x))$$

(1)

Since $SSC_{r_2}(f(\lambda), r)(f(x)) < t$, there exists $r$-fssc $\mu \in I^Y$ with $f(\lambda) \leq \mu$ such that $SSC_{r_2}(f(\lambda), r)(f(x)) \leq \mu(f(x)) < t$. Moreover, $f(\lambda) \leq \mu$ implies $\lambda \leq f^{-1}(\mu)$. From (2), $f^{-1}(\mu)$ is $r$-fsc. Thus $I_{r_1}(C_{r_1}(\lambda, r), r)(x) \leq f^{-1}(\mu)(x) = \mu(f(x)) < t$. It is a contradiction for (1). Hence $f(I_{r_1}(C_{r_1}(\lambda, r), r), r) \leq SSC_{r_2}(f(\lambda), r)$.

(3) $\Rightarrow$ (4): For all $\mu \in I^Y$, $r \in I_0$, put $\lambda = f^{-1}(\mu)$. From (3), we have $f(I_{r_1}(C_{r_1}(f^{-1}(\mu), r), r)) \leq SSC_{r_2}(f(f^{-1}(\mu)), r) \leq SSC_{r_2}(\mu, r)$. It implies $I_{r_1}(C_{r_1}(f^{-1}(\mu), r), r) \leq f^{-1}(I_{r_1}(C_{r_1}(f^{-1}(\mu), r), r)) \leq f^{-1}(SSC_{r_2}(\mu, r))$.

(4) $\Rightarrow$ (3): Let $\lambda \in I^X$ and $r \in I_0$. By (4), we obtain $I_{r_1}(C_{r_1}(\lambda, r), r) \leq I_{r_1}(C_{r_1}(f^{-1}(\lambda), r), r) \leq f^{-1}(SSC_{r_2}(f(\lambda), r))$. Thus $f(I_{r_1}(C_{r_1}(\lambda, r), r), r) \leq SSC_{r_2}(f(\lambda), r)$.

(3) $\Rightarrow$ (1): Let $\mu$ be any $r$-fsso set of $Y$ and $r \in I_0$. Since $f^{-1}(\overline{1} - \mu) = \overline{1} - f^{-1}(\mu)$, by (3), we have $f(I_{r_1}(C_{r_1}(f^{-1}(\overline{1} - \mu), r), r)) \leq SSC_{r_2}(f(f^{-1}(\overline{1} - \mu), r) \leq SSC_{r_2}(\overline{1} - \mu, r) = \overline{1} - SSI_{r_2}(\mu, r) \leq \overline{1} - \mu$ and hence,

$$\overline{1} - C_{r_1}(I_{r_1}(f^{-1}(\mu), r), r) = I_{r_1}(C_{r_1}(f^{-1}(\mu), r), r)
= I_{r_1}(C_{r_1}(f^{-1}(1 - \mu), r), r)
\leq f^{-1}f(I_{r_1}(C_{r_1}(f^{-1}(\overline{1} - \mu), r), r))
\leq f^{-1}(\overline{1} - \mu)
\leq \overline{1} - f^{-1}(\mu).$$

Therefore, we obtain $f^{-1}(\mu) \leq C_{r_1}(I_{r_1}(f^{-1}(\mu), r), r) \Rightarrow f^{-1}(\mu)$ is $r$-fso set in $X$. Hence the mapping $f$ is fuzzy semi-S-irresolute continuous.

**Theorem 3.6.**

Let $(X, \tau)$ and $(Y, \eta)$ be fts’s and $f : X \to Y$ be a bijective mapping. Then the map $f$ is FSS-ICts iff $SSI_{\eta}(f(\lambda), r) \leq f(C_{\tau}(I_{r}(\lambda, r), r))$, for each $\lambda \in I^X$ and $r \in I_0$. 
Proof:

(1) ⇒ (2): Let $f$ be a FSS-ICts mapping and $\lambda \in I^X$ and $r \in I_0$. Then $f^{-1}(SSI_\eta(f(\lambda), r))$ is r-fso set in $X$. By Theorem 3.5 and the fact that $f$ is 1-1, we have,

$$f^{-1}(SSI_\eta(\mu, r)) \leq C_r(I_r(f^{-1}(\mu), r), r).$$

Let $\mu = f(\lambda)$. Then, $f^{-1}(SSI_\eta(f(\lambda), r)) \leq C_r(I_r(f^{-1}(\lambda), r), r) = C_r(I_r(\lambda, r), r)$. Again, since $f$ is onto, we have $SSI_\eta(f(\lambda), r) = f f^{-1}(SSI_\eta(f(\lambda), r) \leq f(C_r(I_r(\lambda, r))$.

Conversely, Let $\mu$ be r-fsso set of $Y$. Then we have $\mu = SSI_\eta(\mu, r)$. By (2), $f(C_r(I_r(f^{-1}(\mu), r), r) \geq SSI_\eta(f(\mu, r)) = SSI_\eta(\mu, r) \geq f^{-1}(C_r(I_r(\mu, r), r))$.)

This implies $C_r(I_r(f^{-1}(\mu), r), r) \geq f^{-1}(\mu)$. Thus, $f^{-1}(\mu)$ is r-fso. Thus (1).

Theorem 3.7.

Let $(X, \tau)$ and $(Y, \eta)$ be fts’s and let $f : X \to Y$ be a mapping. Then the following statements are equivalent.

1. $f$ is called FSS-IO.
2. $f(SSI_\tau(\lambda, r)) \leq C_\eta(I_\eta(f(\lambda), r), r), \text{ for each } \lambda \in I^X$ and $r \in I_0$.
3. $SSI_\tau(f^{-1}(\mu), r) \leq f^{-1}(C_\eta(I_\eta(\mu, r), r), r), \text{ for each } \mu \in I^Y$ and $r \in I_0$.
4. For any $\mu \in I^Y$ and any r-fssc with $f^{-1}(\mu) \leq \lambda$, there exists a r-fsc $r \in I^Y$ with $\mu \leq \rho$ such that $f^{-1}(\rho) \leq \lambda$.

Proof:

(1) ⇒ (2): Let $f$ be FSS-IO. For each $\lambda \in I^X$, since $SSI_\tau(\lambda, r) \leq \lambda$, we have $f(SSI_\tau(\lambda, r)) \leq f(\lambda)$. From (1), $f(SSI_\tau(\lambda, r))$ is r-fso. Hence $f(SSI_\tau(\lambda, r)) \leq C_\eta(I_\eta(f(\lambda), r), r)$. Thus proved (2).

(2) ⇒ (3): For all $\mu \in I^Y$, $r \in I_0$, put $\lambda = f^{-1}(\mu)$ from (2). Then, $f(SSI_\tau(f^{-1}(\mu), r)) \leq C_\eta(I_\eta(f(f^{-1}(\mu)), r), r) \leq C_\eta(I_\eta(\mu, r), r)$. It implies $SSI_\tau(f^{-1}(\mu), r) \leq f^{-1}(C_\eta(I_\eta(\mu, r), r))$.

(3) ⇒ (4): Let $\lambda$ be r-fssc set of $X$ such that $f^{-1}(\mu) \leq \lambda$. Since $\bar{\lambda} - \lambda \leq f^{-1}(\bar{\lambda} - \mu)$ and $SSI_\tau(\bar{\lambda} - \lambda, r) = \bar{\lambda} - \lambda.SSI_\tau(\bar{\lambda} - \lambda, r) = \bar{\lambda} - \lambda \leq SSI_\tau(f^{-1}(\bar{\lambda} - \mu), r)$. From (3),

$$\bar{\lambda} - \lambda \leq SSI_\tau(f^{-1}(\bar{\lambda} - \mu), r) \leq f^{-1}(C_\eta(I_\eta(\bar{\lambda} - \mu, r), r)),$$

It implies $\lambda \geq \bar{\lambda} - f^{-1}(C_\eta(I_\eta(\bar{\lambda} - \mu, r), r)) = f^{-1}(\bar{\lambda} - C_\eta(I_\eta(\bar{\lambda} - \mu, r), r)) \geq f^{-1}(C_\eta(C_\eta(\mu, r), r))$. Hence there exists a r-fsc $\mu \in I^Y$ with $\mu \leq I_\eta(C_\eta(\mu, r))$ such that $f^{-1}(\mu) \leq \lambda$. Thus (4) is proved.

(4) ⇒ (1): Let $\omega$ be r-fsso set of $X$. Now, we have to prove that $f(\omega)$ is r-fso set of $Y$. Put $\mu = \bar{\lambda} - f(\omega)$ and $\lambda = \bar{\lambda} - \omega$ such that $\lambda$ is r-fssc. We obtain $f^{-1}(\mu) = \bar{\lambda} - f(\omega) = \bar{\lambda} - f^{-1}(f(\omega)) \leq \bar{\lambda} - \omega = \lambda = f^{-1}(\mu) \leq \lambda$. From (4), there exists a r-fsc set $\rho$ with $\mu \leq \rho$ such...
that \( f^{-1}(\rho) \leq \lambda = \overline{I} - \omega \). It implies \( \omega \leq \overline{I} - f^{-1}(\rho) = f^{-1}(\overline{I} - \rho) \). Thus,
\[
f(\omega) \leq f(f^{-1}(\overline{I} - \rho)) \leq \overline{I} - \rho \tag{2}
\]
On the other hand, since \( \mu \leq \rho \),
\[
f(\omega) = \overline{I} - \mu \geq \overline{I} - \rho, \tag{3}
\]
Hence from equations (2) and (3), we get, \( f(\omega) = \overline{I} - \rho \). (i.e) \( f(\omega) \) is \( r \)-fso.

\[\square\]

**Theorem 3.8.**

Let \((X, \tau)\) and \((Y, \eta)\) be fts’s and \( f : X \to Y \) be a mapping. The following statements are equivalent.

1. \( f \) is called FSS-IC.
2. \( f(SSC_\tau(\lambda, r)) \geq I_\eta(C_\eta(f(\lambda), r), r) \), for each \( \lambda \in I^X \) and \( r \in I_0 \).
3. \( SSC_\tau(f^{-1}(\mu), r) \geq f^{-1}(I_\eta(C_\eta(\mu, r)), r) \), for each \( \mu \in I^Y \) and \( r \in I_0 \).
4. For any \( \mu \in I^Y \) and \( r \)-fso \( \lambda \in I^X \) with \( f^{-1}(\mu) \leq \lambda \), there exists a \( r \)-fso, \( \rho \in I^Y \) with \( \mu \geq \rho \) such that \( f^{-1}(\rho) \geq \lambda \).

**Proof:**

(1) \( \Rightarrow \) (2): Let \( f \) be fuzzy semi-S-irresolute closed. For each \( \lambda \in I^X \) and \( r \in I_0 \), since \( SSC_\tau(\lambda, r) \geq \lambda \), we have \( f(SSC_\tau(\lambda, r)) \geq f(\lambda) \). From (1), \( f(SSC_\tau(\lambda, r)) \) is \( r \)-fsc. Hence \( f(SSC_\tau(\lambda, r)) \geq I_\eta(C_\eta(f(\lambda), r), r) \).

(2) \( \Rightarrow \) (3): For all \( \mu \in I^Y \), \( r \in I_0 \) put \( \lambda = f^{-1}(\mu) \) from (2). Then,
\[
f(SSC_\tau(f^{-1}(\mu), r)) \geq I_\eta(C_\eta(f(f^{-1}(\mu)), r), r) \geq I_\eta(C_\eta(\mu, r), r).
\]
It implies
\[
SSC_\tau(f^{-1}(\mu), r) \geq f^{-1}(I_\eta(C_\eta(\mu, r), r)).
\]

(3) \( \Rightarrow \) (4): Let \( \lambda \) be \( r \)-fso set of \( X \) such that \( f^{-1}(\mu) \geq \lambda \). Since \( \overline{I} - \lambda \geq f^{-1}(\overline{I} - \mu) \) and \( SSC_\tau(\overline{I} - \lambda, r) = I_\eta(C_\eta(\mu, r), r) \), \( \overline{I} - \lambda \geq SSC_\tau(f^{-1}(\overline{I} - \mu), r) \). From (3), \( \overline{I} - \lambda \geq f^{-1}(I_\eta(C_\eta(\overline{I} - \mu, r), r)) \). It implies \( \lambda \leq \overline{I} - f^{-1}(I_\eta(C_\eta(\overline{I} - \mu, r), r)) \). Hence there exists a \( r \)-fso set \( \mu \in I^Y \) with \( \mu \geq C_\eta(I_\eta(\mu, r), r) \) such that \( f^{-1}(C_\eta(I_\eta(\mu, r), r)) \geq \lambda \).

(4) \( \Rightarrow \) (1): Let \( \omega \) be \( r \)-fssc set of \( X \). Now, we have to prove that \( f(\omega) \) is \( r \)-fsc. Put \( \mu = \overline{I} - f(\omega) \) and \( \lambda = \overline{I} - \omega \) such that \( \lambda \) is \( r \)-fso. We obtain \( f^{-1}(\mu) = f^{-1}(\overline{I} - f(\omega)) = \overline{I} - f^{-1}(f(\omega)) \leq \overline{I} - \omega = \lambda \). From (4), there exists a \( r \)-fso set \( \rho \in I^Y \) with \( \mu \geq \rho \) such that \( f^{-1}(\rho) \geq \lambda = \overline{I} - \omega \). It implies \( \omega \geq \overline{I} - f^{-1}(\rho) = f^{-1}(\overline{I} - \rho) \). Thus,
\[
f(\omega) \geq f(f^{-1}(\overline{I} - \rho)) = \overline{I} - \rho. \tag{4}
\]
On the other hand, since \( \mu \leq \rho \),
\[
f(\omega) = \overline{I} - \mu \leq \overline{I} - \rho \tag{5}
\]
Hence from (4) and (5), we have \( f(\omega) = \overline{I} - \rho \). (i.e) \( f(\omega) \) is \( r \)-fsc.

\[\square\]
Theorem 3.9.

Let \((X, \tau)\) and \((Y, \eta)\) be fts’s and let \(f : X \to Y\) be a bijective mapping. Then the following statements hold:

1. \(f\) is FSS-IC iff \(f^{-1}(I_\eta(C_\eta(\mu, r), r)) \leq SSC_{\tau}(f^{-1}(\mu), r)\), for each \(\mu \in I^Y\) and \(r \in I_0\).
2. \(f\) is FSS-IC iff \(f\) if FSS-IO.

Proof:

(1) Let \(f\) be FSS-IC. From Theorem 3.8(2), for each \(\lambda \in I^X\) and \(r \in I_0\), \(f(SSC_{\tau}(\lambda, r)) \geq I_\eta(C_\eta(f(\lambda), r), r)\). For all \(\mu \in I^Y\), \(r \in I_0\). Put \(\lambda = f^{-1}(\mu)\), since \(f\) is onto, \(f(f^{-1}(\mu)) = \mu\). Thus, \(f(SSC_{\tau}(f^{-1}(\mu), r)) \geq I_\eta(C_\eta(f(f^{-1}(\mu)), r), r) = I_\eta(C_\eta(\mu, r), r)\). It implies \(SSC_{\tau}(f^{-1}(\mu), r) = f^{-1}(f(SSC_{\tau}(f^{-1}(\mu), r))) \geq f^{-1}(I_\eta(C_\eta(\mu, r), r))\).

Conversely, put \(\mu = f(\lambda)\). Since \(f\) is injective, \(f^{-1}(I_\eta(C_\eta(\lambda, r), r)) \leq SSC_{\tau}(f^{-1}(f(\lambda)), r) = SSC_{\tau}(\lambda, r)\). Since \(f\) is onto, \(I_\eta(C_\eta(f(\lambda), r), r) \leq f(SSC_{\tau}(\lambda, r))\). \(\Rightarrow f\) is FSS-IC.

(2) It is easily proved from:

\[
\begin{align*}
    f^{-1}(I_\eta(C_\eta(\mu, r), r)) & \leq SSC_{\tau}(f^{-1}(\mu), r) \\
    \Leftrightarrow f^{-1}(\overline{I} - C_\eta(I_\eta(\overline{I} - \mu, r), r)) & \leq \overline{I} - SSI_{\tau}(\overline{I} - f^{-1}(\mu), r) \\
    \Leftrightarrow \overline{I} - f^{-1}(C_\eta(I_\eta(\overline{I} - \mu, r), r)) & \leq \overline{I} - SSI_{\tau}(f^{-1}(\overline{I} - \mu), r) \\
    \Leftrightarrow f^{-1}(C_\eta(I_\eta(\overline{I} - \mu, r), r)) & \geq SSI_{\tau}(f^{-1}(\overline{I} - \mu), r) \\
    \Leftrightarrow \text{Hence } f \text{ is FSS - IO.}
\end{align*}
\]

Remark 3.10.

For a mapping \(f : X \to Y\), the following statements are valid:

1. \(f\) is fuzzy S-irresolute continuous \(\Rightarrow f\) is fuzzy semi-S-irresolute continuous.
2. \(f\) is fuzzy S-irresolute open \(\Rightarrow f\) is fuzzy semi-S-irresolute open.
3. \(f\) is fuzzy S-irresolute closed \(\Rightarrow f\) is fuzzy semi-S-irresolute closed.
4. \(f\) is fuzzy continuous \(\Rightarrow f\) is fuzzy semi-S-irresolute continuous.

But the converses need not be true as shown by the following examples.

1. In Example 3.2, \(F\) is fuzzy semi-S-irresolute continuous but not fuzzy S-irresolute continuous because \(\mu_2\) is \(\frac{1}{2}\)-fss in \((Y, \eta)\) but \(f^{-1}(\mu_2) = \mu_2\) is not \(\frac{1}{2}\)-fss in \((X, \tau)\).
2. In Example 3.3, \(F\) is fuzzy semi-S-irresolute open but not fuzzy S-irresolute open because \(\mu_1\) is \(\frac{1}{2}\)-fss in \((X, \tau)\) but \(f(\mu_1) = \mu_1\) is not \(\frac{1}{2}\)-fss in \((Y, \eta)\).
3. In Example 3.4, \(F\) is fuzzy semi-S-irresolute closed but not fuzzy S-irresolute closed because \(\mu_1\) is \(\frac{1}{2}\)-fssc in \((X, \tau)\) but \(f(\mu_1) = \mu_1\) is not \(\frac{1}{2}\)-fssc in \((Y, \eta)\).
Theorem 3.11.
Let \( f : X \to Y \) and \( g : Y \to Z \) be mappings. If \( f \) is FS-IO and \( g \) is FSS-IO, then \( g \circ f : X \to Z \) is FSS-IO, but not conversely.

Proof:
Let \( \lambda \) be \( r \)-fss set in \( X \). As \( f \) is FS-IO, \( f(\lambda) \) is \( r \)-fss set of \( Y \). Also since \( g : Y \to Z \) is FSS-IO, \( g(f(\lambda)) \) is \( r \)-fso in \( Z \). (i.e) \((g \circ f)(\lambda) = g(f(\lambda))\) is \( r \)-fso in \( Z \). Thus \( g \circ f \) is FSS-IO.

Theorem 3.12.
Let \( f : X \to Y \) and \( g : Y \to Z \) be mappings. If \( f \) is FS-IC and \( g \) is FSS-IC, then \( g \circ f : X \to Z \) is FSS-IC, but not conversely.

Proof:
Let \( \lambda \) be \( r \)-fssc set in \( X \). As \( f \) is FS-IC, \( f(\lambda) \) is \( r \)-fssc set of \( Y \). Also since \( g : Y \to Z \) is FSS-IC, \( g(f(\lambda)) \) is \( r \)-fsc in \( Z \). (i.e) \((g \circ f)(\lambda) = g(f(\lambda))\) is \( r \)-fsc in \( Z \). Thus \( g \circ f \) is FSS-IC.

Theorem 3.13.
Let \( f : X \to Y \) and \( g : Y \to Z \) be mappings. If \( f \) is FSS-IC and \( g \) is FISC, then \( g \circ f : X \to Z \) is FSS-IC, but not conversely.

Proof:
Let \( \lambda \) be \( r \)-fssc set in \( X \). As \( f \) is FSS-IC, \( f(\lambda) \) is \( r \)-fssc set of \( Y \). Also since \( g : Y \to Z \) is FISC, \( g(f(\lambda)) \) is \( r \)-fsc in \( Z \). (i.e) \((g \circ f)(\lambda) = g(f(\lambda))\) is \( r \)-fsc in \( Z \). Thus \( g \circ f \) is FSS-IC.

Theorem 3.14.
Let \( f : X \to Y \) and \( g : Y \to Z \) be mappings. If \( f \) is fuzzy semi-S-irresolute open and \( g \) is FISO, then \( g \circ f : X \to Z \) is FSS-IO, but not conversely.

Proof:
Let \( \lambda \) be \( r \)-fsso set in \( X \). As \( f \) is FSS-IO, \( f(\lambda) \) is \( r \)-fso set of \( Y \). Also since \( g : Y \to Z \) is FISO, \( g(f(\lambda)) \) is \( r \)-fso in \( Z \). (i.e) \((g \circ f)(\lambda) = g(f(\lambda))\) is \( r \)-fso in \( Z \). Thus \( g \circ f \) is FSS-IO.

Theorem 3.15.
Let \( f : X \to Y \) and \( g : Y \to Z \) be mappings. If \( f \) is FSS-ICts and \( g \) is FsSCts, then \( g \circ f : X \to Z \) is FSCts, but not conversely.

Proof:
Let \( \mu \) be \( r \)-fo set in \( Z \). As \( g \) is FsSCts, \( g^{-1}(\mu) \) is \( r \)-fsso set of \( Y \). Also, since \( f : X \to Y \) is FSS-ICts, \( f^{-1}(g^{-1}(\mu)) \) is \( r \)-fso in \( X \). (i.e) \((g \circ f)^{-1}(\mu) = f^{-1}(g^{-1}(\mu))\) is \( r \)-fso set in \( X \). Thus \( g \circ f \) is FSCts.
Theorem 3.16.
Let \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) be mappings. If \( f \) is FSS-ICts and \( g \) is FS-ICts, then \( g \circ f : X \rightarrow Z \) is FSS-ICts, but not conversely.

**Proof:**
Let \( \mu \) be \( r \)-fsso set in \( Z \). As \( g \) is FS-ICts, \( g^{-1}(\mu) \) is \( r \)-fsso set of \( Y \). Also, since \( f : X \rightarrow Y \) is FSS-ICts, \( f^{-1}(g^{-1}(\mu)) \) is \( r \)-fso set in \( X \). (i.e) \( (g \circ f)^{-1}(\mu) = f^{-1}(g^{-1}(\mu)) \) is \( r \)-fso set in \( X \). Thus \( g \circ f \) is FSS-ICts.

4. Conclusion

Šostak’s fuzzy topology has been recently of major interest among fuzzy topologies. The concepts of fuzzy semi-S-irresolute open, fuzzy semi-S-irresolute closed mappings and fuzzy semi-S-irresolute homeomorphism to the fuzzy topological spaces in Šostak’s sense are introduced and studied. Some of their characteristic properties are considered. Also a comparison between these new types of functions are established and counter examples are also given. These results will help to extend the some generalized continuous mappings, compactness and hence it will help to improve smooth topological and bi-topological spaces.

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