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A Certain Class of Statistical Deferred Weighted $\mathcal{A}$-summability Based on (p, q)-integers and Associated Approximation Theorems

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Abstract

Statistical summability has recently enhanced researchers' substantial awareness since it is more broad than the traditional (ordinary) convergence. The basic concept of statistical weighted $\mathcal{A}$-summability was introduced by Mohiuddine (2016). In this investigation, we introduce the (presumably new) concept of statistical deferred weighted $\mathcal{A}$-summability and deferred weighted $\mathcal{A}$-statistical convergence with respect to the difference sequence of order r involving (p, q)-integers and establish an inclusion relation between them. Furthermore, based upon the proposed methods, we intend to approximate the rate of convergence and to demonstrate a Korovkin type approximation theorem for functions of two variables defined on a Banach space $C_B(\mathcal{D})$. Finally, several illustrative examples are presented in light of our definitions and outcomes established in this paper.

Keywords: Statistical convergence; Statistical deferred weighted $\mathcal{A}$-summability; Deferred weighted $\mathcal{A}$-statistical convergence; Korovkin-type theorem; Positive linear operators; (p, q)-integers; Rate of convergence; Banach space

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1. Introduction, Preliminaries and Motivation

Let ω be the set of all real valued sequences and suppose any subspace of ω be the sequence space. Let (x_k) be a sequence with real and complex terms. Suppose ℓ_∞ be the class of all bounded linear spaces. Let c and c_0 be the respective classes for convergent and null sequences with real and complex terms. We have

$$\|x\|_\infty = \text{sup}_k |x_k| \ \ (k \in \mathbb{N}),$$

and we recall here that under this norm, all the above mentioned spaces are Banach spaces.

Kızmaz demonstrated the initial idea for spaces of difference sequences (Kızmaz) and subsequently he also extended it to the difference sequence of order (natural) r $(r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ by defining

$$\lambda(\Delta^r) = \left\{x = (x_k) : \ \ \Delta^r(x) \in \lambda, \ \ \lambda \in (\ell_\infty, c_0, c)\right\},$$

$$\Delta^0 x = (x_k); \Delta^r x = (\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1}),$$

and

$$\Delta^r x_k = \sum_{i=0}^{r} (-1)^i \binom{r}{i} x_{k+i}.$$

Also, these are all Banach spaces under the norm defined by

$$\|x\|_{\Delta^r} = \sum_{i=1}^{r} |x_i| + \text{sup}_k |\Delta^m x_k|.$$

For more interest in this direction, see the current works by Altay et al. (2006), Bektaş et al. (2012), and Kadak and Baliarsingh (2015).

In the interpretation of sequence spaces, the well-established traditional convergence has got innumerable applications, where the convergence of a sequence demands that almost all elements are to assure the convergence condition; that is, every element of the sequence is required to be in some neighborhood of the limit. Nevertheless, such limitation is there in statistical convergence, where set having a few elements those are not in the neighborhood of the limit are discarded. Fast and Steinhaus presented and considered the preliminary idea of statistical convergence (see Fast (1951) and Steinhaus (1951)). In the past few decades, statistical convergence has been an energetic area of research due essentially to the aspect that it is more broad than customary (ordinary) convergence and such hypothesis is talked about in the investigation in the subjects of Fourier Analysis, Functional Analysis, Number Theory, and Theory of Approximation. In fact, see the current works (Das et al. (2018); Belen and Mohiuddine (2013); Braha et al. (2014); Dutta et al. (2019); Jena et al. (2018); Kadak (2017); Kadak (2016); Paikray et al. (2019); Paikray and Dutta (2019); Pradhan et al. (2018); Srivastava et al. (2018a); Srivastava et al. (2018b); Srivastava et al. (2018c)).

Let the set of natural numbers be $\mathbb{N}$ and suppose that $K \subseteq \mathbb{N}$. Also, let

$$K_n = \{k : k \leqq n \quad \text{and} \quad k \in K\}.$$

The asymptotic density of K is given by

$$d(K) = \lim_{n\to\infty} \frac{|K_n|}{n} = \lim_{n\to\infty} \frac{1}{n}|\{k : k \leqq n \quad \text{and} \quad k \in K\}|,$$

presuming (that) the limit exists, where $|K_n|$ is the cardinality of K_n.

A given sequence (x_n) is statistically convergent (or stat-convergent) to a number L if, for every $\epsilon > 0$,

$$K_\epsilon = \{k : k \leqq n \quad \text{and} \quad |x_k - L| \geqq \epsilon\},$$

has asymptotic density zero (see Fast (1951) and Steinhaus (1951)). That is, for every $\epsilon > 0$,

$$d(K_\epsilon) = \lim_{n\to\infty} \frac{|K_\epsilon|}{n} = \lim_{n\to\infty} \frac{1}{n}|\{k : k \leqq n \quad \text{and} \quad |x_k - L| \geqq \epsilon\}| = 0.$$

Here, we write

$$\text{stat} \lim_{n\to\infty} x_n = L.$$

We present below an example to illustrate that every convergent sequence is statistically convergent; however, the converse is not necessarily true.

Example 1.1.

Let $x = (x_n)$ be a sequence defined by

$$x_n = \begin{cases} \frac{1}{2}, & (n = m^2, \ \ m \in \mathbb{N}), \\ \\ \frac{n^2}{n^2+1}, & (\text{otherwise}). \end{cases}$$

Here, the sequence (x_n) is statistically convergent to 1 even if it is not classically convergent.

In 2009, Karakaya and Chishti introduced the fundamental concept of weighted statistical convergence (see Karakaya and Chishti (2009)) and later the definition was modified by Mursaleen et al. (see Mursaleen et al. (2012)).

Suppose that (x_n) be a sequence of nonnegative numbers with sequence of partial sum (s_n) and let

$$S_n = \sum_{k=0}^{n} s_k \quad (s_0 > 0; \ \ n \to \infty).$$

Then, upon setting

$$\sigma_n = \frac{1}{S_n} \sum_{k=0}^{n} s_k x_k \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

we say (x_n) is weighted statistically convergent (or stat$_{\bar{N}}$-convergent) to a number L if, for every $\epsilon > 0$,

$$\{k : k \leqq S_n \quad \text{and} \quad s_k|x_k - L| \geqq \epsilon\},$$

has weighted density zero (Mursaleen et al. (2012)). That is, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{S_n} |\{k : k \leqq S_n \quad \text{and} \quad s_k|x_k - L| \geqq \epsilon\}| = 0.$$

Here, we write

$$\text{stat}_{\bar{N}} \lim x_n = L.$$

In 2013, Belen and Mohiuddine established a new technique for weighted statistical convergence in terms of the de la Vallée Poussin mean (Belen and Mohiuddine (2013)) and it was subsequently investigated further by Braha et al. (2014) as the Λ_n-weighted statistical convergence. Very recently, a certain class of weighted statistical convergence and associated Korovkin-type approximation theorems involving trigonometric functions have been introduced by Srivastava et al. (see, for details, Srivastava et al. (2018a)).

Suppose X and Y are two sequence spaces and let $\mathcal{A} = (a_{n,k})$ be a non-negative matrix (regular). If for every $x_k \in X$ the series,

$$\mathcal{A}_n x = \sum_{k=1}^{\infty} a_{n,k} x_k,$$

converges for all $n \in \mathbb{N}$ and the sequence $(\mathcal{A}_n x)$ belongs to Y, then the matrix $\mathcal{A}$ maps X into Y. Here, the symbol (X, Y) denote the set of matrices that map X into Y.

Next, as regards to regularity condition, a matrix $\mathcal{A}$ is said to be regular, if

$$\lim_{n\to\infty} \mathcal{A}_n x = L \quad \text{whenever} \quad \lim_{k\to\infty} x_k = L.$$

We recall here, the well-known Silverman-Toeplitz theorem (see Boos (2000) for details), $\mathcal{A} = (a_{n,k})$ is regular if and only if

(a) $\sup\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} |a_{n,k}| < \infty$;

(b) $\lim\limits_{n\to\infty} a_{n,k} = 0$ for each k;

(c) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} a_{n,k} = 1$.

The definition of statistical convergence was extended by Freedman and Sember (1981) just by considering the non-negative regular matrix $\mathcal{A} = (a_{n,k})$, and he termed it as $\mathcal{A}$-statistical convergence. Let for any non-negative regular matrix $\mathcal{A}$, we say that a sequence (x_n) is $\mathcal{A}$-statistically convergent (or stat$_{\mathcal{A}}$-convergent) to a number L if, for each $\epsilon > 0$, we have

$$d_{\mathcal{A}}(K_\epsilon) = 0,$$

where

$$K_\epsilon = \{k : k \in \mathbb{N} \quad \text{and} \quad |x_k - L| \geqq \epsilon\}.$$

That is, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \sum_{k:|x_k-L|\geqq\epsilon} a_{n,k} = 0.$$

Here, we write

$$\text{stat}_{\mathcal{A}} \lim x_n = L.$$

Furthermore, let $(a_{n,k})$ be a nonnegative regular matrix and let (x_n) be a sequence. Then we say that the sequence (x_n) is statistical $\mathcal{A}$-summable to a number L if, for each $\epsilon > 0$, we have

$$d(F_\epsilon) = 0,$$

where

$$F_\epsilon = \{k : k \in \mathbb{N} \ \ \text{and} \ \ |\mathcal{A}_n x - L| \geqq \epsilon\}.$$

Here, we write

$$\text{stat} \lim_{n\to\infty} \mathcal{A}_n x = L.$$

Subsequently, with the development of q-calculus, various researchers worked on certain new generalizations of positive linear operators based on q-integers (Agrawal et al. (2014); Aral and Gupta (2012); Jena et al. (2017); Mursaleen et al. (2013); Srivastava et al. (2017)). Recently, Mursaleen et al. (2015) developed the (p,q)-analogue of Bernstein operators in connection with (p,q)-integers and later on, some results towards the estimation for Baskakov operators and Bernstein-Schurer operators are studied for (p,q)-integers by Acar et al. (2016) and Mursaleen et al. (2015), respectively.

We now recall some definitions and basic notations on (p,q)-integers for our present study.

For any $n \in \mathbb{N}$, the (p,q)-integer $[n]_{p,q}$ is defined by,

$$[n]_{p,q} = \begin{cases} \frac{p^n - q^n}{p-q}, & (n \geq 1), \\ 0, & (n = 0), \end{cases}$$

where $\ 0 < q, \ \ p \leq 1$.

The (p,q)-factorial is defined by,

$$[n]!_{p,q} = \begin{cases} [1]_{p,q}[2]_{p,q}...[n]_{p,q}, & (n \geq 1), \\ 1, & (n = 0). \end{cases}$$

The (p,q)-binomial coefficient is defined by,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]!_{p,q}}{[k]!_{p,q}\,[n-k]!_{p,q}} \qquad \text{for all } \; n,k \in \mathbb{N} \; \text{ and } \; n \geq k.$$

We also recall that, suppose $0 < q < p \leq 1$ and let r be a non-negative integer. Then, the operator

$$\Delta_{p,q}^{[r]} : \omega \to \omega,$$

is defined by

$$\Delta_{p,q}^{[r]}(x_n) = \sum_{i=0}^{r} (-1)^i \begin{bmatrix} r \\ i \end{bmatrix}_{p,q} x_{n-i}.$$

That is,

$$\begin{aligned}
\Delta_{p,q}^{[r]}(x_n) &= \begin{bmatrix} r \\ 0 \end{bmatrix}_{p,q} x_n - \begin{bmatrix} r \\ 1 \end{bmatrix}_{p,q} x_{n-1} + \begin{bmatrix} r \\ 2 \end{bmatrix}_{p,q} x_{n-2} - \begin{bmatrix} r \\ 3 \end{bmatrix}_{p,q} x_{n-3} + \ldots + (-1)^r \begin{bmatrix} r \\ r \end{bmatrix}_{p,q} x_{n-r} \\
&= x_n - [r]_{p,q} x_{n-1} + \frac{[r]_{p,q}[r-1]_{p,q}}{[2]_{p,q}!} x_{n-2} - \frac{[r]_{p,q}[r-1]_{p,q}[r-2]_{p,q}}{[3]!} x_{n-3} + \ldots + (-1)^r x_{n-r} \\
&= x_n - \left(\frac{p^r - q^r}{p-q} \right) x_{n-1} + \left(\frac{(p^r - q^r)(p^{r-1} - q^{r-1})}{(p-q)^2 (p+q)} \right) x_{n-2} \\
&\quad - \left(\frac{(p^r - q^r)(p^{r-1} - q^{r-1})(p^{r-2} - q^{r-2})}{(p-q)^3 (p^2 + pq + q^2)(p+q)} \right) x_{n-3} + \ldots + (-1)^r x_{n-r}.
\end{aligned}$$

Now we present an example to see that a sequence is not convergent; however, the associated difference sequence is convergent.

Example 1.2.

Let us consider a sequence $(x_n) = n + 1$ $(n \in \mathbb{N})$. In fact, it is trivial that the sequence (x_n) is not convergent in the ordinary sense.

Also, we see that

$$\Delta^{[3]}(x_n) = x_n - 3x_{n-1} + 3x_{n-2} - x_{n-3} \quad (x_n = n+1),$$

converges to $0 \quad (n \to \infty)$.

For $r = 3$, we obtain that

$$\Delta_{p,q}^{[3]}(x_n) = x_n - [3]_{p,q}x_{n-1} + [3]_{p,q}x_{n-2} - x_{n-3} \ \ (x_n = n+1)$$

$$= x_n - (p_n^2 + p_nq_n + q_n^2)x_{n-1} + (p_n^2 + p_nq_n + q_n^2)x_{n-2} - x_{n-3}$$

$$= n+1 - (p_n^2 + p_nq_n + q_n^2)n + (p_n^2 + p_nq_n + q_n^2)(n-1) - (n-2) \ \ (x_n = n+1)$$

$$= 3 - (\beta^2 + \alpha\beta + \alpha^2).$$

Clearly, depending on the values of p and q, the difference sequence $\Delta_{p,q}^{[3]}(x_n)$ of order 3 has different limits. This case is mostly due to the definition of (p,q)-integers. However, in order to obtain a convergence criterion for all values of p and q, belonging to the operator $\Delta_{p,q}^{[r]}$, we must have to overcome this difficulty. This type of difficulties can be avoided in the following two ways. The first one is taking $p = q = 1$ and thus the operator reduces to the usual difference sequence. Next, the second way is to replace $p = p_n$ and $q = q_n$ under the limits, $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 \leq \alpha, \beta \leq 1)$ where $0 < q_n < p_n \leq 1$, for all $(n \in \mathbb{N})$. Afterwards, the difference sequence $\Delta_{p,q}^{[3]}(x_n)$ of third order 3, converges to the value $3 - (\beta^2 + \alpha\beta + \alpha^2)$. Thus, if we take $q_n = \left(\frac{n+1}{n+1+s}\right) < \left(\frac{n+1}{n+1+t}\right) = p_n$ such that $0 < q_n < p_n \leq 1$ $(s > t > 0)$, then $\lim_n q_n = 1 = \lim_n p_n$ and hence $\Delta_{p,q}^{[3]}(x_n) \to 0 \quad (n \to \infty)$.

Remark 1.3.

If $r = 1$, $\lim_n q_n = 1$ and $\lim_n p_n = 1$, then the difference operator $\Delta_{p,q}^{[r]}$ reduces to the $\Delta^{[1]}$ (Altay and Başar (2004)). Also, if $r = 0$, $\lim_n q_n = 1$ and $\lim_n p_n = 1$, then the difference operator $\Delta_{p,q}^{[r]}$ reduces to the general sequence (x_n).

Kadak introduced to weighted statistical convergence involving (p,q)-integers to prove related approximation theorems for functions (two variables) (Kadak (2016)). Subsequently, it was extended to the generalized difference sequences involving (p,q)-gamma function and accordingly associated approximation theorems were proved (Kadak (2017)). Furthermore, Mohiuddine introduced the notion of weighted $\mathcal{A}$-summability by using a weighted regular summability matrix (Mohiuddine (2016)). He also gave the definitions of statistically weighted $\mathcal{A}$-summability and weighted $\mathcal{A}$-statistical convergence. In particular, he proved a Korovkin type approximation theorem under the consideration of statistically weighted $\mathcal{A}$-summable sequences of real or complex numbers. Subsequently, Kadak et al. (2017) has investigated the statistical weighted $\mathcal{B}$-summability by using a weighted regular matrix to establish some approximation theorems. Very recently, Srivastava et al. (2018b) introduced the deferred weighted (Nörlund) summability of a sequence and accordingly proved Korvokin type approximation theorems on the basis of equi-statistical convergence.

Motivated essentially by the above-mentioned works, here we would like to introduce the (presumably new) notion of deferred weighted $\mathcal{A}$-statistical convergence and statistical deferred weighted $\mathcal{A}$-summability with respect to the generalized difference sequences of order r involving (p,q)-integers, and to establish certain new approximation results on that basis.

2. Definitions, Notations and Regular Methods

In this section, we introduce some definitions (presumably new) that are required for our proposed study. Also, we present here certain inclusion relations with regard to regular methods.

Let (a_n) and (b_n) be the sequences of non-negative integers fulfilling the conditions: (a) $a_n < b_n \quad (n \in \mathbb{N})$ and (b) $\lim_{n\to\infty} b_n = \infty$. Note that, the conditions (a) and (b) are regularity conditions for deferred weighted mean (Agnew (1932)).

Next, we suppose that (s_n) be the sequence of non-negative numbers (real) such that

$$S_n = \sum_{m=a_n+1}^{b_n} s_m.$$

Now, for defining the deferred weighted mean $D(\bar{N}, s)$ by the difference operator $(\Delta^r_{p,q})$, we first set

$$\Phi_n^{p,q}(\Delta x) = \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} s_m(\Delta^{[r]}_{p,q} x_m) \quad (0 < q < p \leq 1) \;\; (r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

The given sequence (x_n) is said to be deferred weighted summable (or $c^{D(\bar{N})}$-summable) to L involving the difference operator $(\Delta^{[r]}_{p,q})$ if,

$$\lim_{n\to\infty} \Phi_n^{p,q}(\Delta x) = L.$$

In this case, we write

$$c_{\Delta}^{D(\bar{N})} \lim_{n\to\infty} x_n = L.$$

We denote here the set of all sequences that are deferred weighted summable under the difference operator $(\Delta^{[r]}_{p,q})$ by $c_{\Delta}^{D(\bar{N})}$.

Definition 2.1.

Let $\mathcal{A}$ be a nonnegative regular matrix, $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$ and let r is a non-negative integer. Let (a_n) and (b_n) be sequences of non-negative integers. A sequence (x_n) is said to be deferred weighted $\mathcal{A}$-summable (or $[D(\bar{N})_{\mathcal{A}}; s_n]$-summable) to a number L with respect to the difference operator $\Delta^{[r]}_{p,q}$ if the $\mathcal{A}$-transform of (x_n) is deferred weighted summable to the same number L under the difference operator $\Delta^{[r]}_{p,q}$; that is,

$$\lim_{n\to\infty} \Psi_n^{p,q}(\Delta x) = \lim_{n\to\infty} \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} \sum_{k=1}^{\infty} s_m a_{m,k}(\Delta^{[r]}_{p,q} x_k) = L.$$

In this case, we write

$$[D(\bar{N})^{\Delta}_{\mathcal{A}}; s_n] \lim_{n\to\infty} x_n = L.$$

We denote the set of all sequences that are deferred weighted summable using the difference operator $(\Delta^{[r]}_{p,q})$ by $[D(\bar{N})^{\Delta}_{\mathcal{A}}; s_n]$.

Remark 2.2.

If,

$$a_n = 0, \;\; b_n = n, \;\; \lim_n q_n = 1, \;\; \lim_n p_n = 1 \;\text{ and }\; r = 0,$$

then the $\Psi_n^{p,q}(\Delta x)$ mean is identical with the $\mathcal{A}_n^{\bar{N}}$ mean (Mohiuddine (2016)) and if

$$\mathcal{A} = I \;\text{ (identity matrix)}, \;\; a_n + 1 = \alpha(n), \;\; b_n = \beta(n) \;\text{ and }\; s_n = 1,$$

then $\Psi_n^{p,q}(\Delta x)$ mean is same as the $\Lambda_{p,q}^n(x_n)$ mean (Kadak (2016)).

Definition 2.3.

Let $\mathcal{A} = (a_{n,k})$ be a matrix, $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$ and let r is a non-negative integer. Let (a_n) and (b_n) be sequences of non-negative integers. The matrix $\mathcal{A} = (a_{n,k})$ is said to be a deferred weighted regular matrix (or deferred weighted regular method) if,

$$\mathcal{A}x \in c_\Delta^{D(\bar{N})} \quad (\forall\, x_n \in c),$$

with

$$c_\Delta^{D(\bar{N})} \lim \mathcal{A}x_n = \mathcal{A} \lim(x_n),$$

and we denote it by $\mathcal{A} \in \left(c : c_\Delta^{D(\bar{N})}\right)$. This means that $\Psi_n^{p,q}(\Delta x)$ exists for each $n \in \mathbb{N}$, $x_n \in c$ and

$$\lim_{n\to\infty} \Psi_n^{p,q}(\Delta x) \to L \quad \text{whenever} \quad \lim_{n\to\infty} x_n \to L.$$

We here denote the class of all deferred weighted regular matrices (methods) by $\mathcal{R}_{D(w)}^+$.

As a characterization of the deferred weighted regular methods, we present below a theorem as follows.

Theorem 2.4.

Let $\mathcal{A} = (a_{n,k})$ be a sequence of infinite matrices, $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$ and let r is a non-negative integer. Let (a_n) and (b_n) be sequences of non-negative integers. Then $\mathcal{A} \in \left(c : c_\Delta^{D(\bar{N})}\right)$ if and only if

$$\sup_n \sum_{k=1}^{\infty} \frac{1}{S_n} \left| \sum_{m=a_n+1}^{a_n} s_m a_{m,k} \right| < \infty, \tag{1}$$

$$\lim_{n\to\infty} \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} s_m a_{m,k} = 0 \;\; (\text{for each } \; k \in \mathbb{N}), \tag{2}$$

and

$$\lim_{n\to\infty} \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} \sum_{k=1}^{\infty} s_m a_{m,k} = 1. \tag{3}$$

Proof:

Assume that (1)-(3) hold true and that $(\Delta^{[r]}_{p,q}x_k) \to L \quad (n \to \infty)$. Then for each $\epsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that $|(\Delta^{[r]}_{p,q}x_k) - L| \leqq \epsilon \ (m > m_0)$. Thus, we have

$$
\begin{aligned}
&\left|\mathcal{A}^{(a_n,b_n)}_n(\Delta^{[r]}_{p,q}x_k) - L\right| = \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m a_{m,k}(\Delta^{[r]}_{p,q}x_k) - L\right| \\
&= \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m a_{m,k}(\Delta^{[r]}_{p,q}x_k - L) + L\left(\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m a_{m,k} - 1\right)\right| \\
&\leqq \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m a_{m,k}(\Delta^{[r]}_{p,q}x_k - L)\right| + |L|\left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m a_{m,k} - 1\right| \\
&\leqq \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{b_{n-2}} s_m a_{m,k}(\Delta^{[r]}_{p,q}x_k - L)\right| \\
&\qquad + \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=b_{n-1}}^{\infty} s_m a_{m,k}(\Delta^{[r]}_{p,q}x_k - 1)\right| + |L|\left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m a_{m,k} - 1\right| \\
&\leqq \sup_k |\Delta^{[r]}_{p,q}x_k - L| \sum_{k=1}^{b_{n-2}} \frac{1}{S_n}\sum_{m=a_n+1}^{b_n} s_m a_{m,k} + \epsilon\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m a_{m,k} \\
&\qquad + |L|\left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m a_{m,k} - 1\right|.
\end{aligned}
$$

Taking $n \to \infty$ and using (2) and (3), we get

$$
\left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m a_{m,k}(\Delta^{[r]}_{p,q}x_k) - L\right| \leqq \epsilon,
$$

which implies that

$$
\lim_{n\to\infty} \frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m a_{m,k}(\Delta^{[r]}_{p,q}x_k) = L = \lim(x_n),
$$

since $\epsilon > 0$ is arbitrary.

Conversely, let $\mathcal{A} \in \left(c : c^{D(\bar{N})}_{\Delta}\right)$ and $x_n \in c$. Then, since $\mathcal{A}x$ exists, we have the inclusion

$$
\left(c : c^{D(\bar{N})}_{\Delta}\right) \subset (c : L_\infty).
$$

Clearly, there exists a constant M such that

$$
\left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m a_{m,k}\right| \leqq M \quad (\forall\, m, n),
$$

and the corresponding series

$$\frac{1}{P_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m a_{m,k},$$

absolutely converges for each n. Therefore, (1) is valid.

We now choose a sequence $x^{(n)} = (x_k^{(n)}) \in c_0$ given as,

$$x_k^{(n)} = \begin{cases} 1, & (n = k), \\ \\ 0, & (n \neq k), \end{cases}$$

for all $n \in \mathbb{N}$ and $y = (y_n) = (1, 1, 1, ...) \in c$. Then, since $\mathcal{A}x^{(n)}$ and $\mathcal{A}y$ are belong to $c_{\Delta}^{D(\bar{N})}$, thus 2 and 3 are fairly obvious. ■

Next, for the statistical version, we present below the following definitions.

Definition 2.5.

Let $\mathcal{A} \in \mathcal{R}_{D(w)}^{+}$, $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$ and let r is a non-negative integer. Let (a_n) and (b_n) be sequences of non-negative integers and also let $K = (k_i) \subset \mathbb{N}$ $(k_i \leq k_{i+1})$ for all i. Then the deferred weighted $\mathcal{A}$-density of K is defined by

$$d_{D(\bar{N})}^{\mathcal{A}}(K) = \lim_{n\to\infty} \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} \sum_{k\in K} s_m a_{m,k},$$

provided this limit exists. A sequence (x_n) is said to be deferred weighted $\mathcal{A}$-statistical convergence to a number L under the difference operator $\Delta_{p,q}^{[r]}$ for each $\epsilon > 0$, we have

$$d_{D(\bar{N})}^{\mathcal{A}}(K_\epsilon) = 0,$$

where

$$K_\epsilon = \{k : k \in \mathbb{N} \text{ and } |\Delta_{p,q}^{[r]}(x_k) - L| \geqq \epsilon\}.$$

We write here

$$\text{stat}_{\Psi_\Delta}^{p,q} \lim_{n\to\infty}(x_n) = L.$$

Definition 2.6.

Let $\mathcal{A} \in \mathcal{R}_{D(w)}^{+}$, $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$ and let r is a non-negative integer. Let (a_n) and (b_n) be sequences of non-negative integers. Then the sequence is said to be statistically deferred weighted $\mathcal{A}$-summable to a number L under the operator $\Delta_{p,q}^{[r]}$ if, for each $\epsilon > 0$, we have

$$d(E_\epsilon) = 0,$$

where

$$E_\epsilon = \{k : k \in \mathbb{N} \text{ and } |\Psi_n^{p,q}(\Delta x) - L| \geqq \epsilon\}.$$

We write here

$$\text{stat}D(\bar{N})^{p,q}_{\Psi_\Delta} \lim_{n\to\infty}(x_n) = L.$$

We now prove a following theorem that determines a relation between the deferred weighted $\mathcal{A}$ statistical convergence and the statistical deferred weighted $\mathcal{A}$-summability.

Theorem 2.7.

Let $\mathcal{A} \in \mathcal{R}^+_{D(w)}$, (a_n) and (b_n) be sequences of non-negative integers and let $0 < q_n < p_n \leq 1$ $(\forall\, n \in \mathbb{N})$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$. If (x_n) is deferred weighted $\mathcal{A}$-statistical convergent to a number L, then it is statistical deferred weighted $\mathcal{A}$-summable to the same number L, but the converse is not true.

Proof:

Let (x_n) be deferred weighted $\mathcal{A}$-statistical convergent to L under the operator $\Delta^{[r]}_{p,q}$. We have

$$d^{\mathcal{A}}_{D(\bar{N})}(K_\epsilon) = 0,$$

where

$$K_\epsilon = \{k : k \in \mathbb{N} \text{ and } |\Delta^{[r]}_{p,q}(x_k) - L| \geqq \epsilon\}.$$

Thus we have,

$$\begin{aligned}
|\Psi_n^{p,q}(\Delta x) - L| &= \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m a_{m,k}(\Delta^{[r]}_{p,q}x_k - L)\right| \\
&\leqq \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m a_{m,k}\left(\Delta^{[r]}_{p,q}x_k - L\right)\right| + |L|\left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m a_{m,k} - 1\right| \\
&\leqq \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k\in K_\epsilon} s_m a_{m,k}\left(\Delta^{[r]}_{p,q}x_k - L\right)\right| \\
&\quad + \left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k\notin K_\epsilon}^{\infty} s_m a_{m,k}\left(\Delta^{[r]}_{p,q}x_k - L\right)\right| + |L|\left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m a_{m,k} - 1\right| \\
&\leqq \sup_{k\to\infty}\left|\Delta^{[r]}_{p,q}x_k - L\right|\frac{1}{S_n}\sum_{k\in K_\epsilon}\sum_{m=a_n+1}^{b_n} s_m a_{m,k} + \epsilon\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k\notin K_\epsilon} s_m a_{m,k} \\
&\quad + |L|\left|\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=1}^{\infty} s_m a_{m,k} - 1\right| \to \epsilon \quad (n\to\infty),
\end{aligned}$$

which implies that $\Psi_n^{p,q}(\Delta x) \to L$. That is, the sequence (x_n) is deferred weighted $\mathcal{A}$-summable to the number L under the difference operator $\Delta^{[r]}_{p,q}$ and hence the sequence (x_n) is statistically deferred weighted $\mathcal{A}$-summable to the same number L with respect to the same difference operator $\Delta^{[r]}_{p,q}$. ■

In order to show that the converse is not true, we present an example (below).

Example 2.8.

Let us choose an infinite matrix $\mathcal{A}$ as a Cesàro matrix $(\mathcal{C}, 1)$ and is defined by

$$a_{n,k} = \begin{cases} \frac{1}{n}, & (1 \leq k \leq n), \\ \\ 0, & (k > n). \end{cases}$$

For $\lim_n q_n = 1$, $\lim_n p_n = 1$, $s_n = 1$, $a_n = 2n$ and $b_n = 4n$ ($\forall\ n \in \mathbb{N}$), consider a sequence $x = (x_n)$,

$$x_n = \begin{cases} \frac{1}{m^2}, & (n = m^2 - m, m^2 - m + 1, ..., m^2 - 1), \\ \\ -\frac{1}{m^3}, & (n = m^2,\ \ m > 1), \\ \\ 0, & \text{(otherwise)}. \end{cases}$$

We have,

$$\begin{aligned}
\sum_{k=1}^{n} \Delta^{[r]}_{p,q} x_k &= \sum_{k=0}^{n} \sum_{i=0}^{r} (-1)^i \begin{bmatrix} r \\ i \end{bmatrix}_{p,q} x_{n-i} \\
&= \sum_{k=1}^{n} \left\{ x_n - \begin{bmatrix} r \\ 1 \end{bmatrix}_{p,q} x_{n-1} + \begin{bmatrix} r \\ 2 \end{bmatrix}_{p,q} x_{n-2} - \begin{bmatrix} r \\ 3 \end{bmatrix}_{p,q} x_{n-3} + ... + (-1)^r \begin{bmatrix} r \\ r \end{bmatrix}_{p,q} x_{n-r} \right\} \\
&= \sum_{k=1}^{n} \left\{ x_n - [r]_{p,q} x_{n-1} + \frac{[r]_{p,q}[r-1]_{p,q}}{[2]_{p,q}!} x_{n-2} - \frac{[r]_{p,q}[r-1]_{p,q}[r-2]_{p,q}}{[3]!} x_{n-3} + ... + (-1)^r x_{n-r} \right\} \\
&= \left\{ (n) x_n + (1 - [r]_{p,q})(n-1) x_{n-1} + \left(1 - [r]_{p,q} + \frac{[r]_{p,q}[r-1]_{p,q}}{[2]_{p,q}!} (n-2) x_{n-2} \right) \right. \\
&\quad \left. + ... + \left(1 - [r]_{p,q} + \frac{[r]_{p,q}[r-1]_{p,q}}{[2]_{p,q}!} - \frac{[r]_{p,q}[r-1]_{p,q}[m-2]_{p,q}}{[3]_{p,q}!} + ... \right) (n-k) x_{n-k} \right\}.
\end{aligned}$$

Thus,

$$\Psi^{p,q}_n(\Delta x) = \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} \sum_{k=1}^{n} s_m a_{m,k} (\Delta^{[r]}_{p,q} x_k) = \frac{1}{n} \sum_{m=2n+1}^{4n} \frac{1}{n} \sum_{k=1}^{n} \Delta^{[r]}_{p,q} x_k \to 0,$$

which implies that

$$\text{stat}\Psi^{p,q}_n(\Delta x) \to 0.$$

Hence, (x_n) is not deferred weighted $\mathcal{A}$-statistical convergent, even if it is statistical deferred weighted $\mathcal{A}$-summable under the difference operator of order r based on (p,q)-integers. Furthermore, (x_n) is not statistical weighted $\mathcal{A}$-summable; however, it is statistical deferred weighted $\mathcal{A}$-summable with respect to the difference operator of order r based on (p,q)-integers.

3. A Korovkin Type Theorem via Statistical Deferred Weighted $\mathcal{A}$-summability

Recently, a few researchers worked on extending the Korovkin type approximation theorems in various ways based on various aspects, involving (for instance) function spaces, abstract Banach lattices, and so on. This theory is highly valuable in Analysis and many other fields. The main concern of this paper is to introduce the notion of statistical deferred weighted $\mathcal{A}$-summability and deferred weighted $\mathcal{A}$-statistical convergence with respect to the difference sequence of order r involving (p,q)-integers, and then to establish some associated approximation type results in relevance to our presumably defined new concept of statistical deferred weighted $\mathcal{A}$-summability, that will effectively extend and improve most (if not all) of the existing results depending on the choice of deferred weighted $\mathcal{A}$-mean. Furthermore, based upon the proposed methods, we wish to approximate the order of convergence and to investigate a Korovkin type approximation result for a function of two variables. In fact, we extend here the result of Mohiuddine (2016) by using the notion of the statistical deferred weighted $\mathcal{A}$-summability for the generalized difference sequence of order r involving (p,q)-integers and prove the following theorem.

Let $\mathcal{D}$ be any compact subset over $\mathbb{R}^2$. Let, $C_B(\mathcal{D})$ be the space of all real valued continuous functions on $\mathcal{D}$ under the norm:

$$\|f\|_{C_B(\mathcal{D})} = \sup\{|f(x,y)| : (x,y) \in \mathcal{D}\}, \ \ f \in C_B(\mathcal{D}).$$

Let $T : C_B(\mathcal{D}) \to C_B(\mathcal{D})$, we say that T is a positive linear operator, for

$$f \geqq 0 \ \text{ implies } \ T(f) \geqq 0.$$

Also, we use the notation $T(f;x,y)$ for the values of $T(f)$ at the point $(x,y) \in \mathcal{D}$.

Theorem 3.1.

Let $\mathcal{A} \in \mathcal{R}^+_{D(w)}$, (a_n) and (b_n) be a sequences of non-negative integers. Let r be a non-negative integer, and let $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$. Let $T_n \ \ (n \in \mathbb{N})$ be a sequence of positive linear operators from $C_B(\mathcal{D})$ into itself and let $f \in C_B(\mathcal{D})$. Then,

$$\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} \lim_n \|T_n(f(s,t);x,y) - f(x,y)\|_{C_B(\mathcal{D})} = 0, \tag{4}$$

if and only if

$$\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} \lim_n \|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})} = 0, \tag{5}$$

$$\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} \lim_n \|T_n(s;x,y) - x\|_{C_B(\mathcal{D})} = 0, \tag{6}$$

$$\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} \lim_n \|T_n(t;x,y) - y\|_{C_B(\mathcal{D})} = 0, \tag{7}$$

$$\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} \lim_n \|T_n(s^2+t^2;x,y) - (s^2+t^2)\|_{C_B(\mathcal{D})} = 0. \tag{8}$$

Proof:

Since each of the functions given by

$$f_0(s,t) = 1, \quad f_1(s,t) = s, \quad f_2(s,t) = t, \quad \text{and} \quad f_2(s,t) = s^2+t^2,$$

are belong to $C_B(\mathcal{D})$, the following implication

$$(4) \Longrightarrow (5) - (8),$$

is fairly obvious. Now, in order to complete the proof of Theorem 3.1, we first assume that (5)-(8) hold true. Let $f \in C_B(\mathcal{D}), \ \forall (x,y) \in \mathcal{D}$. Since $f(x,y)$ is bounded on $\mathcal{D}$, there exists a constant $M > 0$, such that

$$|f(x,y)| \leqq M \quad (\forall\, x,y \in \mathcal{D}),$$

which implies that

$$|f(s,t) - f(x,y)| \leqq 2M \quad (s,t,x,y \in \mathcal{D}). \tag{9}$$

Clearly, f is a continuous function on $\mathcal{D}$. Thus, for a given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$|f(s,t) - f(x,y)| < \epsilon \quad \text{whenever} \quad |s-x| < \delta \quad \text{and} \quad |t-y| < \delta, \tag{10}$$

for all $s,t,x,y \in \mathcal{D}$.

From equation (9) and (10), we get

$$|f(s,t) - f(x,y)| < \epsilon + \frac{2M}{\delta^2}\left([\varphi(s,x)]^2 + [\varphi(t,y)]^2\right), \tag{11}$$

where

$$\varphi(s,x) = s - x \ \text{ and } \ \varphi(t,y) = t - y.$$

Since the function $f \in C_B(\mathcal{D})$, the inequality (11) holds for $s,t,x,y \in \mathcal{D}$.

Now, the operator $T_n(f;x,y)$ being linear and monotone, so by using this in (11), we obtain

$$\begin{aligned}|T_n(f(s,t);x,y)-f(x,y)| &= |T_n(f(s,t)-f(x,y);x,y)+f(x,y)[T_k(f_0;x,y)-f_0]| \\ &\leqq |T_n(f(s,t)-f(x,y);x,y)+M[T_k(1;x,y)-1]| \\ &\leqq \left|T_n\left(\epsilon+\frac{2M}{\delta^2}\left[\varphi(s,x)^2+\varphi(t,y)^2\right];x,y\right)\right|+M|T_n(1;x,y)-1| \\ &\leqq \epsilon+(\epsilon+M)|T_n(f_0;x,y)-f_0(x,y)|+\frac{2M}{\delta^2}|T_n(f_3;x,y)-f_3(x,y)| \\ &-\frac{4M}{\delta^2}x|T_n(f_1;x,y)-f_1(x,y)|-\frac{4M}{\delta^2}y|T_n(f_2;x,y)-f_2(x,y)| \\ &+\frac{2M}{\delta^2}(x^2+y^2)|T_n(f_0;x,y)-f_0(x,y)| \\ &\leqq \epsilon+\left(\epsilon+M+\frac{4M}{\delta^2}\right)|T_n(1;x,y)-1| \\ &+\frac{4M}{\delta^2}|T_n(f_1;x,y)-f_1(x,y)|+\frac{4M}{\delta^2}|T_n(f_2;x,y)-f_2(x,y)| \\ &+\frac{2M}{\delta^2}|T_n(f_3;x,y)-f_3(x,y)|. \end{aligned} \tag{12}$$

Next, taking $\sup_{x,y\in\mathcal{D}}$, in both side of (12), we get

$$\|T_n(f(s,t);x,y)-f(x,y)\|_{C_B(\mathcal{D})} \leqq \epsilon+N\sum_{j=0}^{3}\|T_n(f_j(s,t);x,y)-f_j(x,y)\|_{C_B(\mathcal{D})}, \tag{13}$$

where

$$N=\left\{\epsilon+M+\frac{4M}{\delta^2}\right\}, \quad (j=0,1,2,3).$$

We now replace $T_n(s,t;x,y)$ by

$$\mathfrak{L}_n(f(s,t);x,y)=\frac{1}{S_n}\sum_{m=a_n+1}^{b_n}\sum_{k=0}^{\infty}s_m a_{m,k}\Delta_{p,q}^{[r]}(T_k(f;x,y)) \ \ (\forall\, m\in\mathbb{N}),$$

in Equation (13).

We now choose $\epsilon'>0$, such that $0<\epsilon'<r$. Then, by setting

$$A_n=|\{n:n\leqq\mathbb{N} \quad \text{and} \quad |\mathfrak{L}_n(f(s,t);x,y)-f(x,y)|\geqq r\}|,$$

and

$$A_{j,n}=\left|\left\{n:n\leqq\mathbb{N} \quad \text{and} \quad |\mathfrak{L}_n(f_j(s,t);x,y)-f_j(x,y)|\geqq\frac{r-\epsilon'}{4N}\right\}\right| \ \ (j=0,1,2,3),$$

we easily find from (13) that

$$A_n\leqq\sum_{j=0}^{3}A_{j,n}.$$

Thus, we have

$$\frac{\|A_n\|_{C_B(\mathcal{D})}}{n} \leqq \sum_{j=0}^{3} \frac{\|A_{j,n}\|_{C_B(\mathcal{D})}}{n}. \tag{14}$$

Consequently, by Definition 2.6 and under the above assumption for the implication in (5)-(8), the right-hand side of (14) tends to zero $(n \to \infty)$. We, thus get

$$\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} - \lim_{n\to\infty} \|T_n(f(s,t);x,y) - f(x,y)\|_{C_B(\mathcal{D})} = 0.$$

Hence, the implication in (4) is true. This completes the proof of Theorem 3.1. ■

Remark 3.2.

If we substitute,

$\mathcal{A} = I$ (identity matrix), $s_n = 1$, $\lim_n q_n = 1$, $\lim_n p_n = 1$, $r = 0$, $a_n = 0$ and $b_n = n$ $(\forall\, n)$,

our Theorem 3.1 gives the statistical version of Korovkin type approximation theorem (Fast (1951)). Also, if we substitute

$$a_n = 0, \;\; b_n = n, \;\; \lim_n q_n = 1, \;\; \lim_n p_n = 1, \;\; \text{and} \;\; r = 0 \;\; (\forall\, n),$$

in our Theorem 3.1, then we obtain statistical weighted $\mathcal{A}$-summability version of Korovkin type approximation theorem (Mohiuddine (2016)).

We now present below an illustrative example for Theorem 3.1 by using (p,q)-analogue of Bernstein operators (for more details, see Mursaleen et al. (2015) for functions of two variables.

Example 3.3.

Let $I = [0,1]$ and for a function $f \in C_B(\mathcal{D})$ on $\mathcal{D} = I \times I$, we have the operators

$$\mathfrak{B}_{n,p,q}(f;x,y) = \sum_{u=0}^{n}\sum_{v=0}^{m} f\left(\frac{[u]_{p,q}}{p_{u-n}[n]_{p,q}}, \frac{[v]_{p,q}}{p_{v-m}[m]_{p,q}}\right) \mathfrak{B}_{u,n}(x)\mathfrak{B}_{v,m}(y), \tag{15}$$

where

$$\mathfrak{B}_{u,n}(x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} p^{\frac{u(u-1)}{2}} x^u \prod_{s=0}^{n-u-1} (p^s - q^s x),$$

and

$$\mathfrak{B}_{v,m}(y) = \frac{1}{p^{\frac{m(m-1)}{2}}} \begin{bmatrix} m \\ v \end{bmatrix}_{p,q} p^{\frac{v(v-1)}{2}} y^v \prod_{s=0}^{m-v-1} (p^s - q^s y).$$

Also, observe that

$$\mathfrak{B}_{n,p,q}(1;x,y) = 1, \quad \mathfrak{B}_{n,p,q}(s;x,y) = x, \quad \mathfrak{B}_{n,p,q}(t;x,y) = y,$$

and

$$\mathfrak{B}_{n,p,q}(s^2+t^2;x,y) = \frac{p^{n-1}}{[n]_{p,q}}x + \frac{p^{m-1}}{[m]_{p,q}}y + \frac{q[n-1]_{p,q}}{[n]_{p,q}}x^2 + \frac{q[m-1]_{p,q}}{[m]_{p,q}}y^2.$$

Now, upon considering the linear operators,

$$T_n : C_B(\mathcal{D}) \to C_B(\mathcal{D}),$$

such that

$$T_n(f; x, y) = (1 + x_n)\mathfrak{B}_{n,p_n,q_n}(f; x, y) \quad (0 < q_n < p_n \leq 1, \ \forall\ n \in \mathbb{N}), \tag{16}$$

where (x_n) be a sequence defined as in Example 2.8. Clearly, (T_n) satisfies the conditions (5)-(8) of our Theorem 3.1, thus we obtain

$$\text{stat } D(\bar{N})^{p;q}_{\Psi_\Delta} \lim_n \|T_n(1; x, y) - 1\|_{C_B(\mathcal{D})} = 0,$$
$$\text{stat } D(\bar{N})^{p;q}_{\Psi_\Delta} \lim_n \|T_n(s; x, y) - x\|_{C_B(\mathcal{D})} = 0,$$
$$\text{stat } D(\bar{N})^{p;q}_{\Psi_\Delta} \lim_n \|T_n(t; x, y) - y\|_{C_B(\mathcal{D})} = 0,$$
$$\text{stat } D(\bar{N})^{p;q}_{\Psi_\Delta} \lim_n \|T_n(s^2 + t^2; x, y) - (s^2 + t^2)\|_{C_B(\mathcal{D})} = 0.$$

Therefore, from Theorem 3.1, we have

$$\text{stat } D(\bar{N})^{p;q}_{\Psi_\Delta} \lim_n \|T_n(f(s,t); x, y) - f(x, y)\|_{C_B(\mathcal{D})} = 0, \quad f \in C_B(\mathcal{D}).$$

However, since (x_n) is not statistical weighted $\mathcal{A}$-summable, so the result of Mohiuddine (Mohiuddine (2016), p. 8, Theorem 3.1) does not hold true for our operators defined by (16). Moreover, since (x_n) is statistical deferred weighted $\mathcal{A}$-summable with respect to the difference operator of order r based on (p, q)-integers, therefore we conclude that our Theorem 3.1 works for the same operators.

4. Rate of the Deferred Weighted $\mathcal{A}$-statistical Convergence

We intend here to investigate the order of deferred weighted $\mathcal{A}$-statistical convergence of the sequence of positive linear operators for functions of two variables defined on $C_B(\mathcal{D})$ into itself under the modulus of continuity.

Definition 4.1.

Let $\mathcal{A} \in \mathcal{R}^+_{D(w)}$, r be a non-negative integer and let (a_n) and (b_n) be sequences of non-negative integers. Suppose, $0 < q_n < p_n \leq 1$ such that $\lim_n q_n = \alpha$ and $\lim_n p_n = \beta$ $(0 < \alpha, \beta \leq 1)$. Also let (u_n) be a positive non-decreasing sequence. Then the sequence (x_n) is deferred weighted $\mathcal{A}$-statistical convergent to a number L with rate $o(u_n)$ if, for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{u_n S_n} \sum_{m=a_n+1}^{b_n} \sum_{k \in K_\epsilon} s_m a_{m,k} = 0,$$

where

$$K_\epsilon = \{k : k \in \mathbb{N} \ \text{ and } \ |(\Delta^{[r]}_{p,q} x)_k - L| \geqq \epsilon\}.$$

Here, we write

$$x_n - L = \text{stat}^{p,q}_{\Psi_\Delta} - o(u_n).$$

We now need to prove the following lemma.

Lemma 4.2.

Let (u_n) and (v_n) be two positive non-decreasing sequences. Assume that $\mathcal{A} \in \mathcal{R}^+_{D(w)}$ and suppose (a_n) and (b_n) be sequences of non-negative integers, and let $x = (x_n)$ and $y = (y_n)$ be two sequences such that

$$x_n - L_1 = \text{stat}^{p,q}_{\Psi_\Delta} - o(u_n),$$

and

$$y_n - L_2 = \text{stat}^{p,q}_{\Psi_\Delta} = o(v_n).$$

Then each of the following assertions hold true:

(i) $(x_n - L_1) \pm (y_n - L_2) = \text{stat}^{p,q}_{\Psi_\Delta} - o(w_n)$,
(ii) $(x_n - L_1)(y_n - L_2) = \text{stat}^{p,q}_{\Psi_\Delta} - o(u_n v_n)$,
(iii) $\gamma(x_n - L_1) = \text{stat}^{p,q}_{\Psi_\Delta} - o(u_n)$ (for any scalar γ),
(iv) $\sqrt{|x_n - L_1|} = \text{stat}^{p,q}_{\Psi_\Delta} - o(u_n)$,

where $w_n = \max\{u_n, v_n\}$.

Proof:

For proving the assertion (i) of Lemma 4.2, we define here the following sets for $\epsilon > 0$ and $x \in \mathcal{D}$:

$$\mathcal{N}_n = \left|\left\{k : k \in \mathbb{N} \text{ and } |\left(\Delta^{[r]}_{p,q} x_k + \Delta^{[r]}_{p,q} y_k\right) - (L_1 + L_2)| \geqq \epsilon\right\}\right|,$$

$$\mathcal{N}_{0;n} = \left|\left\{k : k \in \mathbb{N} \text{ and } |\Delta^{[r]}_{p,q} x_k - L_1| \geqq \frac{\epsilon}{2}\right\}\right|,$$

and

$$\mathcal{N}_{1,n} = \left|\left\{k : k \in \mathbb{N} \text{ and } |\Delta^{[r]}_{p,q} y_k - L_2| \geqq \frac{\epsilon}{2}\right\}\right|.$$

Clearly, we have

$$\mathcal{N}_n \subseteq \mathcal{N}_{0,n} \cup \mathcal{N}_{1,n},$$

which implies, for $n \in \mathbb{N}$, that

$$\lim_{n\to\infty} \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} \sum_{k \in \mathcal{N}_n} s_m a_{m,k} \leqq \lim_{n\to\infty} \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} \sum_{k \in \mathcal{N}_{0,n}} s_m a_{m,k} + \lim_{n\to\infty} \frac{1}{S_n} \sum_{m=a_n+1}^{b_n} \sum_{k \in \infty, \mathcal{N}_n} s_m a_{m,k}. \quad (17)$$

Moreover, since

$$w_n = \max\{u_n, v_n\}, \quad (18)$$

by (17), we get

$$\lim_{n\to\infty}\frac{1}{w_nS_n}\sum_{m=a_n+1}^{b_n}\sum_{k\in\mathcal{N}_n}s_m a_{m,k} \leqq \lim_{n\to\infty}\frac{1}{u_nS_n}\sum_{m=a_n+1}^{b_n}\sum_{k\in\mathcal{N}_{0,n}}s_m a_{m,k} + \lim_{n\to\infty}\frac{1}{v_nS_n}\sum_{m=a_n+1}^{b_n}\sum_{k\in\mathcal{N}_{1,n}}s_m a_{m,k}. \tag{19}$$

Also, by applying Theorem 3.1, we obtain

$$\lim_{n\to\infty}\frac{1}{w_nS_n}\sum_{m=a_n+1}^{b_n}\sum_{k\in\mathcal{N}_n}s_m a_{m,k}=0. \tag{20}$$

Thus, assertion (i) of Lemma 4.2 is proved.

Next, as the assertions (ii) to (iv) of Lemma 4.2 are similar to (i), so these can be proved along similar lines to complete the proof of Lemma 4.2. ■

We now recall the modulus of continuity of a function of two variables $f(x,y)\in C_B(\mathcal{D})$ as,

$$\omega(f;\delta)=\sup_{(s,t),(x,y)\in\mathcal{D}}\left\{|f(s,t)-f(x,y)| : \sqrt{(s-x)^2+(t-y)^2}\leqq\delta\right\} \quad (\delta>0), \tag{21}$$

which implies

$$|f(s,t)-f(x,y)|\leqq\omega\left[f;\sqrt{(s-x)^2+(t-y)^2}\right]. \tag{22}$$

We now introduce a theorem to obtain the rates of deferred weighted $\mathcal{A}$-statistical convergence under the support of modulus of continuity in (21) .

Theorem 4.3.

Let $\mathcal{A}\in\mathcal{R}^+_{D(w)}$, (a_n) and (b_n) be sequences of non-negative integers, r be a non-negative integer, and let $0<q_n<p_n\leq 1$ such that $\lim_n q_n=\alpha$ and $\lim_n p_n=\beta$ $(0<\alpha,\beta\leq 1)$. Let $T_n : C_B(\mathcal{D})\to C_B(\mathcal{D})$ be the sequences of positive linear operators. Also let (u_n) and (v_n) are the positive non-decreasing sequences. Suppose that the following conditions are satisfied:

(i) $\|T_n(1;x,y)-1\|_{C_B(\mathcal{D})}=\text{stat}^{p,q}_{\Psi_\Delta}-o(u_n)$,

(ii) $\omega(f,\lambda_n)=\text{stat}^{p,q}_{\Psi_\Delta}-o(v_n)$ on $\mathcal{D}$,

where

$$\lambda_n=\sqrt{\|T_n(\varphi^2(s,t),x,y)\|_{C_B(\mathcal{D})}} \quad\text{and}\quad \varphi(s,t)=(s-x)^2+(t-y)^2.$$

Then, for all $f\in C_B(\mathcal{D})$, the following assertion holds true:

$$\|T_n(f;x,y)-f(x,y)\|_{C_B(\mathcal{D})}=\text{stat}^{p,q}_{\Psi_\Delta}-o(w_n), \tag{23}$$

where (w_n) is given by (18).

Proof:

Let $f \in C_B(\mathcal{D})$ and $(x,y) \in \mathcal{D}$. Using (22), we have

$$|T_n(f;x,y) - f(x,y)| \leqq T_n(|f(s,t) - f(x,y)|;x,y) + |f(x,y)||T_n(1;x,y) - 1|$$

$$\leqq T_n\left(\frac{\sqrt{(s-x)^2+(t-y)^2}}{\delta} + 1; x, y\right)\omega(f,\delta) + N|T_n(1;x,y) - 1|$$

$$\leqq \left(T_n(1;x,y) + \frac{1}{\delta^2}T_n(\varphi(s,t);x,y)\right)\omega(f,\delta) + N|T_n(1;x,y) - 1|,$$

where

$$\zeta = \|f\|_{C_B(\mathcal{D})}.$$

Now, taking the supremum over both sides, we have

$$\|T_n(f;x,y) - f(x,y)\|_{C_B(\mathcal{D})} \leqq \omega(f,\delta)\left\{\frac{1}{\delta^2}\|T_n(\varphi(s,t);x,y)\|_{C_B(\mathcal{D})} + \|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})} + 1\right\}$$

$$+\zeta\|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})}.$$

Now, putting $\delta = \lambda_n = \sqrt{T_n(\varphi^2;x,y)}$, we get

$$\|T_n(f;x,y) - f(x,y)\|_{C_B(\mathcal{D})} \leqq \omega(f,\lambda_n)\left\{\|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})} + 2\right\} + N\|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})}$$

$$\leqq \omega(f,\lambda_n)\|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})} + 2\omega(f,\lambda_n) + N\|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})}.$$

So, we have

$$\begin{aligned}\|T_n(f;x,y) - f(x,y)\|_{C_B(\mathcal{D})} \leqq \mu\{&\omega(f,\lambda_n)\|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})} \\ &+\omega(f,\lambda_n) + \|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})}\},\end{aligned}$$

where

$$\mu = \max\{2, N\}.$$

For a given $\epsilon > 0$, we choose the following sets:

$$\mathcal{H}_n = \left\{n : n \in \mathbb{N} \text{ and } \|T_n(f;x,y) - f(x,y)\|_{C_B(\mathcal{D})} \geqq \epsilon\right\}, \tag{24}$$

$$\mathcal{H}_{0,n} = \left\{n : n \in \mathbb{N} \text{ and } \omega(f,\lambda_n)\|T_n(f;x,y) - f(x,y)\|_{C_B(\mathcal{D})} \geqq \frac{\epsilon}{3\mu}\right\}, \tag{25}$$

$$\mathcal{H}_{1,n} = \left\{n : n \in \mathbb{N} \text{ and } \omega(f,\lambda_n) \geqq \frac{\epsilon}{3\mu}\right\}, \tag{26}$$

and

$$\mathcal{H}_{2,n} = \left\{n : n \in \mathbb{N} \text{ and } \|T_n(1;x,y) - 1\|_{C_B(\mathcal{D})} \geqq \frac{\epsilon}{3\mu}\right\}. \tag{27}$$

Finally, for conditions (i) and (ii) of Theorem 4.3 along with Lemma 4.2, the last inequalities (24)-(27) lead us to the assertion (23) of Theorem 4.3. The proof of Theorem 4.3 is thus completed. ■

5. Observations and Concluding Remarks

Here, in the last section of our study, we put forth some further concluding remarks and observations connecting to different outcomes which we have demonstrated here.

Remark 5.1.

Let $(x_n)_{n\in\mathbb{N}}$ be the sequence as considered in our Example 2.8. Then, since

$$\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} \lim_{n\to\infty} x_n \to 0 \text{ on } \quad C_{\mathcal{B}}(\mathcal{D}),$$

we have

$$\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} \lim_{n\to\infty} \|T_n(f_j;x,y) - f_j(x,y)\|_{C_{\mathcal{B}}(\mathcal{D})} = 0 \qquad (j = 0,1,2,3). \tag{28}$$

Therefore, by applying Theorem 3.1, we write

$$\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} \lim_{n\to\infty} \|T_n(f;x,y) - f(x,y)\|_{C_{\mathcal{B}}(\mathcal{D})} = 0, \qquad f \in C_{\mathcal{B}}(\mathcal{D}), \tag{29}$$

where

$$f_0(s,t) = 1, \;\; f_1(s,t) = s \;\; f_2(s,t) = t \text{ and } f_3(s,t) = s^2 + t^2.$$

However, since (x_n) is not ordinarily convergent and so also it does not converge uniformly in the ordinary sense. Thus, the traditional (ordinary) Korovkin Theorem is not working here for the operators defined under (16). Thus, clearly this outcome indicates that our Theorem 3.1 is a generalization (non - trivial) of the traditional Korovkin-type theorem (Korovkin (1960)).

Remark 5.2.

Let $(x_n)_{n\in\mathbb{N}}$ be the real sequence as considered in Example 2.8, then, since

$$\text{stat } D(\bar{N})^{p,q}_{\Psi_\Delta} \lim_{n\to\infty} x_n \to 0 \text{ on } C_{\mathcal{B}}(\mathcal{D}),$$

so (28) holds. Now by applying (28) and our Theorem 3.1, condition (29) holds. However, since (x_n) does not weighted $\mathcal{A}$-statistically convergent, so we can say that the result of Mohiuddine (2016, p. 8, Theorem 3.1) does not hold true for our operator defined in (16). Thus, our Theorem 3.1 is also an extension (non-trivial) of Mohiuddine (2016). Based upon the above results, it is concluded here that our proposed method has successfully worked for the operators defined in (16) and therefore it is stronger than the ordinary and the statistical version of the well established Korovkin type approximation theorems (Korovkin (1960); Mohiuddine (2016); Mursaleen et al. (2012)) established earlier.

Remark 5.3.

Suppose in Theorem 4.3, we substitute the conditions (i) and (ii) by the following condition:

$$|T_n(f_j;x,y) - f_j(x,y)|_{C_{\mathcal{B}}(\mathcal{D})} = \text{stat}^{p,q}_{\Psi_\Delta} - o(u_{n_j}) \qquad (j = 0,1,2,3). \tag{30}$$

Now, we can write

$$T_n(\varphi^2;x,y) = \mathcal{M} \sum_{j=0}^{3} \|T_n(f_j(s,t);x,y) - f_j(x,y)\|_{C_B(\mathcal{D})}, \tag{31}$$

where

$$\mathcal{M} = \left\{ \epsilon + M + \frac{2M}{\delta^2} \right\}, \quad (j = 0, 1, 2, 3).$$

It now follows from (30), (31) and Lemma 4.2 that

$$\lambda_n = \sqrt{T_n(\varphi^2)} = \text{stat}^{p;q}_{\Psi_\Delta} - o(d_n) \ \text{ on } \ C_{\mathcal{B}}(\mathcal{D}), \tag{32}$$

where

$$o(d_n) = \max\{u_{n_0}, u_{n_1}, u_{n_2}, u_{n_3}\}.$$

Thus, clearly, we obtain

$$\omega(f, \delta) = \text{stat}^{p;q}_{\Psi_\Delta} - o(d_n) \ \text{ on } \ C_{\mathcal{B}}(\mathcal{D}).$$

By applying (32) in Theorem 4.3, we instantly see that for all $f \in C_{\mathcal{B}}(\mathcal{D})$,

$$T_n(f; x, y) - f(x, y) = \text{stat}^{p,q}_{\Psi_\Delta} - o(d_n) \ \text{ on } \ C_{\mathcal{B}}(\mathcal{D}). \tag{33}$$

Therefore, instead of conditions (i) and (ii) in Theorem 4.3, if we use the condition (30), then we certainly find the rates of the deferred weighted $\mathcal{A}$-statistical convergence for the sequence (T_n) of positive linear operators in Theorem 3.1.

Remark 5.4.

In our present investigation, we have considered a number of fascinating special cases and illustrative examples in relevance to our definitions and also of the outcomes which have been established here.

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