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Alpha-Skew Generalized Normal Distribution and its Applications

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Abstract

The main object of this paper is to introduce a new family of distributions, which is quite flexible to fit both unimodal and bimodal shapes. This new family is entitled alpha-skew generalized normal (ASGN), that skews the symmetric distributions, especially generalized normal distribution through this paper. Here, some properties of this new distribution including cumulative distribution function, survival function, hazard rate function and moments are derived. To estimate the model parameters, the maximum likelihood estimators and the asymptotic distribution of the estimators are discussed. The observed information matrix is derived. Finally, the flexibility of the new distribution, as well as its effectiveness in comparison with other distributions, are shown via an application.

Keywords: Alpha-skew distribution; Generalized normal distribution; Maximum likelihood estimator; Skew-normal distribution

MSC 2010 No.: 60E05, 62F10

1. Introduction

Recently, the skew-symmetric distributions have received considerable amount of attention, for example, the alpha skew normal (ASN) distribution is studied by Elal-Olivero (2010). First, he introduced a new symmetric bimodal-normal distribution with the pdf

$$f(y) = y^2\phi(y),$$

where ϕ is standard normal density function, and then defined alpha-skew-normal distribution as

$$f(y;\alpha) = \frac{(1-\alpha y)^2+1}{2+\alpha^2}\phi(y), \qquad y \in \mathbb{R}, \ \alpha \in \mathbb{R},$$

where α is skew parameter. A random variable with ASN distribution is denoted by $X \sim ASN(\alpha)$.

Also, Shams Harandi and Alamatsaz (2012) introduced the alpha-skew Laplace (ASL) distribution with the pdf

$$f(y;\alpha) = \frac{(1-\alpha y)^2+1}{4(1+\alpha^2)}e^{-|y|}, \qquad y \in \mathbb{R}, \quad \alpha \in \mathbb{R},$$

and investigated some of its properties and denoted this random variable with $X \sim ASL(\alpha)$.

In this paper, a more general case of these two mentioned distributions is presented, which is called as alpha-skew generalized normal (ASGN) distribution which is obtained by adding a skew parameter α to the generalized normal distribution (GND) proposed first by Lee et al. (2013) with pdf

$$f(y;\,\omega) = \frac{1}{2\omega^{\frac{1}{\omega}}\Gamma\left(\frac{1}{\omega}\right)}e^{\frac{-|y|^{\omega}}{\omega}}, \qquad y \in \mathbb{R}, \quad \omega > 0.$$

The motivations for considering this density are:

1. The alpha-skew generalized normal distribution with at most two modes is very flexible and includes five classes of important distributions: Normal, Laplace, Alpha-Skew-Normal, Alpha-Skew-Laplace and Bimodal-Generalized Normal as special cases.
2. It seems that the admissible intervals for the skewness and the kurtosis parameters are $(-0.685,\ 3.803)$ and $(1.250,\ 3.325)$, respectively, which are wider than those of the Azzalini's the ASN distributions, $(-0.811,\ 0.811)$ and $(-1.300,\ 0.749)$, respectively.
3. In many applied studies, data may be skew and bimodal with thinner or thicker tails than normal. We believe that the distribution ASGN can illustrate a better fit with respect to the fitted models for some this kind of real data sets. It is obvious that the new introduced models are not provided best fit in compared to all other models, but of course it is necessary to provide best fit in compared to its sub-models.

The remainder of the paper is organized as follows: In Section 2, we define the ASGN distribution and outline some special cases of the distribution. Also, some properties of the distribution are investigated in this section. cumulative distribution, survival and hazard rate functions are obtained in Section 3. we provide a general expansion for the moments of the ASGN distribution in Section 4. Skewness and kurtosis indices are presented in Section 5. Stochastic representation for this model is discussed in Section 6. We introduce the location-scale version of this distribution in Section 7. In Section 8, we focus on maximum likelihood estimation (MLE) and calculate the elements of the observed information matrix. A simulation study is performed in Section 9. Application of the ASGN distribution using a set of real data is given in Section 10. Finally, Section 11 concludes the paper.

2. Alpha-skew generalized normal distribution

Definition 2.1.

If a random variable Y has the following density function,

$$f(y;\,\omega)=\frac{y^2\,\omega^{\frac{\omega-3}{\omega}}\,e^{-\frac{|y|^{\omega}}{\omega}}}{2\Gamma\left(\frac{3}{\omega}\right)},\qquad y\in\mathbb{R},$$

where $\omega>0$, then we say that Y follows the bimodal-generalized normal BGN distribution.

Definition 2.2.

If a random variable X has the following density function,

$$f(x;\,\alpha,\,\omega)=\frac{\omega^{1-\frac{1}{\omega}}\left[(1-\alpha x)^2+1\right]}{2\left(\alpha^2\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right)\right)}\,e^{-\frac{|x|^{\omega}}{\omega}},\qquad x\in\mathbb{R},\tag{1}$$

where $\alpha\in\mathbb{R}$ and $\omega>0$, then we say that X follows the alpha-skew generalized normal distribution with parameters α and ω, which is denoted by $X\sim ASGN(\alpha,\omega)$.

If $X\sim ASGN(\alpha,\omega)$, the following properties are concluded directly from the definition:

- If $\alpha=0$ and $\omega=1$, then $X\sim Laplas(0,1)$.
- If $\alpha=0$ and $\omega=2$, then $X\sim N(0,1)$.
- If $\omega=1$, then $X\sim ASL(\alpha)$.
- If $\omega=2$, then $X\sim ASN(\alpha)$.
- If $\alpha\to\pm\infty$, then $X\sim BGN(\omega)$.
- $-X\sim ASGN(-\alpha)$.

Theorem 2.1.

The alpha-skew generalized normal density function has at most two modes.

Proof:

Differentiating (1), we have

$$f'(x)=\frac{\omega^{\frac{\omega-1}{\omega}}e^{-\frac{|x|^{\omega}}{\omega}}\left(2\alpha(\alpha x-1)-\left(\alpha^2x^2-2\alpha x+2\right)|x|^{\omega-2}x)\right)}{2\left(\alpha^2\omega^{2/\omega}\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right)\right)}.$$

Due to ω, the above equation is not solved simply. So, it is sufficient to prove the especial cases. Putting $\omega=1$ in the above equation, we get

$$f'(x)=\frac{1}{4\left(1+\alpha^2\right)}\begin{cases}e^{x}\left[\alpha^2x^2-2\alpha\left(1-\alpha\right)x-\left(2\alpha-2\right)\right] & \text{if } x<0,\\ e^{-x}\left[-\alpha^2x^2+2\alpha\left(1+\alpha\right)x-\left(2\alpha+2\right)\right] & \text{if } x\geq 0.\end{cases} \tag{2}$$

It is obvious that each expression in the above has at most two roots. The roots of second expression occur in $x_1=\frac{1+\alpha-\sqrt{\alpha^2-1}}{\alpha}$ and $x_2=\frac{1+\alpha+\sqrt{\alpha^2-1}}{\alpha}$ and the roots of first expression occur in $x_1^*=\frac{1-\alpha-\sqrt{\alpha^2-1}}{\alpha}$ and $x_2^*=\frac{1-\alpha+\sqrt{\alpha^2-1}}{\alpha}$. While $\alpha^2-1\geqslant 0$, we have the following cases

a) for $\alpha\geqslant 1$: it is easily observed that $x_2>0$, $x_1,x_1^*<0$ and $x_2^*>0$;
b) for $\alpha\leqslant -1$: it is observed that $x_2^*<0$, $x_1^*,x_1>0$ and $x_2<0$.

Therefore, there are only three acceptable roots in each case. Hence, $f\left(x\right)$ can have at most two modes. For $-1<\alpha<1$, $f'(x)$ has not real root, but it is positive for $x<0$ and is negative for $x>0$. In this case, $f\left(x\right)$ is unimodal.

Putting $\omega=2$, we obtain

$$f'(x)=\frac{\phi(x)}{2+\alpha^2}\left[-\alpha^2x^3+2\alpha x^2-\left(2-2\alpha^2\right)x-2\alpha\right],$$

where $f'(x)$ has at most three roots and also in this case, $f(x)$ has at most two modes. For $\omega > 2$, the complicated mathematical computations are needed, but the claim can be proved by drawing some plots of density function for different values of ω.

Plots of the alpha-skew generalized normal density function for selected parameter values are given in Figure 1.

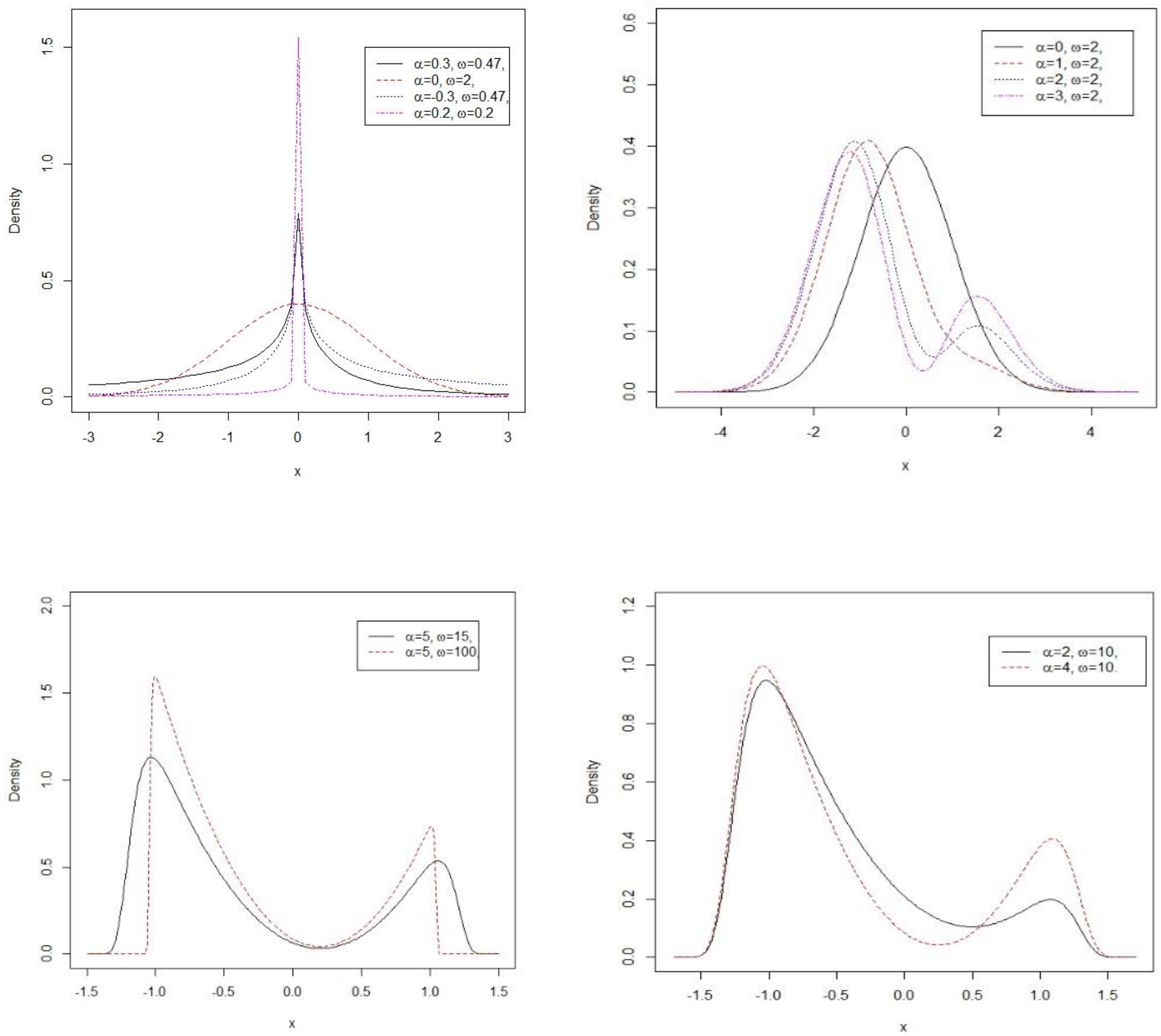

Figure 1. Plots of the alpha skew generalized normal density function for different values of α and ω

3. The cumulative distribution, survival and hazard rate functions

If $X \sim ASGN(\alpha, \omega)$, then cumulative distribution function of X is given as

$$F(x;\alpha,\omega)=\begin{cases}\dfrac{\alpha^2\omega^{2/\omega}\Gamma\left(\dfrac{3}{\omega},\dfrac{(-x)^{\omega}}{\omega}\right)+2\alpha\omega^{1/\omega}\Gamma\left(\dfrac{2}{\omega},\dfrac{(-x)^{\omega}}{\omega}\right)+2\Gamma\left(\dfrac{1}{\omega},\dfrac{(-x)^{\omega}}{\omega}\right)}{2\left(\alpha^2\omega^{2/\omega}\Gamma\left(\dfrac{3}{\omega}\right)+2\Gamma\left(\dfrac{1}{\omega}\right)\right)} & \text{if } x<0,\\[3ex] \dfrac{2\alpha^2\omega^{2/\omega}\Gamma\left(\dfrac{3}{\omega}\right)-\alpha^2\omega^{2/\omega}\Gamma\left(\dfrac{3}{\omega},\dfrac{x^{\omega}}{\omega}\right)+2\alpha\,\omega^{1/\omega}\Gamma\left(\dfrac{2}{\omega},\dfrac{x^{\omega}}{\omega}\right)-2\Gamma\left(\dfrac{1}{\omega},\dfrac{x^{\omega}}{\omega}\right)+4\Gamma\left(\dfrac{1}{\omega}\right)}{2\left(\alpha^2\omega^{2/\omega}\Gamma\left(\dfrac{3}{\omega}\right)+2\Gamma\left(\dfrac{1}{\omega}\right)\right)}, & \text{if } x\geq 0,\end{cases}$$

where the incomplete gamma is defined as

$$\Gamma(a,\ x)=\int_x^{\infty}t^{a-1}\exp(-t)dt\,.$$

The survival and hazard rate functions of random variable X from the ASGN distribution is as following, if $x<0$,

$$S(x)=\frac{2\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\dfrac{3}{\omega}\right)+\alpha\,\omega^{\frac{1}{\omega}+1}\,\Gamma\left(\dfrac{\omega+2}{\omega}\right)-2\alpha\,\omega^{1/\omega}\,\Gamma\left(\dfrac{2}{\omega}\right)-\alpha^2\omega^{2/\omega}\,\Gamma\left(\dfrac{3}{\omega},\dfrac{(-x)^{\omega}}{\omega}\right)}{2\left(\alpha^2\omega^{2/\omega}\,\Gamma\left(\dfrac{3}{\omega}\right)+2\Gamma\left(\dfrac{1}{\omega}\right)\right)}$$
$$-\frac{\alpha\,\omega^{1/\omega}\,\Gamma\left(\dfrac{2}{\omega},\dfrac{(-x)^{\omega}}{\omega}\right)+\Gamma\left(\dfrac{1}{\omega},\dfrac{(-x)^{\omega}}{\omega}\right)-\omega\Gamma\left(1+\dfrac{1}{\omega}\right)-\Gamma\left(\dfrac{1}{\omega}\right)}{\left(\alpha^2\omega^{2/\omega}\,\Gamma\left(\dfrac{3}{\omega}\right)+2\Gamma\left(\dfrac{1}{\omega}\right)\right)},$$

$$H(x)=\left(\frac{2\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\dfrac{3}{\omega}\right)+\alpha\,\omega^{\frac{1}{\omega}+1}\,\Gamma\left(\dfrac{\omega+2}{\omega}\right)-2\alpha\,\omega^{1/\omega}\,\Gamma\left(\dfrac{2}{\omega}\right)-\alpha^2\omega^{2/\omega}\,\Gamma\left(\dfrac{3}{\omega},\dfrac{(-x)^{\omega}}{\omega}\right)}{\alpha\,\omega^{1/\omega}\left(\alpha\,\omega^{1/\omega}\Gamma\left(\dfrac{3}{\omega},\dfrac{(-x)^{\omega}}{\omega}\right)+2\Gamma\left(\dfrac{2}{\omega},\dfrac{(-x)^{\omega}}{\omega}\right)\right)+2\Gamma\left(\dfrac{1}{\omega},\dfrac{(-x)^{\omega}}{\omega}\right)}\right.$$
$$\left.+\frac{-2\alpha\,\omega^{1/\omega}\,\Gamma\left(\dfrac{2}{\omega},\dfrac{(-x)^{\omega}}{\omega}\right)-2\Gamma\left(\dfrac{1}{\omega},\dfrac{(-x)^{\omega}}{\omega}\right)+2\omega\Gamma\left(1+\dfrac{1}{\omega}\right)+2\Gamma\left(\dfrac{1}{\omega}\right)}{\alpha\,\omega^{1/\omega}\left(\alpha\,\omega^{1/\omega}\,\Gamma\left(\dfrac{3}{\omega},\dfrac{(-x)^{\omega}}{\omega}\right)+2\Gamma\left(\dfrac{2}{\omega},\dfrac{(-x)^{\omega}}{\omega}\right)\right)+2\Gamma\left(\dfrac{1}{\omega},\dfrac{(-x)^{\omega}}{\omega}\right)}\right)^{-1},$$

and if $x>0$,

$$S(x)=\frac{\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega},\frac{x^\omega}{\omega}\right)-2\alpha\,\omega^{1/\omega}\,\Gamma\left(\frac{2}{\omega},\frac{x^\omega}{\omega}\right)+2\Gamma\left(\frac{1}{\omega},\frac{x^\omega}{\omega}\right)}{2\left(\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right)\right)},$$

$$H(x)=\frac{2\alpha^2\,\omega^{2/\omega}\Gamma\left(\frac{3}{\omega}\right)-\alpha^2\,\omega^{2/\omega}\Gamma\left(\frac{3}{\omega},\frac{x^\omega}{\omega}\right)+2\alpha\,\omega^{1/\omega}\Gamma\left(\frac{2}{\omega},\frac{x^\omega}{\omega}\right)-2\Gamma\left(\frac{1}{\omega},\frac{x^\omega}{\omega}\right)+4\Gamma\left(\frac{1}{\omega}\right)}{\alpha\,\omega^{1/\omega}\left(\alpha\,\omega^{1/\omega}\Gamma\left(\frac{3}{\omega},\frac{x^\omega}{\omega}\right)-2\Gamma\left(\frac{2}{\omega},\frac{x^\omega}{\omega}\right)\right)+2\Gamma\left(\frac{1}{\omega},\frac{x^\omega}{\omega}\right)}.$$

4. Central moments and moment generating function

If $X \sim ASGN(\alpha,\omega)$, then the r th central moments of X can be obtained as

$$E(X^r)=\frac{\omega^{r/\omega}\left(\alpha\,\omega^{1/\omega}\left(\alpha\left((-1)^r+1\right)\omega^{1/\omega}\,\Gamma\left(\frac{r+3}{\omega}\right)+2\left((-1)^r-1\right)\Gamma\left(\frac{r+2}{\omega}\right)\right)\right)}{2\,\alpha^2\,\omega^{2/\omega}\Gamma\left(\frac{3}{\omega}\right)+4\Gamma\left(\frac{1}{\omega}\right)}+\frac{2\omega^{r/\omega}\left((-1)^r+1\right)\Gamma\left(\frac{r+1}{\omega}\right)}{2\,\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+4\Gamma\left(\frac{1}{\omega}\right)},$$

and the moment generating function of X can be written as

$$\begin{aligned}M_X(t)=E(e^{tX})&=\sum_{n=0}^{\infty}\frac{t^nE(X^n)}{n!}\\&=\sum_{n=0}^{\infty}\frac{t^n}{n!}\left[\frac{\omega^{r/\omega}\left(\alpha\,\omega^{1/\omega}\left(\alpha\left((-1)^r+1\right)\omega^{1/\omega}\,\Gamma\left(\frac{r+3}{\omega}\right)+2\left((-1)^r-1\right)\Gamma\left(\frac{r+2}{\omega}\right)\right)\right)}{2\alpha^2\,\omega^{2/\omega}\Gamma\left(\frac{3}{\omega}\right)+4\Gamma\left(\frac{1}{\omega}\right)}\right.\\&\qquad\left.+\frac{2\alpha\,\omega^{\frac{r+1}{\omega}}\left(\left((-1)^r+1\right)\Gamma\left(\frac{r+1}{\omega}\right)\right)}{2\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+4\Gamma\left(\frac{1}{\omega}\right)}\right].\end{aligned}$$

By applying mathematical calculations and separating even and odd expressions, we get

$$M_X(t)=\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}\left(\frac{4\omega^{\frac{2n}{\omega}}\Gamma\left(\frac{2n+1}{\omega}\right)+2\alpha\,\omega^{\frac{2n+2}{\omega}}\Gamma\left(\frac{2n+3}{\omega}\right)\left(\alpha-\frac{2t}{2n+1}\right)}{2\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+4\Gamma\left(\frac{1}{\omega}\right)}\right).$$

5. The indices of skewness and kurtosis

Suppose $X\sim ASGN(\alpha,\omega)$. Using first to fourth central moments, γ_1 and γ_2 can be obtained which are the indices of skewness and kurtosis of ASGN distribution, respectively.

$$E(X)=\mu_1=-\frac{2\alpha\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)}{\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right)},$$

$$E(X^2)=\mu_2=\frac{\omega^{2/\omega}\left(\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{5}{\omega}\right)+2\Gamma\left(\frac{3}{\omega}\right)\right)}{\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right)},$$

$$E(X^3)=\mu_3=-\frac{2\alpha\,\omega^{4/\omega}\,\Gamma\left(\frac{5}{\omega}\right)}{\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right)},$$

$$E(X^4)=\mu_4=\frac{\omega^{4/\omega}\left(\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{7}{\omega}\right)+2\Gamma\left(\frac{5}{\omega}\right)\right)}{\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right)}.$$

Applying above equations, the indices of skewness and kurtosis are given respectively as

$$\gamma_1=\frac{2\mu_1^2-3\mu_2\,\mu_1+\mu_3}{\left(\mu_2-\mu_1^2\right)^{3/2}}$$

$$=\frac{4\alpha\,\omega^{4/\omega}\left(\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)\Gamma\left(\frac{5}{\omega}\right)+(2\alpha+3)\Gamma\left(\frac{3}{\omega}\right)^2-\Gamma\left(\frac{1}{\omega}\right)\Gamma\left(\frac{5}{\omega}\right)\right)}{\sqrt{\left(\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right)\right)^{-2}}}$$

$$\times \frac{1}{\left(\alpha^4 \omega^{6/\omega} \Gamma\left(\frac{3}{\omega}\right)\Gamma\left(\frac{5}{\omega}\right) - 2\alpha^2 \omega^{4/\omega}\left(\Gamma\left(\frac{3}{\omega}\right)^2 - \Gamma\left(\frac{1}{\omega}\right)\Gamma\left(\frac{5}{\omega}\right)\right) + 4\omega^{2/\omega}\Gamma\left(\frac{1}{\omega}\right)\Gamma\left(\frac{3}{\omega}\right)\right)^{\frac{3}{2}}},$$

$$\gamma_2 = \frac{-3\mu_1^4 + 6\mu_2\mu_1^2 - 4\mu_3\mu_1 + \mu_4}{\left(\mu_2 - \mu_1^2\right)^2}$$

$$= \frac{2\alpha^6 \omega^{6/\omega}\Gamma\left(\frac{3}{\omega}\right)^2\left(5\Gamma\left(\frac{3}{\omega}\right)\Gamma\left(\frac{5}{\omega}\right) + 3\Gamma\left(\frac{1}{\omega}\right)\Gamma\left(\frac{7}{\omega}\right)\right) + 16\Gamma\left(\frac{1}{\omega}\right)^3\Gamma\left(\frac{5}{\omega}\right)}{\left(\alpha^4 \omega^{4/\omega}\Gamma\left(\frac{3}{\omega}\right)\Gamma\left(\frac{5}{\omega}\right) - 2\alpha^2 \omega^{2/\omega}\left(\Gamma\left(\frac{3}{\omega}\right)^2 - \Gamma\left(\frac{1}{\omega}\right)\Gamma\left(\frac{5}{\omega}\right)\right) + 4\Gamma\left(\frac{1}{\omega}\right)\Gamma\left(\frac{3}{\omega}\right)\right)^2}$$

$$+ \frac{8\alpha^2 \omega^{2/\omega}\Gamma\left(\frac{1}{\omega}\right)\left(12\Gamma\left(\frac{3}{\omega}\right)^3 - 5\Gamma\left(\frac{1}{\omega}\right)\Gamma\left(\frac{5}{\omega}\right)\Gamma\left(\frac{3}{\omega}\right) + \Gamma\left(\frac{1}{\omega}\right)^2\Gamma\left(\frac{7}{\omega}\right)\right)}{\left(\alpha^4 \omega^{4/\omega}\Gamma\left(\frac{3}{\omega}\right)\Gamma\left(\frac{5}{\omega}\right) - 2\alpha^2 \omega^{2/\omega}\left(\Gamma\left(\frac{3}{\omega}\right)^2 - \Gamma\left(\frac{1}{\omega}\right)\Gamma\left(\frac{5}{\omega}\right)\right) + 4\Gamma\left(\frac{1}{\omega}\right)\Gamma\left(\frac{3}{\omega}\right)\right)^2}$$

$$+ \frac{\alpha^8 \omega^{8/\omega}\Gamma\left(\frac{3}{\omega}\right)^3\Gamma\left(\frac{7}{\omega}\right) + 4\alpha^4 \omega^{4/\omega}\Gamma\left(\frac{1}{\omega}\right)\Gamma\left(\frac{3}{\omega}\right)\left(3\Gamma\left(\frac{1}{\omega}\right)\Gamma\left(\frac{7}{\omega}\right) - \Gamma\left(\frac{3}{\omega}\right)\Gamma\left(\frac{5}{\omega}\right)\right)}{\left(\alpha^4 \omega^{4/\omega}\Gamma\left(\frac{3}{\omega}\right)\Gamma\left(\frac{5}{\omega}\right) - 2\alpha^2 \omega^{2/\omega}\left(\Gamma\left(\frac{3}{\omega}\right)^2 - \Gamma\left(\frac{1}{\omega}\right)\Gamma\left(\frac{5}{\omega}\right)\right) + 4\Gamma\left(\frac{1}{\omega}\right)\Gamma\left(\frac{3}{\omega}\right)\right)^2}.$$

6. Stochastic representation

In this section, some theorems are expressed to obtain a stochastic representation for the BGN and ASGN models.

Theorem 6.1.

Let T and V be independent random variables, where $T \sim Gamma(3/\omega, \omega)$ and $P[V=1] = P[V=-1] = \frac{1}{2}$. If $Y = T^{1/\omega}V$, then Y has the bimodal-generalized normal distribution.

Proof:

If $h(y)$ is the density function of $H = T^{1/\omega}$, then

$$h(y)=\frac{y^2\,e^{-\frac{y^{\omega}}{\omega}}}{\Gamma\left(\frac{3}{\omega}\right)\,\omega^{3/\omega-1}}, \qquad y \geqslant 0,$$

on the other hand, if $Y = HV$, it is easy to show that the density function of Y is

$$f_Y(y)=\frac{1}{2}[h(y)+h(-y)]=\begin{cases}\dfrac{y^2\,e^{-\frac{(-y)^{\omega}}{w}}}{2\Gamma\left(\frac{3}{\omega}\right)\,\omega^{3/\omega-1}} & \text{if } y<0,\\[2ex] \dfrac{y^2\,e^{-\frac{y^{\omega}}{\omega}}}{2\Gamma\left(\frac{3}{\omega}\right)\,\omega^{3/\omega-1}} & \text{if } y\geq 0,\end{cases} = \frac{y^2\,e^{-\frac{|y|^{\omega}}{\omega}}}{2\,\Gamma\left(\frac{3}{\omega}\right)\,\omega^{3/\omega-1}}.$$

Hence, the proof is completed.

Remark 6.1.

As shown following, the density function (1) of $ASGN(\alpha,\omega)$ model can be represented as sum of two functions:

$$\begin{aligned} f(x) &= \frac{\omega^{1-\frac{1}{\omega}}\left[(1-\alpha x)^2+1\right]\,e^{-\frac{|x|^{\omega}}{\omega}}}{2\left(\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right)\right)} \\ &= \frac{\omega^{1-\frac{1}{\omega}}\left(2+\alpha^2\,x^2\right)e^{-\frac{|x|^{\omega}}{\omega}}}{2\left(\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right)\right)}+\frac{\omega^{1-\frac{1}{\omega}}\left(-2\alpha\,x\right)e^{-\frac{|x|^{\omega}}{\omega}}}{2\left(\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right)\right)}. \end{aligned} \tag{3}$$

In equation (3), the right expression shows a symmetric density function which defined as following.

Definition 6.3.

If random variable S has density function

$$f_S(x)=\frac{\omega^{1-\frac{1}{\omega}}\left(2+\alpha^2\,x^2\right)e^{-\frac{|x|^{\omega}}{\omega}}}{2\left(\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right)\right)}, \qquad x\in ¡\ , \tag{4}$$

where $\alpha, \omega \in \mathbb{R}$, then random variable S is the symmetric component of $ASGN(\alpha,\omega)$ distribution and denoted as $S \sim SCASGN(\alpha,\omega)$.

If $S \sim SCASGN(\alpha,\omega)$, then the following properties are satisfied:

- If $\alpha = \omega = 0$, then $S \sim N(0,1)$.
- If $\alpha \to \pm\infty$, then $S \sim BGN$.

Remark 6.2.

Note that the density function (4) is a mixture of two generalized normal density and bimodal-generalized normal density, as following

$$f_S(x) = \frac{\omega^{1-\frac{1}{\omega}}\left(2+\alpha^2 x^2\right) e^{-\frac{|x|^\omega}{\omega}}}{2\left(\alpha^2 \omega^{\frac{2}{\omega}}\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right)\right)}$$

$$= \frac{2\Gamma\left(\frac{1}{\omega}\right)}{\alpha^2\omega^{\frac{2}{\omega}}\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right)} \times \frac{e^{-\frac{|x|^\omega}{\omega}}}{2\omega^{\frac{1}{\omega}-1}\Gamma\left(\frac{1}{\omega}\right)} + \frac{\alpha^2 \omega^{\frac{2}{\omega}}\Gamma\left(\frac{3}{\omega}\right)}{\alpha^2 \omega^{\frac{2}{\omega}}\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right)} \times \frac{x^2 e^{-\frac{|x|^\omega}{\omega}}}{2\omega^{\frac{3}{\omega}-1}\Gamma\left(\frac{3}{\omega}\right)}.$$

Lemma 6.1.

Let $f_X(x)$ be the density function of $X \sim ASGN(\alpha,\omega)$ and $f_S(x)$ be the density function of $S \sim SCASGN(\alpha,\omega)$, then we have

$$\sup_{x\in\mathbb{R}} \frac{f_X(x)}{f_S(x)} = \frac{2+\sqrt{2}}{2}.$$

6.1. Generate random sample from ASGN distribution

A random sample from the ASGN distribution can be generated by following steps.

1) Y is generated from bimodal-generalized normal distribution as below.
   Generate $V \sim DU\{-1,1\}$ and $T \sim Gamma\left(\frac{3}{\omega},\omega\right)$, put $Y = T^{\frac{1}{\omega}} V$.
2) Generate Z from generalized normal distribution, independent of Y.
3) Generate $S \sim SCASGN(\alpha,\omega)$ by using mixture distribution of generalized normal and bimodal-generalized normal, with probabilities

$$w_1 = \frac{2\Gamma\left(\frac{1}{\omega}\right)}{2\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+4\Gamma\left(\frac{1}{\omega}\right)}, \qquad w_2 = \frac{\alpha^2\,\omega^{2/\omega}\,\Gamma(\frac{3}{\omega})}{2\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+4\Gamma\left(\frac{1}{\omega}\right)}.$$

4) Generate $U \sim U(0,1)$, independent of S.
5) Using the acceptance-rejection algorithm in order to generate random sample X from $ASGN$ distribution as:

If

$$U < \frac{1}{M}\,\frac{f_X(S)}{f_S(S)} = \frac{2[(1-\alpha S)^2+1]}{(2+\sqrt{2})(2+\alpha^2 S^2)},$$

put $X = S$ and otherwise repeat previous steps, where

$$M = \sup_x \frac{f_X(x)}{f_S(x)} = \frac{2+\sqrt{2}}{2}.$$

7. Location-scale Family

The location-scale Family of $ASGN(\alpha,\omega)$ distribution is obtained by adding location and scale parameters. Suppose $X \sim ASGN(\alpha,\omega)$, then the density of $W=\mu+\sigma X$ for $\mu\in\mathbb{R}$ and $\sigma>0$ is given as

$$f(w;\boldsymbol{\theta}) = \frac{\omega^{1-\frac{1}{\omega}}\left(\left(1-\frac{\alpha(x-\mu)}{\sigma}\right)^2+1\right)e^{-\frac{\left|\frac{x-\mu}{\sigma}\right|^{\omega}}{\omega}}}{2\sigma\left(\alpha^2\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right)\right)}, \qquad x\in\mathbb{R},$$

where $\boldsymbol{\theta}=(\mu,\sigma,\alpha,\omega)$ and denoted by $W \sim ASGN(\boldsymbol{\theta})$.

8. Maximum likelihood estimation

In this section, the maximum likelihood estimators (MLEs) of four parameters ASGN distribution are discussed. Let $X_1,\ldots,X_n$ denote a random sample of size n from the $ASGN(\mu,\sigma,\alpha,\omega)$ distribution. So, the associated log-likelihood function is

$$\mathrm{l}(\mu,\sigma,\alpha,\omega)=\sum_{i=1}^{n}\log\left[\left(1-\alpha\left(\frac{x_i-\mu}{\sigma}\right)\right)^2+1\right]-\sum_{i=1}^{n}\frac{\left|\frac{(x_i-\mu)}{\sigma}\right|^{\omega}}{\omega}-n\log\sigma$$
$$-\,n\log\left[2\left(\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+2\,\Gamma\left(\frac{1}{\omega}\right)\right)\right]-n\left(1-\frac{1}{\omega}\right)\log\omega.$$

By differentiating the log-likelihood function with respect to μ, σ, α and ω, respectively, components of score vector $\left(\frac{\partial\,\mathrm{l}}{\partial\mu},\frac{\partial\,\mathrm{l}}{\partial\sigma},\frac{\partial\,\mathrm{l}}{\partial\alpha},\frac{\partial\,\mathrm{l}}{\partial\omega}\right)$ are derived as

$$\frac{\partial}{\partial\mu}\ell(\mu,\sigma,\alpha,\omega)=\frac{1}{\sigma}\left[\sum_{i=1}^{n}\frac{2\alpha\left(1-\alpha\,z_i\right)}{(1-\alpha\,z_i)^2+1}+\sum_{i=1}^{n}|\,z_i\,|^{\,\omega-2}\,z\right],$$
$$\frac{\partial}{\partial\sigma}\ell(\mu,\sigma,\alpha,\omega)=\frac{1}{\sigma}\left[\sum_{i=1}^{n}\frac{2\alpha\,z_i(1-\alpha\,z_i)}{(1-\alpha\,z_i)^2+1}-n+\sum_{i=1}^{n}|\,z_i\,|^{\omega}\right],$$

$$\frac{\partial}{\partial\alpha}\mathrm{l}(\mu,\sigma,\alpha,\omega)=\sum_{i=1}^{n}\frac{2\,z_i(1-\alpha\,z_i)}{(1-\alpha\,z_i)^2+1}+\frac{2\,n\,\alpha\,\Gamma\left(\frac{3}{\omega}\right)}{\alpha^2\,\Gamma\left(\frac{3}{\omega}\right)+2\,\omega^{-2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)},$$

$$\frac{\partial}{\partial\omega}\mathrm{l}(\mu,\sigma,\alpha,\omega)=\sum_{i=1}^{n}\frac{|\,z_i\,|^{\omega}-\omega\,|\,z_i\,|^{\omega}\,\log|\,z_i\,|}{\omega^2}-\frac{2\,n\,\Gamma\left(\frac{1}{\omega}\right)\left(\omega-1+\log\omega+\psi^0\left(\frac{1}{\omega}\right)\right)}{\omega^2\left(\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+2\,\Gamma\left(\frac{1}{\omega}\right)\right)}$$
$$-\frac{n\,\alpha^2\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)\left(3\log\omega+3\psi^0\left(\frac{3}{\omega}\right)-3+\omega\right)}{\omega^2\left(\alpha^2\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right)\right)},$$

where $\psi^0(x)=\dfrac{d\Gamma(x)}{dx}\Big/\Gamma(x)$ and $z_i=\dfrac{x_i-\mu}{\sigma}$. Generally, to obtain MLEs of μ, σ, α and ω, four above equations should be set equal to zero and solved simultaneously. Since there exists no closed form for these results, we used existing numerical methods in software.

8.1. Asymptotic distribution of the MLEs

The elements of the Fisher information matrix for ASGN distribution are as

$$I_{11} = -E\left[\frac{\partial^2 l}{\partial \mu^2}\right] = \frac{-2\alpha^2 A_0 + 4\alpha^2 E_0 + (\omega-1)E\left[|Z|^{\omega-2}\right]}{\sigma^2},$$

$$I_{12} = I_{21} = -E\left[\frac{\partial^2 l}{\partial \mu \partial \sigma}\right] = \frac{-2\alpha^2 A_1 + 4\alpha^2 E_1 + 2\alpha B_0 + E[Z|Z|^{\omega}] + (\omega-1)E[Z|Z|^{\omega-2}]}{\sigma^2},$$

$$I_{13} = I_{31} = -E\left[\frac{\partial^2 l}{\partial \mu \partial \alpha}\right] = \frac{2\alpha A_1 - 4\alpha E_1 - 2B_0}{\sigma},$$

$$I_{14} = I_{41} = -E\left[\frac{\partial^2 l}{\partial \mu \partial \omega}\right] = \frac{-E[Z|Z|^{\omega-2}\log(|Z|)]}{\sigma},$$

$$I_{22} = -E\left[\frac{\partial^2 l}{\partial \sigma^2}\right] = \frac{2\alpha^2 A_2 + 4\alpha^2 E_2 + 4\alpha B_1 - 1 + (\omega+1)E[|Z|^{\omega}]}{\sigma^2},$$

$$I_{23} = I_{32} = -E\left[\frac{\partial^2 l}{\partial \sigma \partial \alpha}\right] = \frac{2\alpha A_2 - 4\alpha E_2 - 2B_1}{\sigma},$$

$$I_{24} = I_{42} = -E\left[\frac{\partial^2 l}{\partial \sigma \partial \omega}\right] = \frac{-E[Z^2|Z|^{\omega-2}\log(|Z|)]}{\sigma},$$

$$I_{33} = -E\left[\frac{\partial^2 l}{\partial \alpha^2}\right] = -2A_2 + 4E_2 + \frac{2\Omega(\alpha,\omega)\,\omega^{2/\omega}\,\Gamma\left(\frac{3}{\omega}\right) - 4\alpha^2\omega^{4/\omega}\,\Gamma\left(\frac{3}{\omega}\right)^2}{\Omega^2(\alpha,\omega)},$$

$$I_{34} = I_{43} = -E\left[\frac{\partial^2 l}{\partial \alpha \partial \omega}\right] = \frac{4\alpha\,\omega^{\frac{2}{\omega}-2}\,\Gamma\left(\frac{1}{\omega}\right)\Gamma\left(\frac{3}{\omega}\right)\left(-2\log(\omega) + \psi^{(0)}\left(\frac{1}{\omega}\right) - 3\psi^{(0)}\left(\frac{3}{\omega}\right) + 2\right)}{\Omega^2(\alpha,\omega)},$$

$$I_{44} = -E\left[\frac{\partial^2 l}{\partial \omega^2}\right] = \frac{9\alpha^2\omega^{\frac{2}{\omega}-4}\,\Gamma\left(\frac{3}{\omega}\right)\psi^{(0)}\left(\frac{3}{\omega}\right)^2 + 9\alpha^2\omega^{\frac{2}{\omega}-4}\,\Gamma\left(\frac{3}{\omega}\right)\psi^{(1)}\left(\frac{3}{\omega}\right)}{\Omega(\alpha,\omega)}$$

$$+\frac{\alpha^2\omega^{2/\omega}\left(\frac{4\log(\omega)}{\omega^3} - \frac{6}{\omega^3}\right)\Gamma\left(\frac{3}{\omega}\right) + \alpha^2\omega^{2/\omega}\left(\frac{2}{\omega^2} - \frac{2\log(\omega)}{\omega^2}\right)^2\Gamma\left(\frac{3}{\omega}\right)}{\Omega(\alpha,\omega)}$$

$$+\frac{-3\alpha^2\omega^{\frac{2}{\omega}-2}\left(\frac{4(1-\log(\omega))}{\omega^2} - \frac{2}{\omega}\right)\Gamma\left(\frac{3}{\omega}\right)\psi^{(0)}\left(\frac{3}{\omega}\right)}{\Omega(\alpha,\omega)}$$

$$+\frac{2\Gamma\left(\frac{1}{\omega}\right)\psi^{(0)}\left(\frac{1}{\omega}\right)^2 + 2\Gamma\left(\frac{1}{\omega}\right)\psi^{(1)}\left(\frac{1}{\omega}\right) + 4\omega\Gamma\left(\frac{1}{\omega}\right)\psi^{(0)}\left(\frac{1}{\omega}\right)}{\omega^4\,\Omega(\alpha,\omega)}$$

$$-\frac{\left(-3\alpha^2\omega^{\frac{2}{\omega}-2}\Gamma\left(\frac{3}{\omega}\right)\psi^{(0)}\left(\frac{3}{\omega}\right)+\alpha^2\omega^{2/\omega}\left(\frac{2}{\omega^2}-\frac{2\log(\omega)}{\omega^2}\right)\Gamma\left(\frac{3}{\omega}\right)-\frac{2\Gamma\left(\frac{1}{\omega}\right)\psi^{(0)}\left(\frac{1}{\omega}\right)}{\omega^2}\right)^2}{\Omega^2(\alpha,\omega)}$$

$$+\frac{2|Z|^{\omega}-2\omega|Z|^{\omega}\log\left(|Z|\right)+\omega^2|Z|^{\omega}\log^2\left(|Z|\right)}{\omega^3}+\frac{\omega-3+2\log(\omega)}{\omega^3},$$

where

$$\Omega(\alpha,\omega)=\alpha^2\omega^{2/\omega}\Gamma\left(\frac{3}{\omega}\right)+2\Gamma\left(\frac{1}{\omega}\right),\qquad A_i=E\left[\frac{Z^i}{1+(1-\alpha Z)^2}\right],\qquad i=0,1,2,$$

$$B_i=E\left[\frac{Z^i(1-\alpha Z)}{1+(1-\alpha Z)^2}\right],\qquad i=0,1,\qquad E_i=E\left[\frac{Z^i(1-\alpha Z)^2}{(1+(1-\alpha Z)^2)^2}\right],\qquad i=0,1,2,$$

and $Z\sim N(0,1)$.

In $\boldsymbol{\theta}^*=(\mu^*,\sigma^*,0,2)$, the Fisher information matrix is

$$I_0=\begin{pmatrix}\frac{1}{\sigma^2} & 0 & \frac{-1}{\sigma} & 0\\ 0 & \frac{2}{\sigma^2} & 0 & -0.36\\ \frac{-1}{\sigma} & 0 & 1 & 0\\ 0 & -0.36 & 0 & 0.62\end{pmatrix}$$

Note from this matrix that the first and third column corresponding to the parameters μ and α are linearly dependent, implying that it is a singular matrix. As observed, first row of matrix I is equal to third row multiplying by $-1/\sigma$ and implying that for $\alpha=0$ and $\omega=2$, $ASGN(\alpha,\omega)$ distribution does not satisfies the regularity conditions, in other words, the information matrix I is a singular matrix that means it has not inverse. Thus, in this case the MLEs of μ, σ, α and ω distribution cannot be reached based on usual procedure. Considering singularity problem for Fisher information matrix, the general result in Rotnitzky et al. (2000) can be applied in order to establish asymptotic distribution of MLEs as follow.

At $\boldsymbol{\theta}=\boldsymbol{\theta}^*$, we consider first order partial derivatives.

$$S_\mu = \frac{Z}{\sigma},$$
$$S_\sigma = \frac{Z^2 - 1}{\sigma},$$
$$S_\alpha = -Z,$$
$$S_\omega = -\frac{Z^2}{4}\left(2\log|Z| - 1\right) - 0.067,$$

where $S_\alpha = -\sigma \times S_\mu$, then we use reparametrization.

$$\left.\frac{\partial\, l\,(Y;\boldsymbol{\theta})}{\partial\alpha}\right|_{\boldsymbol{\theta}^*} = Z = \mathbf{K}_1 \begin{pmatrix} \frac{Z}{\sigma} \\ \frac{Z^2-1}{\sigma} \\ -\frac{Z^2}{4}\left(2\log|Z|-1\right)-0.067 \end{pmatrix},$$

So,

$$\mathbf{K}_1 = (-\sigma, 0, 0),$$

and

$$\mathbf{A}_1 = (-\sigma, 0, 0, 0)^T \alpha.$$

By applying following reparametrization and substituting in condition $C1$ in Rotnitzky et al. (2000), we have

$$\tilde{\boldsymbol{\theta}} = \begin{pmatrix} \mu \\ \sigma \\ \alpha \\ \omega \end{pmatrix} - \begin{pmatrix} -\sigma\alpha \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\tilde{\mu} = \mu + \sigma\alpha, \qquad (\tilde{\sigma}, \tilde{\alpha}, \tilde{\omega}) = (\sigma, \alpha, \omega),$$

$$\S_\alpha^{(1)} = 0, \qquad \S_\alpha^{(2)} = 0, \qquad \S_\alpha^{(3)} = Z^3,$$

where $\S_\alpha^{(3)}$ is neither zero nor a linear combination of $\S_\mu, \S_\sigma, \S_\omega$ and at $s = 3$, we used Theorem 5 in Rotnitzky et al. (2000). Using Theorem 5, the following results are gotten for $\boldsymbol{\theta} = (\mu, \sigma, \alpha, \omega)$.

When $\boldsymbol{\theta} = \boldsymbol{\theta}^*$, MLE of $\hat{\boldsymbol{\theta}}$ exists and is unique with probability tending to 1 and is consistent for $\boldsymbol{\theta}$;

$$\begin{bmatrix} n^{1/2}(\hat{\mu}-\mu^*-\sigma^*\hat{\alpha}) \\ n^{1/2}(\hat{\sigma}-\sigma^*) \\ n^{1/6}\hat{\alpha} \\ n^{1/2}(\hat{\omega}-\omega^*) \end{bmatrix} \xrightarrow{d} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3^{\frac{1}{3}} \\ Z_4 \end{bmatrix},$$

where (Z_1, Z_2, Z_3, Z_4) is a random vector of normal with mean zero and covariance matrix equal to inverse of vector covariance matrix

$$\left[\frac{Z}{\sigma^*}, \frac{Z^2-1}{\sigma^*}, \frac{Z^3}{6}, \frac{Z^2}{4}(1-2\log|Z|)-0.067\right],$$

where obtained as

$$\begin{pmatrix} \frac{1}{\sigma^2} & 0 & \frac{1}{2\sigma} & 0 \\ 0 & \frac{2}{\sigma^2} & 0 & \frac{-0.36}{\sigma} \\ \frac{1}{2\sigma} & 0 & \frac{5}{12} & 0 \\ 0 & \frac{-0.36}{\sigma} & 0 & 0.11 \end{pmatrix}^{-1},$$

and

$$2\{L_n(\hat{\boldsymbol{\theta}}) - L_n(\boldsymbol{\theta}^*)\} \xrightarrow{d} \chi_4^2 .$$

9. Simulation

In this section, the aim is to evaluate the maximum likelihood estimation of μ, σ, α and ω by using numerical methods and to obtain some quantities such as bias and the mean square error (MSE) to verify usefulness of these estimators. The acceptance-rejection algorithm is applied in simulation. The simulation results are presented to verify the consistency properties of the MLEs. All computations were implemented by using the Mathematica and R software. The simulation results of MLEs, bias and MSE of the ASGN distribution parameters are reported for $\mu = 0$, $\sigma = 1$ and sample sizes $n = 100$, 300, 500 with 10000 iterations in Table 1. As the sample size increases the estimated values of parameters tend to their true values that means the bias and the mean squared errors decrease.

10. Data Analysis

In this section, to illustrate the applicability of normal, alpha skew normal and alpha skew generalized normal models, a real data sets are analyzed. The data are the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England. Unfortunately, the units of measurement are not given in the paper. To study previous works on this data, one can see Barreto-Souza et al. (2010) and Smith and Naylor (1987).

The models $N(\mu,\sigma)$ and $ASN(\mu,\sigma,\alpha)$ are fitted to the data set using the maximum likelihood approach. Results are reported in Table 2. The standard errors of estimators being estimated by using the observed information matrix. Table 2 reports the MLEs of the parameters, AIC, AICc and BIC for the N, the ASN and the ASGN distributions. Also, the corresponding Anderson-Darling test statistic (AD), the Cramér-von Mises test statistic (CM), the Kolmogorov-Smirnov test statistic (K-S) and P-Value are provided in this table.

Table 1. MLEs of the parameters, AIC, AICc, BIC, AD, CM, K-S and \$ P \$-Value for the N, the ASN and the ASGN distributions

	N	ASN	$ASGN$
$\hat{\mu}$	1.5068	1.3200	1.4900
$\hat{\sigma}$	0.3215	0.2560	0.0393
$\hat{\alpha}$		−1.5775	–1.2276
$\hat{\omega}$			0.57931
$-\log L$	17.9118	10.5191	8.7047
AIC	39.8236	27.0383	25.4095
$AICc$	40.0236	27.4451	26.0992
BIC	40.4341	27.9540	26.6305
AD	1.8991	0.4349	0.5304
CM	0.4274	0.1644	0.1396
$K-S$	0.1810	0.1039	0.0721
$P-Value$	0.0321	0.5061	0.8984

Table 2. Bias and MSE of the MLE of μ, σ, α and ω for the ASGN distribution.

			Estimates of μ		Estimates of σ		Estimates of α		Estimates of ω	
α	ω	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
–1	0.5	100	0.0480	0.0421	0.4863	0.4497	0.6370	0.6365	−0.2498	0.0626
		300	0.0394	0.0243	0.4731	0.3657	0.7646	0.6174	−0.2506	0.0629
		500	0.0366	0.0207	0.4851	0.3828	0.7755	0.6338	−0.2501	0.0627
	1	100	−0.0118	0.0628	0.0133	0.1102	−0.0139	0.0578	0.0252	0.0680
		300	−0.0031	0.0135	0.0054	0.0240	−0.0017	0.0146	0.0047	0.0116
		500	−0.0026	0.0066	0.0029	0.0134	−0.0016	0.0083	0.0031	0.0063

	1.5	100	0.1112	0.0788	−0.1484	0.0711	0.0976	0.1898	0.4209	0.5769
		300	0.0431	0.0236	−0.1275	0.0364	0.0597	0.0632	0.3827	0.2449
		500	0.0300	0.0084	−0.1147	0.0236	0.0675	0.0288	0.3899	0.2000
	2	100	0.0958	0.0629	−0.2415	0.0970	0.1801	0.2574	0.1598	0.5235
		300	0.0361	0.0150	−0.2282	0.0711	0.1323	0.0905	0.1323	0.1402
		500	0.0194	0.0051	−0.2130	0.0556	0.1407	0.0527	0.1123	0.0842
1	0.5	100	0.0015	0.0047	−0.0007	0.0153	−0.0015	0.0026	0.0235	0.0076
		300	−0.0009	0.0015	−0.0037	0.0046	−0.0023	0.0005	0.0075	0.0019
		500	−0.0006	0.0005	−0.0028	0.0027	−0.0004	0.0002	0.0040	0.0010
	1	100	−0.0083	0.0400	−0.0230	0.0176	−0.0074	0.0267	0.0382	0.0505
		300	−0.0001	0.0104	−0.0074	0.0055	−0.0001	0.0058	0.0110	0.0126
		500	−0.0001	0.0050	−0.0029	0.0033	0.0003	0.0025	0.0079	0.0073
	1.5	100	−0.0116	0.1066	−0.0241	0.0193	−0.0171	0.0948	0.0879	0.2647
		300	0.0033	0.0609	−0.0164	0.0052	0.0030	0.0545	0.0094	0.0398
		500	0.0055	0.0461	−0.0110	0.0028	0.0061	0.0393	0.0091	0.2090
	2	100	0.0147	0.1103	0.0027	0.0213	0.0176	0.1433	0.2279	0.6768
		300	−0.0139	0.0819	0.0008	0.0059	−0.0128	0.0981	0.0731	0.1243
		500	0.0033	0.0777	0.0022	0.0034	0.0019	0.0907	0.0462	0.0662
0	0.5	100	−0.0470	0.0483	0.4997	0.5202	−0.7693	0.6497	0.2499	0.0625
		300	−0.0500	0.0262	0.4774	0.3632	−0.7851	0.6536	−0.2506	0.0629
		500	−0.0387	0.0192	0.4425	0.2827	−0.7731	0.6298	−0.2502	0.0627
	1	100	0.0006	0.0464	0.0209	0.0865	−0.0062	0.0487	0.0236	0.0490
		300	−0.0001	0.0131	0.0031	0.0240	0.0028	0.0144	0.0055	0.0123
		500	0.0065	0.0071	0.0056	0.0124	0.0058	0.0077	0.0050	0.0062
	1.5	100	−0.1057	0.0723	−0.1446	0.0698	−0.0826	0.1764	0.4317	0.5506
		300	−0.0423	0.0169	−0.1165	0.0308	−0.0708	0.0480	0.4091	0.2582
		500	−0.0284	0.0067	−0.1149	0.0220	−0.0714	0.0241	0.3898	0.1969
	2	100	−0.0938	0.0624	−0.2503	0.0996	−0.1640	0.2532	0.1220	0.4060
		300	−0.0381	0.0168	−0.2204	0.0658	−0.1495	0.0944	0.1102	0.1450
		500	−0.0199	0.0056	−0.2143	0.0566	−0.1401	0.0549	0.1094	0.0860

From the values of AIC, AICc, BIC, AD, CM and K-S statistics, the ASGN distribution displays a better fit to this data than the N and the ASN distributions. P-Value corresponding to K-S statistic implies that both the ASGN and the ASN distributions are the suitable the fit to the data, but the N distribution is not a suitable the fit to the data with 95% confidence.

In the following to assure the results, the likelihood radio test (LRT) is performed for the hypothesis

$$\begin{cases} H_0 : X \sim N \\ H_1 : X \sim ASGN \end{cases} \equiv \begin{cases} H_0 : \alpha = 0, \omega = 2 \\ H_1 : \alpha \neq 0, \omega \neq 2 \end{cases} \quad (5)$$

To verify test (5), it is sufficient to compare the likelihood radio test statistic with the 95% critical value, $\chi^2(4)$. Since the value of statistics are 18.42 and 9.41, respectively, so the null hypothesis is rejected, concluding that ASGN gives a better fit to the data than the normal model.

11. Conclusion

In recent years, many new distributions are introduced by statistical researchers to fit the real data sets in around us. The main object of this paper is to introduce a symmetric new distribution which is quite flexible to fit the both unimodal and bimodal shapes. we introduce the alpha-skew generalized normal distribution that skews the symmetric distributions, especially generalized normal distribution. Some properties of the new distribution including cumulative distribution function, survival function, hazard rate function and moments are derived. To estimate the model parameters, the maximum likelihood estimators and the asymptotic distribution of the estimators are discussed. The observed information matrix is derived which is singular. Finally, the flexibility of the new distribution, as well as its effectiveness in comparison with other distributions, are shown via an application.

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