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Numerical Solution of the Lane-Emden Equations with Moving Least Squares Method

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Abstract

In the present paper, we have used moving least squares (MLS) method to solve the integral form of the Lane-Emden equations with initial conditions. The Volterra integral form of the Lane-Emden equations overcomes their singular behavior at $x = 0$, and the MLS method leads to a satisfactory solution for the equation. The convergence of the method is investigated and finally its applicability is displayed through numerical examples.

Keywords: Lane-Emden equations; Volterra integral equation; Moving least squares method

MSC 2010 No.: 34K28, 65L05

1. Introduction

The Lane-Emden equations were first developed by the two astrophysicists Jonathan Homer Lane and Robert Emden while they were engaged in the analysis of the behavior of a spherical cloud of gas acting under the mutual attraction of its molecules based on thermodynamic rules (Richardson (1921)). These equations are generally defined in the following form:

$$u''(x) + \frac{k}{x}u'(x) + f(u(x)) = 0, \quad x \neq 0, \tag{1}$$

where $f(u)$ is some given function of u. Equation (1) constantly has an analytic solution in the neighborhood of the singular point $x = 0$ with the following initial conditions (Davis (1962)):

$$u(0) = \alpha, \;\; u^{'}(0) = 0. \tag{2}$$

Equation (1) with various $f(u)$s can be observed in different phenomena within mathematical physics including the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres and the theory of thermionic currents (Wazwaz (2011)). The most known form of $f(u)$ is introduced as the following:

$$f(u) = u^{\xi}, \tag{3}$$

where ξ is a constant parameter. Beside this, other forms of $f(u)$ such as exponential functions are as the following:

$$f(u) = e^{u(x)}, \qquad f(u) = e^{-u(x)}.$$

A complete explanation of the formation of these models and the physical structure of their solutions can be found in the references (Wazwaz (2002); Wazwaz (2005); Yildirim et al. (2009)). Also, other different ways of solving these equations can be seen in the literature (Adomian et al. (1995); Belyschko et al. (1994); Canuto et al. (2006); Davis (1962); Levin (1998); Ravi et al. (2010); Shang (2009); Yildirim et al. (2009)), most of which use algorithms based on series solutions or perturbation techniques. The purpose of the present paper is to use the MLS method to solve the Lane-Emden equation. In so doing, we will first change the Lane-Emden equation into its Volterra integral form to overcome its singular point in the neighborhood of $x = 0$, and then solve it with MLS method.

2. The Volterra integral form of Lane-Emden equation

In this section, in order to overcome the singularity of the equation its Volterra integral form will be presented. This form has been offered by Wazwaz et al. to solve the Lane-Emden equation in their article (Wazwaz et al. (2013)).

According to Equation (1), if $k \neq 1$, in order to transform it to integral form we set:

$$u = \alpha - \frac{1}{k-1} \int_0^x t \left(1 - \frac{t^{k-1}}{x^{k-1}}\right) f(u(t))dt. \tag{4}$$

By two times derivating Equation (4) and applying the Leibniz's rule, we have:

$$u^{'}(x) = -\int_0^x \left(\frac{t^k}{x^k}\right) f(u(t))dt,$$

$$u^{''}(x) = -f(u(x)) + \int_0^x k \left(\frac{t^k}{x^{k+1}}\right) f(u(t))dt.$$

Multiplying $u^{'}(x)$ by $\frac{k}{x}$ and adding the result to $u^{''}(x)$ gives the generalized Lane-Emden Equation (1). This shows that the Volterra integral form (4) is the equivalent integral form for the generalized Lane-Emden equation. If $k = 1$, the integral form will be:

$$u = \alpha + \int_0^x t \ln \left(\frac{t}{x}\right) f(u(t))dt, \tag{5}$$

which can be obtained in the limit as $k \to 1$ in Equation (4) by L'Hopital's rule. Therefore, the integral form of Lane-Emden equation is as the following:

$$u(x) = \begin{cases} \alpha + \int_0^x t \ln(\frac{t}{x}) f(u(t))dt, & k = 1, \\ \\ \alpha - \frac{1}{k-1} \int_0^x t(1 - \frac{t^{k-1}}{x^{k-1}}) f(u(t))dt, & k > 0, \quad k \neq 1. \end{cases} \tag{6}$$

3. A summary of MLS method

The MLS, as an approximation method, has been introduced by Shepard (Shepard (1968)) in the lowest order case and have been generalized to a higher degree by Lancaster and Salkauskas (Lancaster et al. (1986)). The use of MLS in solving PDEs was pioneered by the works of B. Nayroles, T. Belytschko and others (Belyschko et al. (1994); Nayroles et al. (1991); Nayroles et al. (1992)).

Suppose that discrete values of a function u are given at certain data sites $X = \{x_1, x_2, \cdots, x_N\} \subseteq \Omega \subseteq \mathbb{R}$. In MLS method, approximating function in each certain data site $x \in \Omega$ is written according to the value of functions which are local data sites, and in order to determine the influence of each data point, a weight function $\omega : \Omega \times \Omega \to \mathbb{R}$ is used which the further it goes from data site x, the more the value tends towards zero and for data sites $x, y \in \Omega$ which $|x - y|$ is greater than a certain threshold, it is zero. Let $\mathcal{P}_q$ be the space of polynomials of maximum degree q, $q \ll N$ and $q \leq s$, and suppose $\{p_0, p_1, \cdots, p_m\}$ are basis for $\mathcal{P}_q$, where $m = q$ (Wendland (2005)). The MLS approximation $\hat{u}(x)$ of $u(x)$, $\forall x \in \overline{\Omega}$, can be defined as

$$\hat{u}(x) = \mathbf{p}^T(x)\mathbf{a}(x), \quad \forall x \in \overline{\Omega}, \tag{7}$$

where $\mathbf{p}^T(x) = [p_0(x), p_1(x), \cdots, p_m(x)]$ and $\mathbf{a}(x)$ is a vector with components $a_i(x), i = 0, 1, \cdots, m$. We can use different basis such as monomials, Chebyshev and Legendre polynomials for this method, but the MLS approximation can be implemented in a more stable fashion, if a shifted and scaled polynomial basis function is used as a basis for $\mathcal{P}_q$. In this paper, we use the basis

$$p_\eta(x) = \left\{ \frac{(x_i - x)^\eta}{h^\eta_{X,\Omega}} \right\}, \quad \eta = 0, 1, \ldots, m,$$

where the fill distance is

$$h_{X,\Omega} = \sup_{x \in \Omega} \min_{1 \leq i \leq n} |x - x_i|.$$

Components of vector $\mathbf{a}(x)$ are achieved by functional minimizing as follows:

$$\begin{aligned} \mathcal{J}(x) &= \sum_{i=1}^{n} \omega_{h_i}(x - x_i)(\mathbf{p}^T(x_i)\mathbf{a}(x) - u_i)^2 \\ &= [\mathbf{p}.\mathbf{a}(x) - \mathbf{u}]^T.\mathcal{W}.[\mathbf{p}.\mathbf{a}(x) - \mathbf{u}], \end{aligned} \tag{8}$$

where $\omega_{h_i}(x - x_i)$ is the weight function associated with the node i, n is the number of nodes in $\overline{\Omega}$ for which the weight function $\omega_{h_i}(x - x_i) > 0$ and u_i are the fictitious nodal values, but not the

nodal values of the unknown trial function $\hat{u}(x)$ i.e. $\hat{u}(x_i) \neq u_i$. In this study, we use the Gaussian weight functions:

$$\omega_{h_i}(x-x_i) = \begin{cases} \frac{exp(-(\frac{d_i}{\beta})^2)-exp(-(\frac{h_i}{\beta})^2)}{1-exp(-(\frac{h_i}{\beta})^2)}, & 0 \leq d_i \leq h_i, \\ \\ 0, & d_i > h_i, \end{cases}$$

where $d_i = |x - x_i|$, β is a constant controlling the shape of the weight function $\omega_{h_i}(x - x_i)$ and h_i is the size of the support domain. The matrices P and $\mathcal{W}$ are defined:

$$P = \begin{bmatrix} \mathbf{p}^T(x_1) \\ \mathbf{p}^T(x_2) \\ \vdots \\ \mathbf{p}^T(x_n) \end{bmatrix}, \qquad \mathcal{W} = diag(\omega_{h_1}(x-x_1), \omega_{h_2}(x-x_2), \ldots, \omega_{h_n}(x-x_n)).$$

From the functional solution of $\mathcal{J}$ with regard to $\mathbf{a}(x)$, the linear relation between $\mathbf{a}(x)$ and $\mathbf{u}$ is achieved as follows:

$$A(x)\mathbf{a}(x) = B(x)\mathbf{u}, \tag{9}$$

where matrices $A(x)$ and $B(x)$ are defined as follows :

$$A(x) = P^T\mathcal{W}P = B(x)P = \sum_{i=1}^{n} \omega_{h_i}(x-x_i)\mathbf{p}(x_i)\mathbf{p}^T(x_i),$$

$$B(x) = P^T\mathcal{W} = [\omega_{h_1}(x-x_1)\mathbf{p}(x_1), \omega_{h_2}(x-x_2)\mathbf{p}(x_2), \ldots, \omega_{h_n}(x-x_n)\mathbf{p}(x_n)].$$

The matrix A is often called the moment matrix and its size equals $(m+1) \times (m+1)$. If we select the nodal points such that $A(x)$ is non-singular, then Equation (9) has the unique solution

$$\mathbf{a}(x) = A^{-1}(x)B(x)\mathbf{u}, \tag{10}$$

by putting $\mathbf{a}(x)$ in Equation (7) we have

$$\hat{u}(x) = \mathbf{p}^T(x)A^{-1}(x)B(x)\mathbf{u} = \sum_{i=1}^{n} \varphi_i(x)u_i, \quad x \in \overline{\Omega}, \tag{11}$$

where

$$\varphi_i(x) := \sum_{j=0}^{m} p_j(x)[A^{-1}(x)B(x)]_{ji},$$

are called the shape functions of the MLS approximation, corresponding to nodal point x_i. If $\omega_{h_i}(x - x_i) \in C^r(\Omega)$ and $p_j(x) \in C^s(\Omega)$, $i = 1, 2, \ldots, n$, $j = 0, 1, \ldots, m$, then $\varphi_i(x) \in C^{min(r,s)}(\Omega)$.

4. The details of the suggested method

As already introduced in Section 1, and with regard to the information in section 3, the Volterra integral form of Equation (1) for $k > 0,\ k \neq 0$ is written as follows:

$$u(x) = \alpha - \frac{1}{k-1}\int_0^x t\left(1 - \frac{t^{k-1}}{x^{k-1}}\right) f(u(t))dt, \quad x \in [0,1].$$

Now, by changing the variable $t \longrightarrow \mu x$ the above Volterra integral form is transformed to Fredholm integral form:

$$u(x) = \alpha - \frac{x^2}{k-1} \int_0^1 \mu(1-\mu^{k-1}) f(u(\mu x)) d\mu, \quad x \in [0,1]. \tag{12}$$

To apply the MLS method, at first N evaluation points $\{x_i\}$ are selected on the interval $[0,1]$ where $0 \leq x_1 < x_2 < \ldots < x_N \leq 1$. The distribution of nodes could be selected regularly or randomly. Then we can replace u with $\hat{u} = \sum_{i=1}^{N} \varphi_i(x) u_i$. So Equation (12) becomes:

$$\sum_{i=1}^{N} \varphi_i(x) u_i = \alpha - \frac{x^2}{k-1} \int_0^1 \mu(1-\mu^{k-1}) f\left(\sum_{i=1}^{N} \varphi_i(\mu x) u_i\right) d\mu, \quad x \in [0,1]. \tag{13}$$

Since for $N-n$ nodes $\varphi_i(x) = 0$, n is replaced by N in Equation (13). Now, by replacing x with the evaluation points of $x_1, x_2, \ldots, x_N$ in Equation (13), the following system of equations will be achieved:

$$\sum_{i=1}^{n} \varphi_i(x_j) u_i + \frac{x_j^2}{k-1} \int_0^1 \mu(1-\mu^{k-1}) f\left(\sum_{i=1}^{n} \varphi_i(\mu x_j) u_i\right) g(\mu x_j) d\mu = \alpha, \tag{14}$$

using a n_1-point quadrature formula with the coefficients $\{\tau_\ell\}$ and weights $\{\omega_\ell\}$ in the interval $[0,1]$ for numerically solving the integration in Equation (14) yields

$$\sum_{i=1}^{n} \varphi_i(x_j) \hat{u}_i + \frac{x_j^2}{k-1} \sum_{\ell=1}^{n_1} \tau_\ell (1-{\tau_\ell}^{k-1}) f\left(\sum_{i=1}^{n} \varphi_i(\tau_\ell x_j) \hat{u}_i\right) g(\tau_\ell x_j) \omega_\ell = \alpha.$$

Here, $\hat{u}_i$s are an estimate for the u_is and by using the Levenberg-Marquardt algorithm, $\hat{u}$ is finally achieved. Then the values of $u(x)$ at any point of $x \in [0,1]$ can be approximated by Equation (11) as

$$u(x) \approx u_N(x) = \sum_{i=1}^{N} \varphi_i(x) \hat{u}_i, \quad x \in [0,1]. \tag{15}$$

5. Convergence Analysis

This section covers the convergence analysis of the proposed method. At first, the error estimate of MLS approximation is presented in terms of the parameter R which plays the role of the mesh-size. In Levin (1998), Levin analyzed the MLS method for a particular weight function obtaining error estimate in the uniform norm for the approximation of a regular function in N dimensions. In (Armentano et al. (2001)) Armentano and Duran proved error estimates in L^∞ for the function and its derivatives in the one-dimensional case. In Armentano (2001), Armentano obtained the error estimates in L^∞ and L^2 norms for one and higher dimensions. In Zuppa (2003), Zuppa proved error estimates for approximation of the function and the first and second order derivatives in L^∞ norm. The error estimate of the method, proposed in this work is based on those obtained in Mirzaei et al. (2010) and Zuppa (2003) for the one-dimensional cases.

Let Ω be an open bounded domain in $\mathbb{R}$ and $\mathcal{Q}_N$ denotes an arbitrarily chosen set of N points $x_i \in \overline{\Omega}$ referred to as nodes:

$$\mathcal{Q}_N = \{x_1, x_2, \ldots, x_N\}, \quad x_i \in \overline{\Omega}.$$

Let $\mathcal{I}_N := \{\Upsilon_i\}_{i=1}^N$ denotes a finite open covering of $\overline{\Omega}$ consisting of N clouds Υ_i such that $x_i \in \Upsilon_i$ and Υ_i is centered around x_i in some way, and

$$\overline{\Omega} \subset \bigcup_{i=1}^{N} \Upsilon_i.$$

Define the radius h_i of Υ_i as $max_{x \in \partial\Upsilon_i}\{|x_i - x|\}$.

A function u is said of class $C^{q,1}$ in $\overline{\Omega}$ if and only if u is of class C^q in $\overline{\Omega}$ and the partial derivatives $D^s u$ of u of order $q(|s| = q)$ are Lipschitz continuous in $\overline{\Omega}$. The semi-norm $|\ .\ |_{q,1}$ is defined as (Zuppa (2003)):

$$|u|_{q,1} = sup\left\{ \frac{|D^s u(x) - D^s u(y)|}{|x - y|} \ :\ x, y \in \overline{\Omega},\ x \neq y, |s| = q \right\}.$$

In order to have the MLS approximation well defined we need that the minimization problem has a unique solution at every point $x \in \overline{\Omega}$ and this is equivalent to the non-singularity of matrix $A(x)$. In (Zuppa (2003)) the error estimate was obtained with the following assumption about the system of nodes and weight functions$\{\mathcal{Q}_N, \mathcal{S}_N = \{\omega_{h_i}\}_{i=1}^N\}$:

Proposition 5.1. (Zuppa (2003))

For any $x \in \overline{\Omega}$, the matrix $A(x)$ defined in (9) is non-singular.

Definition 5.2.

Given $x \in \overline{\Omega}$, the set $\mathcal{ST}(x) = \{i : \omega_{h_i}(x_i - x) \neq 0\}$ will be called the star of x.

Theorem 5.3. (Zuppa (2003))

A necessary condition for the Property (5.1) to be satisfied is that for any $x \in \overline{\Omega}$,

$$n = card(\mathcal{ST}(x)) \geq card(\mathcal{P}_q) = m + 1.$$

For a sample point $c \in \overline{\Omega}$, if $\mathcal{ST}(c) = \{i_1, \ldots, i_s\}$, the mesh-size of the star $\mathcal{ST}(c)$ is defined by the number $h(\mathcal{ST}(c)) := max\{h_{i_1}, \ldots, h_{i_s}\}$. Consider the following global assumptions about parameters. There exist

(1) An upper bound of the overlap of clouds:

$$E = \sup_{c \in \overline{\Omega}} \Big\{ card\Big(\mathcal{ST}(c)\Big) \Big\}.$$

(2) Upper bounds of the condition number:

$$CB_q = \sup_{c \in \overline{\Omega}} \Big\{ CN_q\Big(\mathcal{ST}(c)\Big) \Big\}, \quad q = 1, 2,$$

where the numbers $CN_q(\mathcal{ST}(c))$ are computable measures of the quality of the star $\mathcal{ST}(c)$ which defined in Theorem 7 of Zuppa (2003).

(3) An upper bound of the mesh-size of stars:

$$R = \sup_{c\in\overline{\Omega}} \Big\{ h\Big(\mathcal{ST}(c)\Big)\Big\}.$$

(4) A uniform bound of the derivatives of $\{\omega_{h_i}\}$, that is the constant $G_q > 0,\ q = 1, 2$, such that

$$\|D^s\omega_{h_i}\|_{L^\infty} \le \frac{G_q}{R^{|s|}}, \quad 1 \le |s| \le q.$$

(5) There exist the number $\gamma \ge 0$ such that any two points $x, y \in \overline{\Omega}$ can be jointed by a rectifiable curve Γ in $\overline{\Omega}$ with length $|\Gamma| \le \gamma|x - y|$.

Assuming all these conditions, Zuppa (Zuppa (2003)) proved:

Theorem 5.4.

There exist constants $C_q,\ q = 1$ or 2,

$$C_1 = C_1(\gamma, d, E, G_1, CB_1), \qquad C_2 = C_2(\gamma, d, E, G_2, CB_1, CB_2),$$

such that for each $u \in C^{q,1}(\overline{\Omega})$.

$$\|D^s u - D^s\hat{u}\|_{L^\infty(\Omega)} \le C_q R^{q+1-|s|}|u|_{q,1}, \quad 0 \le |s| \le q. \tag{16}$$

As highlighted in Zuppa (2003), the number $CN_2(\mathcal{ST}(c))$ can be very high near the boundary points. This drawback can degrade appreciatively the global error estimate when $q = 2$.

Now we write Equation (12) in abstract form as

$$(\lambda - \mathcal{F})u = g, \tag{17}$$

where

$$\mathcal{F}u = \int_\Omega k(x, \mu) f(u(\mu x))d\mu.$$

Similarly, Equation (13) can be written as

$$(\lambda - \mathcal{F})\hat{u} = g. \tag{18}$$

Assume that $\mathcal{F}$ is a compact operator (for more details about the compact integral operators see Chapter 1 of Atkinson (1997)).

Lemma 5.5.

If (17) is uniquely solvable and $\|u - \hat{u}\| \to 0$ then (18) is uniquely solvable.

Proof:

See Mirzaei et al. (2010), Lemma 4.1. ∎

Suppose that

$$\mathcal{F}_M u = \sum_{i=1}^{M} k(x,\tau_i)u(x\tau_i)\omega_i,$$

then

$$\|\mathcal{F}_M\| = max_{x\in\Omega}\sum_{i=1}^{M}|k(x,\tau_i)\omega_i|.$$

At the first view, the error analysis of method depends on showing $\|\mathcal{F}-\mathcal{F}_M\| \to 0$ as M increases. This cannot be done here; in fact, (Atkinson (1997)),

$$\|\mathcal{F}-\mathcal{F}_M\| \geq \|\mathcal{F}\|.$$

We begin by looking at quantities which do converge to zero as $M \to \infty$.

Lemma 5.6.

Let Ω be a closed, bounded set in $\mathbb{R}$, and let $k(x,\mu)$ be continuous for $x,\mu \in \Omega$. Let the quadrature scheme

$$\int_\Omega f(y)dy = \sum_{i=1}^{M} f(y_i)\omega_i,$$

be convergent for all continuous functions on Ω. Define

$$e_M(x,\mu) = \int_\Omega k(x,\nu)k(\nu,\mu)d\nu - \sum_{i=1}^{M} k(x,\nu_i)k(\nu_i,\mu)\omega_i, \quad x,\mu\in\Omega,\ M\geq 1,$$

as the numerical integration error for the integrand $k(x,.)k(.,\mu)$. Then for $u \in C(\Omega)$,

$$(\mathcal{F}-\mathcal{F}_M)\mathcal{F}u(x) = \int_\Omega e_M(x,\mu)u(\mu x)d\mu,$$
$$(\mathcal{F}-\mathcal{F}_M)\mathcal{F}_M u(x) = \sum_{i=1}^{M} e_M(x,\tau_i)u(x\tau_i)\omega_i.$$

In addition,

$$\|(\mathcal{F}-\mathcal{F}_M)\mathcal{F}\| = \max_{x\in\Omega}\int_\Omega |e_M(x,\mu)|d\mu,$$
$$\|(\mathcal{F}-\mathcal{F}_M)\mathcal{F}_M\| = \max_{x\in\Omega}\sum_{i=1}^{M}|e_M(x,\tau_i)\omega_i|.$$

Finally, the numerical integration error converges to zero uniformly on Ω,

$$\lim_{M\to\infty}\max_{x,\mu\in\Omega}|e_M(x,\mu)| = 0,$$

and thus

$$\|(\mathcal{F}-\mathcal{F}_M)\mathcal{F}\|,\ \|(\mathcal{F}-\mathcal{F}_M)\mathcal{F}_M\| \to 0 \ \text{ as } \ M\to\infty. \tag{19}$$

Proof:

See Atkinson (1997), Chapter 4, Lemma 4.1.1. ∎

To carry out an error analysis, we need the following perturbation theorem.

Theorem 5.7.

Let $\mathcal{X}$ be a Banach space, let $\mathcal{S}$, $\mathcal{T}$ be bounded operators on $\mathcal{X}$ to $\mathcal{X}$, and let $\mathcal{S}$ be compact. For given $\lambda \neq 0$, assume $\lambda - \mathcal{T} : \mathcal{X} \to \mathcal{X}$ is one to one and onto, which implies $(\lambda - \mathcal{T})^{-1}$ exists as a bounded operator on $\mathcal{X}$ to $\mathcal{X}$. Finally, assume

$$\|(\mathcal{T} - \mathcal{S})\mathcal{S}\| < \frac{|\lambda|}{\|(\lambda - \mathcal{T})^{-1}\|}.$$

Then $(\lambda - \mathcal{S})^{-1}$ exists and is bounded on $\mathcal{X}$ to $\mathcal{X}$, with

$$\|(\lambda - \mathcal{S})^{-1}\| \leq \frac{1 + \|(\lambda - \mathcal{T})^{-1}\|\|\mathcal{S}\|}{|\lambda| - \|(\lambda - \mathcal{T})^{-1}\|\|(\mathcal{T} - \mathcal{S})\mathcal{S}\|}.$$

If $(\lambda - \mathcal{T})\omega = g$ and $(\lambda - \mathcal{S})z = g$, then

$$\|\omega - z\| \leq \|(\lambda - \mathcal{S})^{-1}\|\|\mathcal{T}\omega - \mathcal{S}\omega\|.$$

Proof:

See Atkinson (1997), Chapter 4, Theorem 4.1.1. ∎

Finally, the following theorem completes the convergence analysis of the method in L^∞ norm. Before that, we note that the approximation scheme in one dimension could be written in compact form as

$$(\lambda - \mathcal{F}_M)u_N = g.$$

Theorem 5.8.

Let $u \in C^{q,1}(\overline{\Omega})$ $(q = 1, 2)$ where Ω be a closed, bounded set in $\mathbb{R}$; let $k(x, \mu)$ be continuous for $x, \mu \in \Omega$. Assume the quadrature scheme is convergent for all continuous functions on Ω. Further, assume that the integral equation (17) is uniquely solvable for given $g \in C(\Omega)$ with $\lambda \neq 0$. Moreover take a suitable approximation $\hat{u}$ of u. Then, for all sufficiently large M, the approximate inverses $(\lambda - \mathcal{F}_M)^{-1}$ exist and are uniformly bounded,

$$\|(\lambda - \mathcal{F}_M)^{-1}\| \leq \frac{1 + \|(\lambda - \mathcal{F})^{-1}\|\|\mathcal{F}_M\|}{|\lambda| - \|(\lambda - \mathcal{F})^{-1}\|\|(\mathcal{F} - \mathcal{F}_M)\mathcal{F}_M\|} \leq \mathcal{Q},$$

with a suitable constant $\mathcal{Q} < \infty$. For the equations $(\lambda - \mathcal{F})u = g$ and $(\lambda - \mathcal{F}_M)u_N = g$, we have

$$\|u - u_N\|_{L^\infty(\Omega)} \leq C_q R^{q+1}|u|_{q,1}(1 + \mathcal{Q}\|(\mathcal{F} - \mathcal{F}_M)\|_{L^\infty(\Omega)}) + \|(\mathcal{F} - \mathcal{F}_M)\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)}.$$

where $\hat{u}$, R and C_q are introduced in Theorem 5.4.

Proof:

See Mirzaei et al. (2010), Theorem 4.4. ∎

6. Numerical examples

In this section, some examples are provided to show the strength of the proposed method in approximating the solution of Lane-Emden equations. For computational details and the numerical implementation of the method we take $h_i = \frac{2}{N-1}$, for the linear case, $h_i = \frac{2.5}{N-1}$ for the quadratic case, $h_i = \frac{3}{N-1}$ for the degree 3 case and $\beta = \frac{0.6}{N-1}$ for all of them to ensure the invertibility of the matrix A in MLS method. Also in our computations we use the 5-point Gauss-Legendre quadrature rule for numerical integration.

Consider the following Lane-Emden equation with the values of $\xi = 0, 1, 5$ for $f(u(x)) = u^{\xi}$ and $k = 2$, which is a basic equation in the theory of stellar structure (Wazwaz (2001)):

$$u^{''}(x) + \frac{2}{x}u^{'}(x) + u^{\xi} = 0, \;\; u(0) = 1, \;\; u^{'}(0) = 0. \tag{20}$$

It was physically shown that interesting values of ξ lie in the interval $[0, 5]$, moreover, the exact answer has been available only for $\xi = 0, 1, 5$ and for the other values of ξ series solutions are available (Canuto et al. (2006); Ravi et al. (2010)).

It is important to note that Equation (20) for $\xi = 0, 1$ is linear and for the other values of ξ it is nonlinear. According to Section 2, the integral form of Equation (20) will be as follows:

$$u(x) = 1 - \int_0^x t\left(1 - \frac{t}{x}\right) u^{\xi} dt.$$

After replacing $\xi = 0$ the above equation will transform to the following:

$$u(x) = 1 - \int_0^x t\left(1 - \frac{t}{x}\right) dt,$$

after solving that equation, we will come up with the exact solution of the $u(x) = 1 - \frac{x^2}{3!}$.

For $\xi = 1$ we have:

$$u(x) = 1 - \int_0^x t\left(1 - \frac{t}{x}\right) u(t) dt,$$

which has been solved with the MLS method. The results of the process and the comparison of its answer with the exact solution of $u(x) = \frac{sin(x)}{x}$ can be found in Table (1) and Figures (1), (2).

Also for $\xi = 5$ we have:

$$u(x) = 1 - \int_0^x t\left(1 - \frac{t}{x}\right) u^5(t) dt,$$

Table 1. Maximum error for different m (*degree of basis)*, N (*points number)*

N	$m = 1$	$m = 2$	$m = 3$
11	7.4×10^{-3}	3.08×10^{-4}	9.91×10^{-5}
21	2.3×10^{-3}	4.01×10^{-5}	7.50×10^{-6}
41	2.6×10^{-4}	9.47×10^{-7}	4.05×10^{-7}
201	8.0×10^{-6}	1.7×10^{-8}	1.27×10^{-10}

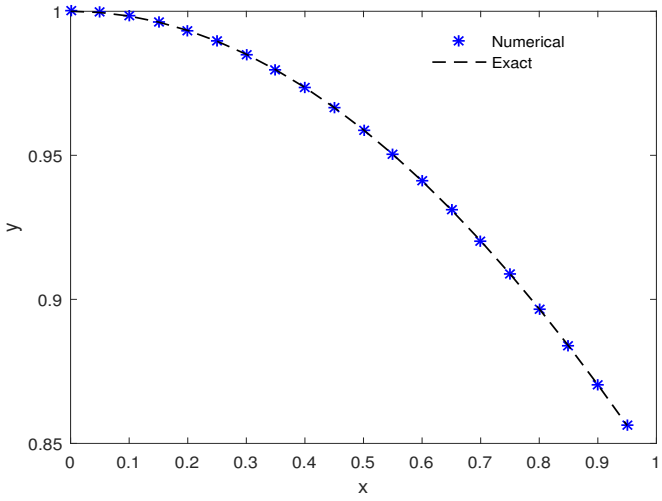

Figure 1. Numerical approximation of degree 2 with 41 points

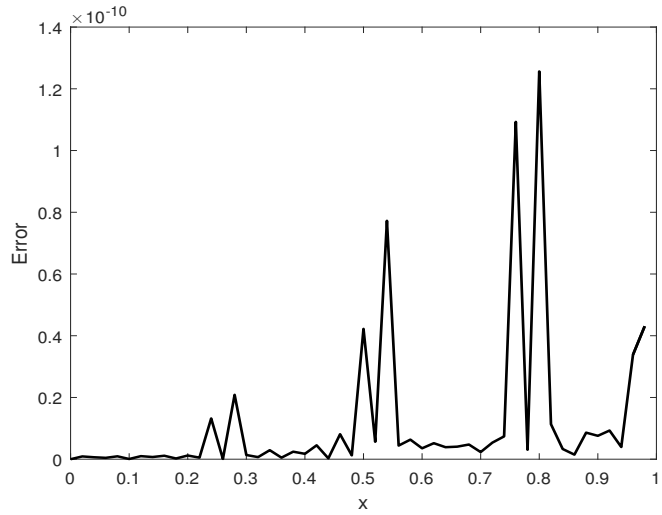

Figure 2. The MLS approximation error of degree 3 with 201 points

the exact solution of this problem is $u(x) = (1 + \frac{x^2}{3})^{\frac{-1}{2}}$. We have achieved the following answers using the MLS method through comparisons with the exact solution. Refer to Table (2) and Figure (3) for details.

Table 2. Maximum error for different m, N

N	$m=1$	$m=2$	$m=3$
5	1.04×10^{-1}	2.13×10^{-2}	2.13×10^{-2}
9	9.5×10^{-3}	2.00×10^{-3}	9.39×10^{-4}
21	2.5×10^{-3}	1.26×10^{-4}	1.23×10^{-4}
51	1.9×10^{-3}	4.27×10^{-5}	2.42×10^{-5}

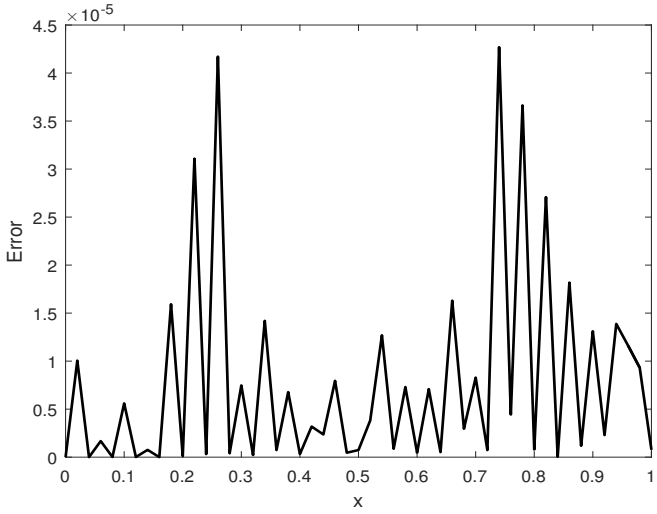

Figure 3. The MLS approximation error of degree 2 with 51 points

Now, consider the following Lane-Emden equation:

$$u''(x)+\frac{2}{x}u'(x)+e^{u(x)}=0,\ \ u(0)=0,\ \ u'(0)=0. \tag{21}$$

This equation models the distribution of mass in an isothermal sphere (Momoniat et al. (2006)). In order to solve Equation (21), first its integral form is considered,

$$u(x)=-\int_0^x t(1-\frac{t}{x})e^{u(t)}dt.$$

Now, by applying the MLS method, we will solve it and compare the results with the following series answer achieved through the Adomian Decomposition Method (ADM) (Wazwaz (2001)). See Table (3) and Figure (4).

$$u(x)=-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{8x^6}{3.7!}+\frac{122x^8}{9.9!}-\frac{4087x^{10}}{45.11!}+e(O^{12}).$$

Similarly, acceptable results can be achieved for the following equation:

$$u''(x)+\frac{2}{x}u'(x)+e^{-u(x)}=0,\ \ u(0)=0,\ \ u'(0)=0,$$

that is used in Richardson's theory (Richardson (1921)) of thermionic currents which is related to the emission of electricity from hot bodies.

Table 3. Maximum error for different m, N

N	$m = 1$	$m = 2$
5	3.15×10^{-2}	2.49×10^{-2}
9	1.14×10^{-2}	3.11×10^{-4}
11	2.5×10^{-3}	4.34×10^{-5}
41	2.94×10^{-4}	6.23×10^{-6}

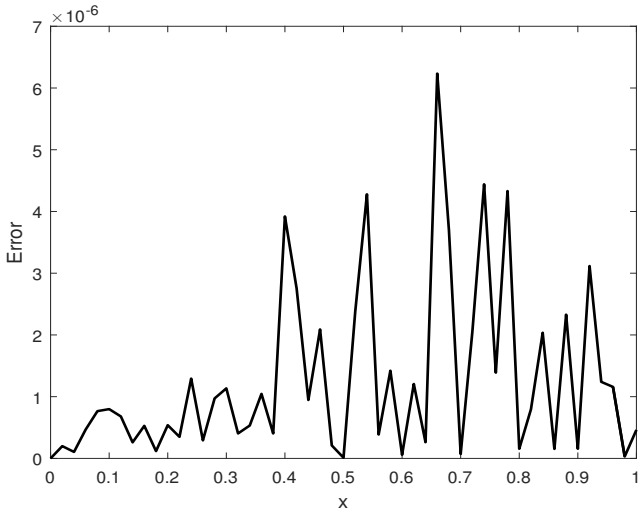

Figure 4. The MLS approximation error of degree 2 with 41 points compared to solution of ADM

7. Conclusion

In this paper, the moving least squares method was used to solve the integral form of the Lane-Emden equations in various forms and at the end an acceptable level of accuracy in answering the equation was achieved. The Lane-Emden equation has been investigated before in multiple occasions and different approaches have been offered to solve it. This paper, however, was meant to come up with a new approach to solve the equation with an emphasis on the efficiency of the offered approach.

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