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Analysis of Two Stage $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ Retrial G-queue with Discretionary Priority Services, Working Breakdown, Bernoulli Vacation, Preferred and Impatient Units

[1]G. Ayyappan and [2]B. Somasundaram

Department of Mathematics
Pondicherry Engineering College
Puducherry-605014, India
[1]ayyappanpec@hotmail.com; [2]somu.b92@gmail.com

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Abstract

In this paper, we study $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ retrial queueing system with discretionary priority services. There are two stages of service for the ordinary units. During the first stage of service of the ordinary unit, arriving priority units can have an option to interrupt the service, but, in the second stage of service it cannot interrupt. When ordinary units enter the system, they may get the service even if the server is busy with the first stage of service of an ordinary unit or may enter into the orbit or leave the system. Also, the system may breakdown at any point of time when the server is in regular service period. During the breakdown period, the interrupted priority unit will get the fresh service at a slower rate but the ordinary unit can not get the service and the server will go for repair immediately. During the ordinary unit service period, the arrival of negative unit will interrupt the service and it may enter into an orbit or leave the system. After completion of each priority unit's service, the server goes for a vacation with a certain probability. We allow reneging to happen during repair and vacation periods. Using the supplementary variable technique, the Laplace transforms of time-dependent probabilities of system state are derived. From this, we deduce the steady-state results. Also, the expected number of units in the respective queues and the expected waiting times, are computed. Finally, the numerical results are graphically expressed.

Keywords: Batch arrivals; Discretionary priority queues; Working breakdown; Negative arrival; Bernoulli vacation

MSC 2010 No.: 60K25, 68M30, 90B22

1. Introduction

In queueing system, retrial queues have been pretty impressive, in which the arriving units which find the server busy upon entry, leave the service space and attempt to get their service when the server is idle. In between, awaiting unit (blocked unit) which remains in a retrial group is said to be in orbit. Such a retrial queue plays a significant part in communication systems, computer networks, call center networks, etc.

Choudhury and Deka (2015) discussed $M^X/G/1$ retrial queueing system with two phases of service and Bernoulli vacation. Rajadurai et al. (2015) studied a batch arrival feedback G-retrial queue with two phases of service and server breakdown. Montazer-Haghighi and Mishev (2013) examined the three-stage hiring model with batch arrival and bulk service queueing system.

The priority queueing system has received significant attention in the history of queueing analysis. The priority disciplines in queuing systems can be classified into preemptive and non-preemptive. Under the non-preemptive discipline, in case of the arrival of priority units when an ordinary unit is being served, arriving priority unit will wait until the service is completed. Under the preemptive discipline, the arriving priority unit will always interrupt the ordinary unit's service. Rajadurai et al. (2016) considered an $M/G/1$ preemptive priority feedback retrial queue with working vacations and vacation interruption. More than that, in some situations both disciplines have been considered, which is termed as discretionary priority service. Fajardo and Drekic (2016) studied $M/G/1$ mixed priority queue with discretionary service.

The idea of working breakdown is introduced by Kalidass and Ramanath (2012), in which the server can provide the service during the breakdown period. Ayyappan and Udayageetha (2017) proposed $M^X/G/1$ mixed priority feedback retrial G-queue with two way-communication, working breakdown under Bernoulli vacation. Recently, Ammar and Rajadurai (2019) presented a priority retrial queueing system with disaster and working breakdown service.

Discretionary priority discipline was first proposed by Avi-Itzhak et al. (1964). Kim and Chae (2010) studied a discrete - time discretionary priority queueing system with a single-stage service. Zhao and Lian (2010) analyzed a two-stage $MAP/M/1$ discretionary priority queueing system in which the first stage assumes the preemptive and the second stage assumes non-preemptive service. Zhao et al. (2015) explained a two-stage $MAP/PH/1$ queue with discretionary priority service. Drekic and Woolford (2005) described the $M/M/1$ preemptive priority queue with balking. Montazer-Haghighi et al. (2013) investigated $M/M/c$ queueing system with balking and reneging. Wu et al. (2013) examined a discrete-time $Geo/G/1$ retrial queue with preferred and impatient units.

Our model has potential applications in computer networking systems. For example, the messages (positive units) arrive at the router (server) according to a Poisson process. The router may subject to breakdown during the service period and receive repair immediately. Such a system is affected by a virus (negative units), destroying the message in transmission. This destructed message may be put in the buffer (orbit) or may be cancelled for transmission.

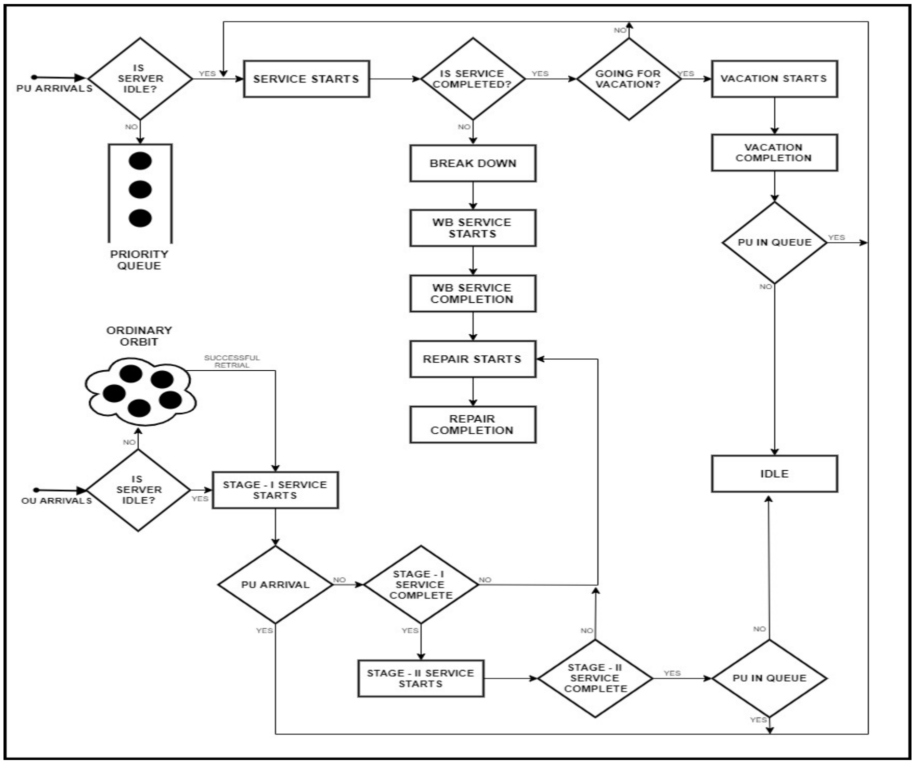

Figure 1. Schematic Diagram

In this article, we analyze a single server batch arrival discretionary priority-based retrial G-queue with two stages of service, working breakdown, Bernoulli vacation, preferred and impatient units. We assume that units arrive according to compound Poisson process in which priority units are assigned as type-1 units and ordinary units (retrial units) as type-2. The type-3 units are called as negative units. If the server is busy with the ordinary unit, the arriving type-3 unit will interrupt the service and remove the unit under service. The removed unit can enter the orbit or leave the system. The server provides two stages of service for ordinary units. During the first stage of service, arriving priority unit can interrupt or it will wait in the queue. But during the second stage of service it cannot interrupt. If the server is busy with the first stage of service at the ordinary unit arrival epoch, the arriving ordinary unit has an option to push them out and receive the service (preferred unit) or enter into an orbit or leave the system. Assume that the server is subject to active breakdowns with parameter α. When the server gets breakdown during priority unit's service period, it will complete the service at slower rate for current unit. But during an ordinary service period, it is sent for repair immediately. If there is breakdown during the ordinary unit's service period, it will wait for remaining service to complete. After each priority unit's service, the server has an option to go on vacation. The priority units leave the system after joining the queue due to server being on repair or vacation.

The article is prepared as follows. The representation of the mathematical model is stated in Section 2, equations describing the model and the time - dependent solutions are obtained in Section 3. The steady - state results are determined in Section 4. The expected queue size and expected waiting time are derived in Sections 5 and 6, sequentially. Remarkable particular cases are discussed in

Section 7 and in Section 8, numerical outcomes and their graphical illustrations are shown.

2. Mathematical Description of the Model

(1) Priority and ordinary units arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda_1\ c_i dt\ (i = 1, 2, 3, ...)$ and $\lambda_2\ c_j dt\ (j = 1, 2, 3, ...)$ be the first order probabilities that a batch of i and j units arrives at the system during a short interval of time $(t, t + dt)$, where $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$, $0 \leq c_j \leq 1$, $\sum_{j=1}^{\infty} c_j = 1$, and $\lambda_1 > 0$, $\lambda_2 > 0$ are the mean arrival rate for priority and ordinary units entering into the system.
(2) The retrial units are recognized as ordinary units. A new batch of ordinary units finds the server idle and one unit among the batch gets the service quickly and the remaining units join the orbit ultimately. An ordinary unit in the orbit always reverts to the orbit when it finds the server busy on its retrial attempt.
(3) Consider that the server renders two stages of service for ordinary units. The first stage of service can either be disrupted with probability r or continue the service with probability $(1 - r)$. But the second stage of service cannot be disrupted by the arrival of the priority unit.
(4) The negative unit appears according to a Poisson arrival rate λ^-. If the server is busy with the ordinary unit, the arriving negative unit interferes the service and eliminates the unit under the service. The removed unit either joins the orbit with probability q or leaves the system with probability $(1 - q)$.
(5) If an ordinary unit arrives during first stage of service, the arriving unit has an option to push out the unit in service and initiates its service with probability bp (this unit is called preferred unit) or enters the orbit with probability $b(1-p)$ or leaves the system with probability $(1-b)$.
(6) The system may become breakdown during the regular busy period and breakdowns are assumed to occur according to Poisson stream with parameter α. When the server gets breakdown during the priority unit's service period, the server will complete the service at a slower rate $\omega(\ell)$ for the current unit, but during the ordinary unit's service period, it is sent for repair instantly. The interrupted ordinary unit waits till the repair completion of the server to complete its remaining service.
(7) The priority units can decide to renege the queue during repair and vacation period and it follows exponentially with rate ξ.
(8) After completion of each priority service, the server either goes for a vacation with probability θ or serves the next unit with probability $(1 - \theta)$.

We defined the following notations:
$N_1(t)$ = The number of units in the priority queue at time t,
$N_2(t)$ = The number of units in the orbit at time t,
$Y(t)$ = The state of the server at time t.

In addition, let $M^0(t)$, $B_i^0(t)$, $W^0(t)$, $V^0(t)$ and $R_i^0(t)$, $i = 1, 2, 3$ be the elapsed time for retrial, service of priority unit, ordinary unit first stage, ordinary unit second stage, working breakdown, vacation and repair respectively at time t.

Assume that $M(0)=0$, $M(\infty)=1$, $B_i(0)=0$, $B_i(\infty)=1$, $W(0)=0$, $W(\infty)=1$, $V(0)=1$, $V(\infty)=1$ are continuous at $\ell=0$ and $R_i(0)=0$, $R_i(\infty)=1$ are continuous at $y=0$ for $i=1,2,3$.

Then the functions $\beta(\ell)$, $\mu_i(\ell)$, $\omega(\ell)$, $\gamma(\ell)$ and $\eta_i(\ell)$ are hazard rate for retrial, service of priority unit, ordinary unit first stage, ordinary unit second stage, working breakdown, vacation and repair respectively. i.e,

$$\beta(\ell)=\frac{dM(\ell)}{1-M(\ell)};\quad \mu_i(\ell)=\frac{dB_i(\ell)}{1-B_i(\ell)};\quad \omega(\ell)=\frac{dW(\ell)}{1-W(\ell)};$$
$$\gamma(\ell)=\frac{dV(\ell)}{1-V(\ell)};\quad \eta_i(\ell)=\frac{dR_i(\ell)}{1-R_i(\ell)}.$$

Further, in the multivariate Markov process $\{N_1(t),N_2(t),Y(t),t\geq 0\}, Y(t)$ denotes the server's state (0, 1, 2, 3, 4, 5, 6, 7, 8) depending on whether the server is free, busy with priority unit, busy with first stage ordinary unit, busy with second stage ordinary unit, working breakdown service, vacation, repair after working breakdown service, repair for ordinary first stage and repair for ordinary second stage respectively.

Now define the probability $I_{0,0}(t)=\Pr\{N_1(t)=0,N_2(t)=0,Y(t)=0\}$ and probability densities are as follows:

$$I_{0,n}(t,\ell)d\ell=\Pr\{N_1(t)=0,N_2(t)=n,Y(t)=0;\ell\leq M^0(t)<\ell+d\ell\},\ n\geq 1,$$

$$P^{(1)}_{m,n}(\ell,t)d\ell=\Pr\{N_1(t)=m,N_2(t)=n,Y(t)=1;\ell\leq B^0_1(t)<\ell+d\ell\},$$

$$P^{(2)}_{m,n}(\ell,t)d\ell=\Pr\{N_1(t)=m,N_2(t)=n,Y(t)=2;\ell\leq B^0_2(t)<\ell+d\ell\},$$

$$P^{(3)}_{m,n}(\ell,t)d\ell=\Pr\{N_1(t)=m,N_2(t)=n,Y(t)=3;\ell\leq B^0_3(t)<\ell+d\ell\},$$

$$Q^{(1)}_{m,n}(\ell,t)d\ell=\Pr\{N_1(t)=m,N_2(t)=n,Y(t)=4;\ell\leq W^0(t)<\ell+d\ell\},$$

$$V_{m,n}(\ell,t)d\ell=\Pr\{N_1(t)=m,N_2(t)=n,Y(t)=5;\ell\leq V^0(t)<\ell+d\ell\},$$

$$R^{(1)}_{m,n}(\ell,t)d\ell=\Pr\{N_1(t)=m,N_2(t)=n,Y(t)=6;\ell\leq R^0_1(t)<\ell+d\ell\},$$

$$R^{(2)}_{m,n}(\ell,y,t)dy=\Pr\{N_1(t)=m,N_2(t)=n,Y(t)=7;y\leq R^0_2(t)<y+dy/B^0_2(t)=\ell\},$$

$$R^{(3)}_{m,n}(\ell,y,t)dy=\Pr\{N_1(t)=m,N_2(t)=n,Y(t)=8;y\leq R^0_3(t)<y+dy/B^0_3(t)=\ell\},$$

for $\ell\geq 0$, $y\geq 0$ $t\geq 0$, $m\geq 0$ and $n\geq 0$.

3. Equation Governing the System

The set of differential - difference equations governing the system as follows:

The server is in idle state,

$$\begin{aligned}\frac{d}{dt}I_{0,0}(t) =& -(\lambda_1+\lambda_2)I_{0,0}(t) + (1-\theta)\int_0^\infty P_{0,0}^{(1)}(\ell,t)\mu_1(\ell)d\ell + \int_0^\infty P_{0,0}^{(3)}(\ell,t)\mu_3(\ell)d\ell \\ &+ \int_0^\infty R_{0,0}(\ell,t)\eta(\ell)d\ell + \int_0^\infty V_{0,0}(\ell,t)\gamma(\ell)d\ell \\ &+ \lambda^-(1-q)\Big\{\int_0^\infty P_{0,0}^{(2)}(\ell,t)d\ell + \int_0^\infty P_{0,0}^{(3)}(\ell,t)d\ell\Big\},\end{aligned} \tag{1}$$

The server is in retrial state,

$$\frac{\partial}{\partial t}I_{0,n}(\ell,t) + \frac{\partial}{\partial \ell}I_{0,n}(\ell,t) = -(\lambda_1+\lambda_2+\beta(\ell))I_{0,n}(\ell,t), \tag{2}$$

The server is providing priority service,

$$\begin{aligned}\frac{\partial}{\partial t}P_{m,n}^{(1)}(\ell,t) + \frac{\partial}{\partial \ell}P_{m,n}^{(1)}(\ell,t) =& -(\lambda_1+\lambda_2+\alpha+\mu_1(\ell))P_{m,n}^{(1)}(\ell,t) \\ &+ (1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_iP_{m-i,n}^{(1)}(\ell,t) + (1-\delta_{n0})\lambda_2\sum_{j=1}^{n}c_jP_{m,n-j}^{(1)}(\ell,t),\end{aligned} \tag{3}$$

The server is providing ordinary first stage service,

$$\begin{aligned}\frac{\partial}{\partial t}P_{m,n}^{(2)}(\ell,t) + \frac{\partial}{\partial \ell}P_{m,n}^{(2)}(\ell,t) =& -(\lambda_1+\lambda_2+\alpha+\lambda^-+\mu_2(\ell))P_{m,n}^{(2)}(\ell,t) \\ &+ (1-\delta_{m0})\lambda_1\overline{r}\sum_{i=1}^{m}c_iP_{m-i,n}^{(2)}(\ell,t) + (1-\delta_{n0})\lambda_2 b\overline{p}\sum_{j=1}^{n}c_jP_{m,n-j}^{(2)}(\ell,t) \\ &+ \lambda_2(1-b)P_{m,n}^{(2)}(\ell,t) + \int_0^\infty R_{m,n}^{(2)}(\ell,y,t)\eta_2(y)dy,\end{aligned} \tag{4}$$

The server is providing ordinary second stage service,

$$\begin{aligned}\frac{\partial}{\partial t}P_{m,n}^{(3)}(\ell,t) + \frac{\partial}{\partial \ell}P_{m,n}^{(3)}(\ell,t) =& -(\lambda_1+\lambda_2+\alpha+\lambda^-+\mu_3(\ell))P_{m,n}^{(3)}(\ell,t) \\ &+ (1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_iP_{m-i,n}^{(3)}(\ell,t) + (1-\delta_{n0})\lambda_2\sum_{j=1}^{n}c_jP_{m,n-j}^{(3)}(\ell,t) \\ &+ \int_0^\infty R_{m,n}^{(3)}(\ell,y,t)\eta_3(y)dy,\end{aligned} \tag{5}$$

The server is providing working breakdown service,

$$\frac{\partial}{\partial t}Q^{(1)}_{m,n}(\ell,t)+\frac{\partial}{\partial \ell}Q^{(1)}_{m,n}(\ell,t)=-(\lambda_1+\lambda_2+\omega(\ell))Q^{(1)}_{m,n}(\ell,t)$$
$$+(1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_iQ^{(1)}_{m-i,n}(\ell,t)+(1-\delta_{n0})\lambda_2\sum_{j=1}^{n}c_jQ^{(1)}_{m,n-j}(\ell,t), \tag{6}$$

The server is in repair process for priority,

$$\frac{\partial}{\partial t}R^{(1)}_{m,n}(\ell,t)+\frac{\partial}{\partial \ell}R^{(1)}_{m,n}(\ell,t)=-(\lambda_1+\lambda_2+\xi+\eta_1(\ell))R^{(1)}_{m,n}(\ell,t)+\xi R^{(1)}_{m+1,n}(\ell,t)$$
$$+(1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_iR^{(1)}_{m-i,n}(\ell,t)+(1-\delta_{n0})\lambda_2\sum_{j=1}^{n}c_jR^{(1)}_{m,n-j}(\ell,t), \tag{7}$$

The server is in repair process for ordinary first stage,

$$\frac{\partial}{\partial t}R^{(2)}_{m,n}(\ell,y,t)+\frac{\partial}{\partial \ell}R^{(2)}_{m,n}(\ell,y,t)=-(\lambda_1+\lambda_2+\xi+\eta_2(\ell))R^{(2)}_{m,n}(\ell,y,t)+\xi R^{(2)}_{m+1,n}(\ell,y,t)$$
$$+(1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_iR^{(2)}_{m-i,n}(\ell,y,t)+(1-\delta_{n0})\lambda_2\sum_{j=1}^{n}c_jR^{(2)}_{m,n-j}(\ell,y,t), \tag{8}$$

The server is in repair process for ordinary second stage,

$$\frac{\partial}{\partial t}R^{(3)}_{m,n}(\ell,y,t)+\frac{\partial}{\partial \ell}R^{(3)}_{m,n}(\ell,y,t)=-(\lambda_1+\lambda_2+\xi+\eta_3(\ell))R^{(3)}_{m,n}(\ell,y,t)+\xi R^{(3)}_{m+1,n}(\ell,y,t)$$
$$+(1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_iR^{(3)}_{m-i,n}(\ell,y,t)+(1-\delta_{n0})\lambda_2\sum_{j=1}^{n}c_jR^{(3)}_{m,n-j}(\ell,y,t), \tag{9}$$

The server is on vacation,

$$\frac{\partial}{\partial t}V_{m,n}(\ell,t)+\frac{\partial}{\partial \ell}V_{m,n}(\ell,t)=-(\lambda_1+\lambda_2+\xi+\gamma(\ell))V_{m,n}(\ell,t)+\xi V_{m+1,n}(\ell,t)$$
$$+(1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_iV_{m-i,n}(\ell,t)+(1-\delta_{n0})\lambda_2\sum_{j=1}^{n}c_jV_{m,n-j}(\ell,t). \tag{10}$$

The boundary conditions at $\ell=0$ are

$$I_{0,n}(0,t)=(1-\theta)\int_0^\infty P^{(1)}_{0,n}(\ell,t)\mu_1(\ell)d\ell+\int_0^\infty P^{(3)}_{0,n}(\ell,t)\mu_3(\ell)d\ell+\int_0^\infty R_{0,n}(\ell,t)\eta(\ell)d\ell$$
$$+\int_0^\infty V_{0,n}(\ell,t)\gamma(\ell)d\ell+\lambda^-(1-q)\Big\{\int_0^\infty P^{(2)}_{0,n}(\ell,t)d\ell+\int_0^\infty P^{(3)}_{0,n}(\ell,t)d\ell\Big\}$$
$$+\lambda^-q\Big\{\int_0^\infty P^{(2)}_{0,n-1}(\ell,t)d\ell+\int_0^\infty P^{(3)}_{0,n-1}(\ell,t)d\ell\Big\}, \tag{11}$$

$$
\begin{aligned}
P_{m,n}^{(1)}(0,t) &= \lambda_1 c_{m+1} I_{0,n}(t) + (1-\delta_{0,n})\lambda_1 r c_{m+1} \int_0^\infty P_{0,n-1}^{(2)}(\ell,t)d\ell \\
&\quad + (1-\theta)\int_0^\infty P_{m+1,n}^{(1)}(\ell,t)\mu_1(\ell)d\ell + \int_0^\infty P_{m+1,n}^{(3)}(\ell,t)\mu_3(\ell)d\ell \\
&\quad + \int_0^\infty V_{m+1,n}(\ell,t)\gamma(\ell)d\ell + \int_0^\infty R_{m+1,n}^{(1)}(\ell,t)\eta_1(\ell)d\ell, \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
P_{0,n}^{(2)}(0,t) &= \lambda_2 c_{n+1} I_{0,0}(t) + (1-\delta_{0,n})\lambda_2 bp \sum_{j=1}^{n} c_j \int_0^\infty P_{0,n-j}^{(2)}(\ell,t)d\ell \\
&\quad + \lambda_2 \sum_{j=1}^{n} c_j \int_0^\infty I_{0,n+1-j}(\ell,t)d\ell + \int_0^\infty I_{0,n+1}(\ell,t)\beta(\ell)d\ell, \qquad (13)
\end{aligned}
$$

$$
P_{0,n}^{(3)}(0,t) = \int_0^\infty P_{0,n}^{(2)}(\ell,t)\mu_2(\ell)d\ell, \qquad (14)
$$

$$
Q_{m,n}^{(1)}(0,t) = \alpha \int_0^\infty P_{m,n}^{(1)}(\ell,t)d\ell, \qquad (15)
$$

$$
R_{m,n}^{(1)}(0,t) = \int_0^\infty Q_{m,n}^{(1)}(\ell,t)\omega(\ell)d\ell, \qquad (16)
$$

$$
R_{m,n}^{(2)}(\ell,0,t) = \alpha \int_0^\infty P_{m,n}^{(2)}(\ell,t)d\ell, \qquad (17)
$$

$$
R_{m,n}^{(3)}(\ell,0,t) = \alpha \int_0^\infty P_{m,n}^{(3)}(\ell,t)d\ell, \qquad (18)
$$

$$
V_{m,n}(0,t) = \theta \int_0^\infty P_{m,n}^{(1)}(\ell,t)\mu_1(\ell)d\ell. \qquad (19)
$$

The initial conditions are

$$
\begin{gathered}
P_{m,n}^{(i)}(0) = R_{m,n}^{(i)}(0) = Q_{m,n}^{(1)}(0) = V_{m,n}(0) = 0, \\
i = 1,2,3,\ \ m,n \geq 0,\ I_{0,n}(0) = 0,\ \ n \geq 1 \ \text{and} \ I_{0,0}(0) = 1.
\end{gathered} \qquad (20)
$$

The Probability Generating Function (PGF) of this model:

$$
\begin{aligned}
I(\ell,t,z_2) &= \sum_{n=1}^{\infty} z_2^n I_{0,n}(\ell,t), \qquad A(\ell,t,z_1,z_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_1^m z_2^n A_{m,n}(\ell,t), \\
A(\ell,t,z_1) &= \sum_{m=0}^{\infty} z_1^m A_m(\ell,t), \qquad A(\ell,t,z_2) = \sum_{n=0}^{\infty} z_2^n A_n(\ell,t),
\end{aligned}
$$

where $A = P^{(i)},\ Q^{(1)},\ V,\ R^{(i)}$. Now taking Laplace transforms for Equation (1) to (19) and using (20), we get

$$\begin{aligned}(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s)-1 &= (1-\theta)\int_0^\infty \overline{P}^{(1)}_{0,0}(\ell,s)\mu_1(\ell)d\ell + \int_0^\infty \overline{P}^{(3)}_{0,0}(\ell,s)\mu_3(\ell)d\ell \\ &+ \int_0^\infty \overline{R}_{0,0}(\ell,s)\eta(\ell)d\ell + \int_0^\infty \overline{V}_{0,0}(\ell,s)\gamma(\ell)d\ell \\ &+ \lambda^-(1-q)\Big\{\int_0^\infty \overline{P}^{(2)}_{0,0}(\ell,t)d\ell + \int_0^\infty \overline{P}^{(3)}_{0,0}(\ell,s)d\ell\Big\},\end{aligned} \tag{21}$$

$$\frac{\partial}{\partial\ell}\overline{I}_{0,n}(\ell,s) + (s+\lambda_1+\lambda_2+\beta(\ell))\overline{I}_{0,n}(\ell,s) = 0, \tag{22}$$

$$\begin{aligned}&\frac{\partial}{\partial\ell}\overline{P}^{(1)}_{m,n}(\ell,s) + (s+\lambda_1+\lambda_2+\alpha+\mu_1(\ell))\overline{P}^{(1)}_{m,n}(\ell,s) \\ &\quad = (1-\delta_{m0})\lambda_1\sum_{i=1}^{m} c_i\overline{P}^{(1)}_{m-i,n}(\ell,s) + (1-\delta_{n0})\lambda_2\sum_{j=1}^{n} c_j\overline{P}^{(1)}_{m,n-j}(\ell,s),\end{aligned} \tag{23}$$

$$\begin{aligned}&\frac{\partial}{\partial\ell}\overline{P}^{(2)}_{m,n}(\ell,t) + (s+\lambda_1+\lambda_2 b+\alpha+\lambda^-+\mu_2(\ell))\overline{P}^{(2)}_{m,n}(\ell,s) = \int_0^\infty \overline{R}^{(2)}_{m,n}(\ell,y,s)\eta_2(y)dy \\ &\quad + (1-\delta_{m0})\lambda_1\overline{r}\sum_{i=1}^{m} c_i\overline{P}^{(2)}_{m-i,n}(\ell,s) + (1-\delta_{n0})\lambda_2 b\overline{p}\sum_{j=1}^{n} c_j\overline{P}^{(2)}_{m,n-j}(\ell,s),\end{aligned} \tag{24}$$

$$\begin{aligned}&\frac{\partial}{\partial\ell}\overline{P}^{(3)}_{m,n}(\ell,s) + (s+\lambda_1+\lambda_2+\alpha+\lambda^-+\mu_3(\ell))\overline{P}^{(3)}_{m,n}(\ell,s) = \int_0^\infty \overline{R}^{(3)}_{m,n}(\ell,y,s)\eta_3(y)dy \\ &\quad + (1-\delta_{m0})\lambda_1\sum_{i=1}^{m} c_i\overline{P}^{(3)}_{m-i,n}(\ell,s) + (1-\delta_{n0})\lambda_2\sum_{j=1}^{n} c_j\overline{P}^{(3)}_{m,n-j}(\ell,s),\end{aligned} \tag{25}$$

$$\begin{aligned}&\frac{\partial}{\partial\ell}\overline{Q}^{(1)}_{m,n}(\ell,s) + (s+\lambda_1+\lambda_2+\omega(\ell))\overline{Q}^{(1)}_{m,n}(\ell,s) \\ &\quad = (1-\delta_{m0})\lambda_1\sum_{i=1}^{m} c_i\overline{Q}^{(1)}_{m-i,n}(\ell,s) + (1-\delta_{n0})\lambda_2\sum_{j=1}^{n} c_j\overline{Q}^{(1)}_{m,n-j}(\ell,s),\end{aligned} \tag{26}$$

$$\begin{aligned}&\frac{\partial}{\partial\ell}\overline{R}^{(1)}_{m,n}(\ell,s) + (s+\lambda_1+\lambda_2+\xi+\eta_1(\ell))\overline{R}^{(1)}_{m,n}(\ell,s) = \xi\overline{R}^{(1)}_{m+1,n}(\ell,s) \\ &\quad + (1-\delta_{m0})\lambda_1\sum_{i=1}^{m} c_i\overline{R}^{(1)}_{m-i,n}(\ell,s) + (1-\delta_{n0})\lambda_2\sum_{j=1}^{n} c_j\overline{R}^{(1)}_{m,n-j}(\ell,s),\end{aligned} \tag{27}$$

$$\begin{aligned}&\frac{\partial}{\partial\ell}\overline{R}^{(2)}_{m,n}(\ell,y,s) + (s+\lambda_1+\lambda_2+\xi+\eta_2(\ell))\overline{R}^{(2)}_{m,n}(\ell,y,s) = \xi\overline{R}^{(2)}_{m+1,n}(\ell,y,s) \\ &\quad + (1-\delta_{m0})\lambda_1\sum_{i=1}^{m} c_i\overline{R}^{(2)}_{m-i,n}(\ell,y,s) + (1-\delta_{n0})\lambda_2\sum_{j=1}^{n} c_j\overline{R}^{(2)}_{m,n-j}(\ell,y,s),\end{aligned} \tag{28}$$

$$\frac{\partial}{\partial \ell}\overline{R}^{(3)}_{m,n}(\ell,y,s)+(s+\lambda_1+\lambda_2+\xi+\eta_3(\ell))\overline{R}^{(3)}_{m,n}(\ell,y,s)=\xi\overline{R}^{(3)}_{m+1,n}(\ell,y,s)$$
$$+(1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_i\overline{R}^{(3)}_{m-i,n}(\ell,y,s)+(1-\delta_{n0})\lambda_2\sum_{j=1}^{n}c_j\overline{R}^{(3)}_{m,n-j}(\ell,y,s), \tag{29}$$

$$\frac{\partial}{\partial \ell}\overline{V}_{m,n}(\ell,s)+(s+\lambda_1+\lambda_2+\xi+\gamma(\ell))\overline{V}_{m,n}(\ell,s)=\xi\overline{V}_{m+1,n}(\ell,s)$$
$$+(1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_i\overline{V}_{m-i,n}(\ell,s)+(1-\delta_{n0})\lambda_2\sum_{j=1}^{n}c_j\overline{V}_{m,n-j}(\ell,s), \tag{30}$$

$$\overline{I}_{0,n}(0,s)=(1-\theta)\int_0^\infty\overline{P}^{(1)}_{0,n}(\ell,s)\mu_1(\ell)d\ell+\int_0^\infty\overline{P}^{(3)}_{0,n}(\ell,s)\mu_3(\ell)d\ell+\int_0^\infty\overline{R}_{0,n}(\ell,s)\eta(\ell)d\ell$$
$$+\int_0^\infty\overline{V}_{0,n}(\ell,s)\gamma(\ell)d\ell+\lambda^-(1-q)\Big\{\int_0^\infty\overline{P}^{(2)}_{0,n}(\ell,s)d\ell+\int_0^\infty\overline{P}^{(3)}_{0,n}(\ell,s)d\ell\Big\}$$
$$+\lambda^-q\Big\{\int_0^\infty\overline{P}^{(2)}_{0,n-1}(\ell,s)d\ell+\int_0^\infty\overline{P}^{(3)}_{0,n-1}(\ell,s)d\ell\Big\}, \tag{31}$$

$$\overline{P}^{(1)}_{m,n}(0,s)=\lambda_1c_{m+1}\overline{I}_{0,n}(s)+(1-\delta_{0,n})\lambda_1rc_{m+1}\int_0^\infty\overline{P}^{(2)}_{0,n-1}(\ell,s)d\ell$$
$$+(1-\theta)\int_0^\infty\overline{P}^{(1)}_{m+1,n}(\ell,s)\mu_1(\ell)d\ell+\int_0^\infty\overline{P}^{(3)}_{m+1,n}(\ell,s)\mu_3(\ell)d\ell$$
$$+\int_0^\infty\overline{V}_{m+1,n}(\ell,s)\gamma(\ell)d\ell+\int_0^\infty\overline{R}^{(1)}_{m+1,n}(\ell,s)\eta_1(\ell)d\ell, \tag{32}$$

$$\overline{P}^{(2)}_{0,n}(0,s)=\lambda_2c_{n+1}\overline{I}_{0,0}(s)+(1-\delta_{0,n})\lambda_2bp\sum_{j=1}^{n}c_j\int_0^\infty\overline{P}^{(2)}_{0,n-j}(\ell,s)d\ell$$
$$+\lambda_2\sum_{j=1}^{n}c_j\int_0^\infty\overline{I}_{0,n+1-j}(\ell,s)d\ell+\int_0^\infty\overline{I}_{0,n+1}(\ell,s)\beta(\ell)d\ell, \tag{33}$$

$$\overline{P}^{(3)}_{0,n}(0,s)=\int_0^\infty\overline{P}^{(2)}_{0,n}(\ell,s)\mu_2(\ell)d\ell, \tag{34}$$

$$\overline{Q}^{(1)}_{m,n}(0,s)=\alpha\int_0^\infty\overline{P}^{(1)}_{m,n}(\ell,s)d\ell, \tag{35}$$

$$\overline{R}^{(1)}_{m,n}(0,s)=\int_0^\infty\overline{Q}^{(1)}_{m,n}(\ell,s)\omega(\ell)d\ell, \tag{36}$$

$$\overline{R}^{(2)}_{m,n}(\ell,0,s)=\alpha\int_0^\infty\overline{P}^{(2)}_{m,n}(\ell,s)d\ell, \tag{37}$$

$$\overline{R}^{(3)}_{m,n}(\ell,0,s)=\alpha\int_0^\infty\overline{P}^{(3)}_{m,n}(\ell,s)d\ell, \tag{38}$$

$$\overline{V}_{m,n}(0,s) = \theta \int_0^\infty \overline{P}^{(1)}_{m,n}(\ell,s)\mu_1(\ell)d\ell. \tag{39}$$

The equations from (22) to (30) are multiplied by z_2^n, summation over n ($n = 0$ to ∞) and using the PGF, we have

$$(\frac{\partial}{\partial \ell} + s + \lambda_1 + \lambda_2 + \beta(\ell))\overline{I}_0(\ell,s,z_2) = 0, \tag{40}$$

$$(\frac{\partial}{\partial \ell} + s + \lambda_1 + \lambda_2(1 - C(z_2)) + \alpha + \mu_1(\ell))\overline{P}^{(1)}_m(\ell,s,z_2) = (1-\delta_{m0})\lambda_1 \sum_{i=1}^{m} c_i \overline{P}^{(1)}_{m-i}(\ell,s,z_2), \tag{41}$$

$$\begin{aligned}(\frac{\partial}{\partial \ell} + s + \lambda_1 + \lambda_2 b(1 - \overline{p}C(z_2)) + \alpha + \lambda^- + \mu_2(\ell))\overline{P}^{(2)}_m(\ell,s,z_2)\\ = (1-\delta_{m0})\lambda_1 \overline{r} \sum_{i=1}^{m} c_i \overline{P}^{(2)}_{m-i}(\ell,s,z_2) + \int_0^\infty \overline{R}^{(2)}_m(\ell,y,s,z_2)\eta_2(y)dy,\end{aligned} \tag{42}$$

$$\begin{aligned}(\frac{\partial}{\partial \ell} + s + \lambda_1 + \lambda_2(1 - C(z_2)) + \alpha + \lambda^- + \mu_3(\ell))\overline{P}^{(3)}_m(\ell,s,z_2)\\ = (1-\delta_{m0})\lambda_1 \sum_{i=1}^{m} c_i \overline{P}^{(3)}_{m-i}(\ell,s,z_2) + \int_0^\infty \overline{R}^{(3)}_m(\ell,y,s,z_2)\eta_3(y)dy,\end{aligned} \tag{43}$$

$$(\frac{\partial}{\partial \ell} + s + \lambda_1 + \lambda_2(1 - C(z_2)) + \omega(\ell))\overline{Q}^{(1)}_m(\ell,s,z_2) = (1-\delta_{m0})\lambda_1 \sum_{i=1}^{m} c_i \overline{Q}^{(1)}_{m-i}(\ell,s,z_2), \tag{44}$$

$$\begin{aligned}(\frac{\partial}{\partial \ell} + s + \lambda_1 + \lambda_2(1 - C(z_2)) + \xi + \eta_1(\ell))\overline{R}^{(1)}_m(\ell,s,z_2)\\ = (1-\delta_{m0})\lambda_1 \sum_{i=1}^{m} c_i \overline{R}^{(1)}_{m-i}(\ell,s,z_2) + \xi \overline{R}^{(1)}_{m+1}(\ell,s,z_2),\end{aligned} \tag{45}$$

$$\begin{aligned}(\frac{\partial}{\partial \ell} + s + \lambda_1 + \lambda_2(1 - C(z_2)) + \xi + \eta_2(\ell))\overline{R}^{(2)}_m(\ell,s,z_2)\\ = (1-\delta_{m0})\lambda_1 \sum_{i=1}^{m} c_i \overline{R}^{(2)}_{m-i}(\ell,s,z_2) + \xi \overline{R}^{(2)}_{m+1}(\ell,s,z_2),\end{aligned} \tag{46}$$

$$\begin{aligned}(\frac{\partial}{\partial \ell} + s + \lambda_1 + \lambda_2(1 - C(z_2)) + \xi + \eta_3(\ell))\overline{R}^{(3)}_m(\ell,s,z_2)\\ = (1-\delta_{m0})\lambda_1 \sum_{i=1}^{m} c_i \overline{R}^{(3)}_{m-i}(\ell,s,z_2) + \xi \overline{R}^{(3)}_{m+1}(\ell,s,z_2),\end{aligned} \tag{47}$$

$$
\begin{aligned}
&(\frac{\partial}{\partial \ell}+s+\lambda_1+\lambda_2(1-C(z_2))+\xi+\gamma(\ell))\overline{V}_m(\ell,s,z_2)\\
&\qquad\qquad = (1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_i\overline{V}_{m-i}(\ell,s,z_2)+\xi\overline{V}_{m+1}(\ell,s,z_2).
\end{aligned}
\tag{48}
$$

The equations from (41) to (48) are multiplied by z_1^m, summation over m ($m = 0$ to ∞) and using PGF, we get

$$(\frac{\partial}{\partial \ell}+s+\lambda_1(1-C(z_1))+\lambda_2(1-C(z_2))+\alpha+\mu_1(\ell))\overline{P}^{(1)}(\ell,s,z_1,z_2)=0, \tag{49}$$

$$
\begin{aligned}
&(\frac{\partial}{\partial \ell}+s+\lambda_1(1-\overline{r}C(z_1))+\lambda_2 b(1-\overline{p}C(z_2))+\alpha+\lambda^-+\mu_2(\ell))\overline{P}^{(2)}(\ell,s,z_1,z_2)\\
&\qquad\qquad = \int_0^\infty \overline{R}^{(2)}(\ell,y,s,z_1,z_2)\eta_2(y)dy,
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
&(\frac{\partial}{\partial \ell}+s+\lambda_1(1-C(z_1))+\lambda_2(1-C(z_2))+\alpha+\lambda^-+\mu_3(\ell))\overline{P}^{(3)}(\ell,s,z_1,z_2)\\
&\qquad\qquad = \int_0^\infty \overline{R}^{(3)}(\ell,y,s,z_1,z_2)\eta_3(y)dy,
\end{aligned}
\tag{51}
$$

$$(\frac{\partial}{\partial \ell}+s+\lambda_1(1-C(z_1))+\lambda_2(1-C(z_2))+\omega(\ell))\overline{Q}^{(1)}(\ell,s,z_1,z_2)=0, \tag{52}$$

$$(\frac{\partial}{\partial \ell}+s+\lambda_1(1-C(z_1))+\lambda_2(1-C(z_2))+\xi(1-\frac{1}{z_1})+\eta_1(\ell))\overline{R}^{(1)}(\ell,s,z_1,z_2)=0, \tag{53}$$

$$(\frac{\partial}{\partial \ell}+s+\lambda_1(1-C(z_1))+\lambda_2(1-C(z_2))+\xi(1-\frac{1}{z_1})+\eta_2(\ell))\overline{R}^{(2)}(\ell,s,z_1,z_2)=0, \tag{54}$$

$$(\frac{\partial}{\partial \ell}+s+\lambda_1(1-C(z_1))+\lambda_2(1-C(z_2))+\xi(1-\frac{1}{z_1})+\eta_3(\ell))\overline{R}^{(3)}(\ell,s,z_1,z_2)=0, \tag{55}$$

$$(\frac{\partial}{\partial \ell}+s+\lambda_1(1-C(z_1))+\lambda_2(1-C(z_2))+\xi(1-\frac{1}{z_1})+\gamma(\ell))\overline{V}(\ell,s,z_1,z_2)=0. \tag{56}$$

Next, Equation (31) is multiplied by z_2^n and summation over n ($n = 1$ to ∞), and we obtain

$$
\begin{aligned}
\overline{I}_0(0,s,z_2) &= 1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}+(1-\theta)\int_0^\infty \overline{P}_0^{(1)}(\ell,s,z_2)\mu_1(\ell)d\ell\\
&\quad+\int_0^\infty \overline{P}_0^{(3)}(\ell,s,z_2)\mu_3(\ell)d\ell+\int_0^\infty \overline{R}_0(\ell,s,z_2)\eta(\ell)d\ell+\int_0^\infty \overline{V}_0(\ell,s,z_2)\gamma(\ell)d\ell\\
&\quad+\lambda^-(1-q(1-z_2))\Big\{\int_0^\infty \overline{P}_0^{(2)}(\ell,s,z_2)d\ell+\int_0^\infty \overline{P}_0^{(3)}(\ell,s,z_2)d\ell\Big\}.
\end{aligned}
\tag{57}
$$

Equation (33) is multiplied by z_2^{n+1} and summation over n ($n = 0$ to ∞). We have

$$z_2\overline{P}_0^{(2)}(0,s,z_2) = \lambda_2 C(z_2)\overline{I}_{0,0}(s) + \lambda_2 bpC(z_2)\int_0^\infty \overline{P}_0^{(2)}(\ell,s,z_2)d\ell + \int_0^\infty \overline{I}_0(\ell,s,z_2)\beta(\ell)d\ell + \lambda_2 C(z_2)\int_0^\infty \overline{I}_0(\ell,s,z_2)d\ell, \tag{58}$$

Equations (32) and (34) to (39) are multiplied by z_2^n and summation over n ($n = 0$ to ∞). We have

$$\begin{aligned}\overline{P}_m^{(1)}(0,s,z_2) &= \lambda_1 c_{m+1}\overline{I}_0(s,z_2) + \lambda_1 r c_{m+1} z_2\int_0^\infty \overline{P}_0^{(2)}(\ell,s,z_2)d\ell \\ &\quad + \int_0^\infty \overline{V}_{m+1}(\ell,s,z_2)\gamma(\ell)d\ell + (1-\theta)\int_0^\infty \overline{P}_{m+1}^{(1)}(\ell,s,z_2)\mu_1(\ell)d\ell \\ &\quad + \int_0^\infty \overline{P}_{m+1}^{(3)}(\ell,s,z_2)\mu_3(\ell)d\ell + \int_0^\infty \overline{R}_{m+1}^{(1)}(\ell,s,z_2)\eta_1(\ell)d\ell,\end{aligned} \tag{59}$$

$$\overline{P}_0^{(3)}(0,s,z_2) = \int_0^\infty \overline{P}_0^{(2)}(\ell,s,z_2)\mu_2(\ell)d\ell, \tag{60}$$

$$\overline{Q}_m^{(1)}(0,s,z_2) = \alpha\int_0^\infty \overline{P}_m^{(1)}(\ell,s,z_2)d\ell, \tag{61}$$

$$\overline{R}_m^{(1)}(0,s,z_2) = \int_0^\infty \overline{Q}_m^{(1)}(\ell,s,z_2)\omega(\ell)d\ell, \tag{62}$$

$$\overline{R}_m^{(2)}(\ell,0,s,z_2) = \alpha\int_0^\infty \overline{P}_m^{(2)}(\ell,s,z_2)d\ell, \tag{63}$$

$$\overline{R}_m^{(3)}(\ell,0,s,z_2) = \alpha\int_0^\infty \overline{P}_m^{(3)}(\ell,s,z_2)d\ell, \tag{64}$$

$$\overline{V}_m(0,s,z_2) = \theta\int_0^\infty \overline{P}_m^{(1)}(\ell,s,z_2)\mu_1(\ell)d\ell. \tag{65}$$

Now, Equation (59) is multiplied by z_1^{m+1} and summation over m ($m = 0$ to ∞). We have

$$\begin{aligned}z_1\overline{P}^{(1)}(0,s,z_1,z_2) &= \lambda_1 C(z_1)\overline{I}_0(s,z_2) + \lambda_1 rC(z_1)z_2\int_0^\infty \overline{P}_0^{(2)}(\ell,s,z_2)d\ell \\ &\quad + (1-\theta)\int_0^\infty \overline{P}^{(1)}(\ell,s,z_1,z_2)\mu_1(\ell)d\ell + \int_0^\infty \overline{P}^{(3)}(\ell,s,z_1,z_2)\mu_3(\ell)d\ell \\ &\quad + \int_0^\infty \overline{V}(\ell,s,z_1,z_2)\gamma(\ell)d\ell + \int_0^\infty \overline{R}^{(1)}(\ell,s,z_1,z_2)\eta_1(\ell)d\ell \\ &\quad - \Big\{(1-\theta)\int_0^\infty \overline{P}_0^{(1)}(\ell,s,z_2)\mu_1(\ell)d\ell + \int_0^\infty \overline{P}_0^{(3)}(\ell,s,z_2)\mu_3(\ell)d\ell \\ &\quad + \int_0^\infty \overline{V}_0(\ell,s,z_2)\gamma(\ell)d\ell + \int_0^\infty \overline{R}_0^{(1)}(\ell,s,z_2)\eta_1(\ell)d\ell\Big\},\end{aligned} \tag{66}$$

Equations (61), (62), (64) and (65) are multiplied by z_1^m, and summation over m ($m = 0$ to ∞). We have

$$\overline{Q}^{(1)}(0, s, z_1, z_2) = \alpha \int_0^{\infty} \overline{P}^{(1)}(\ell, s, z_1, z_2) d\ell, \tag{67}$$

$$\overline{R}^{(1)}(0, s, z_1, z_2) = \int_0^{\infty} \overline{Q}^{(1)}(\ell, s, z_1, z_2)\omega(\ell) d\ell, \tag{68}$$

$$\overline{R}^{(2)}(\ell, 0, s, z_1, z_2) = \alpha \int_0^{\infty} \overline{P}^{(2)}(\ell, s, z_1, z_2) d\ell, \tag{69}$$

$$\overline{R}^{(3)}(\ell, 0, s, z_1, z_2) = \alpha \int_0^{\infty} \overline{P}^{(3)}(\ell, s, z_1, z_2) d\ell, \tag{70}$$

$$\overline{V}(0, s, z_1, z_2) = \theta \int_0^{\infty} \overline{P}^{(1)}(\ell, s, z_1, z_2)\mu_1(\ell) d\ell. \tag{71}$$

Integration of Equations (40), (49) and (52) to (56) between 0 and ℓ give

$$\overline{I}_0(\ell, s, z_2) = \overline{I}_0(0, s, z_2) e^{-(s+\lambda_1+\lambda_2)\ell - \int_0^{\ell} \beta(t)dt}, \tag{72}$$

$$\overline{P}^{(1)}(\ell, s, z_1, z_2) = \overline{P}^{(1)}(0, s, z_1, z_2) e^{-\phi_1(s,z)\ell - \int_0^{\ell} \mu_1(t)dt}, \tag{73}$$

$$\overline{Q}^{(1)}(\ell, s, z_1, z_2) = \overline{Q}^{(1)}(0, s, z_1, z_2) e^{-\phi(s,z)\ell - \int_0^{\ell} \omega(t)dt}, \tag{74}$$

$$\overline{R}^{(1)}(\ell, s, z_1, z_2) = \overline{R}^{(1)}(0, s, z_1, z_2) e^{-\phi_4(s,z)\ell - \int_0^{\ell} \eta_1(t)dt}, \tag{75}$$

$$\overline{R}^{(2)}(\ell, y, s, z_1, z_2) = \overline{R}^{(2)}(\ell, 0, s, z_1, z_2) e^{-\phi_4(s,z)y - \int_0^{y} \eta_2(t)dt}, \tag{76}$$

$$\overline{R}^{(3)}(\ell, y, s, z_1, z_2) = \overline{R}^{(3)}(\ell, 0, s, z_1, z_2) e^{-\phi_4(s,z)y - \int_0^{y} \eta_3(t)dt}, \tag{77}$$

$$\overline{V}(\ell, s, z_1, z_2) = \overline{V}(0, s, z_1, z_2) e^{-\phi_4(s,z)\ell - \int_0^{\ell} \gamma(t)dt}. \tag{78}$$

We multiply Equations (72) to (78) by $\beta(\ell)$, $\mu_1(\ell)$, $\omega(\ell)$, $\eta_1(\ell)$, $\eta_2(\ell)$, $\eta_3(\ell)$ and $\gamma(\ell)$, respectively,

$$\int_0^{\infty} \overline{I}_0(\ell, s, z_2)\beta(\ell) d\ell = \overline{I}_0(0, s, z_2)\overline{M}(s + \lambda_1 + \lambda_2), \tag{79}$$

$$\int_0^{\infty} \overline{P}^{(1)}(\ell, s, z_1, z_2)\mu_1(\ell) d\ell = \overline{P}^{(1)}(0, s, z_1, z_2)\overline{B}_1(\phi_1(s, z)), \tag{80}$$

$$\int_0^{\infty} \overline{Q}^{(1)}(\ell, s, z_1, z_2)\omega(\ell) d\ell = \overline{Q}^{(1)}(0, s, z_1, z_2)\overline{W}(\phi(s, z)), \tag{81}$$

$$\int_0^{\infty} \overline{R}^{(1)}(\ell, s, z_1, z_2)\eta_1(\ell) d\ell = \overline{R}^{(1)}(0, s, z_1, z_2)\overline{R}_1(\phi_4(s, z)), \tag{82}$$

$$\int_0^\infty \overline{R}^{(2)}(\ell, y, s, z_1, z_2)\eta_2(y)dy = \overline{R}^{(2)}(\ell, 0, s, z_1, z_2)\overline{R}_2(\phi_4(s, z)), \tag{83}$$

$$\int_0^\infty \overline{R}^{(3)}(\ell, y, s, z_1, z_2)\eta_3(y)dy = \overline{R}^{(3)}(\ell, 0, s, z_1, z_2)\overline{R}_3(\phi_4(s, z)), \tag{84}$$

$$\int_0^\infty \overline{V}(\ell, s, z_1, z_2)\gamma(\ell)d\ell = \overline{V}(0, s, z_1, z_2)\overline{V}(\phi_4(s, z)), \tag{85}$$

where

$$\begin{aligned}
\phi(s, z) &= s + \lambda_1(1 - C(z_1)) + \lambda_2(1 - C(z_2)),\\
\phi_1(s, z) &= s + \lambda_1(1 - C(z_1)) + \lambda_2(1 - C(z_2)) + \alpha,\\
\phi_4(s, z) &= s + \lambda_1(1 - C(z_1)) + \lambda_2(1 - C(z_2)) + \xi\left(1 - \frac{1}{z_1}\right).
\end{aligned}$$

Now, using Equations (69) and (70), we get

$$\begin{aligned}
(\frac{\partial}{\partial \ell} + s + \lambda_1(1 - \overline{r}C(z_1)) + \lambda_2 b(1 - \overline{p}C(z_2)) + \alpha(1 - \overline{R}_2(\phi_4(s, z))) + \lambda^- + \mu_2(\ell))&\\
\times \overline{P}^{(2)}(\ell, s, z_1, z_2) = 0,&
\end{aligned} \tag{86}$$

$$\begin{aligned}
(\frac{\partial}{\partial \ell} + s + \lambda_1(1 - C(z_1)) + \lambda_2(1 - C(z_2)) + \alpha(1 - \overline{R}_3(\phi_4(s, z))) + \lambda^- + \mu_3(\ell))&\\
\times \overline{P}^{(3)}(\ell, s, z_1, z_2) = 0.&
\end{aligned} \tag{87}$$

Integration of Equations (86) and (87) between 0 to ℓ gives

$$\overline{P}^2(\ell, s, z_1, z_2) = \overline{P}_0^{(2)}(0, s, z_2)e^{-\phi_2(s,z)\ell - \int_0^\ell \mu_2(t)dt}, \tag{88}$$

$$\overline{P}^3(\ell, s, z_1, z_2) = \overline{P}_0^{(3)}(0, s, z_2)e^{-\phi_3(s,z)\ell - \int_0^\ell \mu_3(t)dt}. \tag{89}$$

Solving Equations (88) and (89), we get

$$\int_0^\infty \overline{P}^{(2)}(\ell, s, z_1, z_2)\mu_2(\ell)d\ell = \overline{P}_0^{(2)}(0, s, z_2)\overline{B}_2(\phi_2(s, z)), \tag{90}$$

$$\int_0^\infty \overline{P}^{(3)}(\ell, s, z_1, z_2)\mu_3(\ell)d\ell = \overline{P}_0^{(3)}(0, s, z_2)\overline{B}_3(\phi_3(s, z)), \tag{91}$$

at $z_1 = 0$,

$$\int_0^\infty \overline{P}_0^{(2)}(\ell, s, z_2)\mu_2(\ell)d\ell = \overline{P}_0^{(2)}(0, s, z_2)\overline{B}_2(\psi_2(s, z)), \tag{92}$$

$$\int_0^\infty \overline{P}_0^{(3)}(\ell, s, z_2)\mu_3(\ell)d\ell = \overline{P}_0^{(3)}(0, s, z_2)\overline{B}_3(\psi_3(s, z)), \tag{93}$$

where

$$\phi_2(s,z) = s + \lambda_1(1-\bar{r}C(z_1)) + \lambda_2 b(1-\overline{p}C(z_2)) + \alpha(1-\overline{R}_2(\phi_4(s,z))) + \lambda^-,$$

$$\phi_3(s,z) = s + \lambda_1(1-C(z_1)) + \lambda_2(1-C(z_2)) + \alpha(1-\overline{R}_3(\phi_4(s,z))) + \lambda^-,$$

$$\psi_2(s,z) = s + \lambda_1 + \lambda_2 b(1-\overline{p}C(z_2)) + \alpha(1-\overline{R}_2(\psi_4(s,z))) + \lambda^-,$$

$$\psi_3(s,z) = s + \lambda_1 + \lambda_2(1-C(z_2)) + \alpha(1-\overline{R}_3(\psi_4(s,z))) + \lambda^-,$$

$$\psi_4(s,z) = s + \lambda_1 + \lambda_2(1-C(z_2)).$$

Now substitute Equations (79) to (85) and (90) to (93) into (57), (58) and (66). We get

$$\begin{aligned}\overline{I}_0(0,s,z_2) &= 1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s) + \overline{P}_0^{(1)}(0,s,z_2)\Big\{\overline{B}_1(\psi_1(s,z))[1+\theta\overline{V}(\psi_4(s,z))\\ &-\theta] + \alpha[\frac{1-\overline{B}_1(\psi_1(s,z))}{\psi_1(s,z)}]\overline{W}(\psi_4(s,z))\overline{R}_1(\psi_4(s,z))\Big\} + \overline{P}_0^{(2)}(0,s,z_2)\\ &\times\Big\{\overline{B}_2(\psi_2(s,z))\overline{B}_3(\psi_3(s,z)) + \lambda^-(1-q(1-z_2))\big\{[\frac{1-\overline{B}_2(\psi_2(s,z))}{\psi_2(s,z)}]\\ &+\overline{B}_2(\psi_2(s,z))[\frac{1-\overline{B}_3(\psi_3(s,z))}{\psi_3(s,z)}]\big\}\Big\},\end{aligned} \tag{94}$$

$$\begin{aligned}\overline{P}^{(1)}(0,s,z_1,z_2)&\Big\{z_1-\overline{B}_1(\phi_1(s,z))[1-\theta+\theta\overline{V}(\phi_4(s,z))] - \alpha[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}]\\ &\times\overline{W}(\phi(s,z))\overline{R}_1(\phi_4(s,z))\Big\} = \lambda_1 C(z_1)\overline{I}_0(0,s,z_2)[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}]\\ &+\overline{P}_0^{(2)}(0,s,z_2)\Big\{\lambda_1 rC(z_1)z_2[\frac{1-\overline{B}_2(\psi_2(s,z))}{\psi_2(s,z)}] + \overline{B}_2(\psi_2(s,z))\\ &\times[\overline{B}_3(\phi_3(s,z)) - \overline{B}_3(\psi_3(s,z))]\Big\} - \overline{P}_0^{(1)}(0,s,z_2)\Big\{\overline{B}_1(\psi_1(s,z))[1-\theta\\ &+\theta\overline{V}(\psi_4(s,z))] + \alpha[\frac{1-\overline{B}_1(\psi_1(s,z))}{\psi_1(s,z)}]\overline{W}(\psi_4(s,z))\overline{R}_1(\psi_4(s,z))\Big\},\end{aligned} \tag{95}$$

$$\begin{aligned}\overline{P}_0^{(2)}(0,s,z_2)&\big\{z_2-\lambda_2 bpC(z_2)[\frac{1-\overline{B}_2(\psi_2(s,z))}{\psi_2(s,z)}]\big\} = \lambda_2 C(z_2)\overline{I}_{0,0}(s) + \overline{I}_0(0,s,z_2)\\ &\times\big\{\overline{M}(s+\lambda_1+\lambda_2) + \lambda_2 C(z_2)[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}]\big\}.\end{aligned} \tag{96}$$

We have to solve Equations (94), (95) and (96). Letting $z_1 = g(z_2)$ in (95), we get

$$\begin{aligned}
&\overline{P}_0^{(1)}(0,s,z_2)\Big\{\overline{B}_1(\psi_1(s,z))[1-\theta+\theta\overline{V}(\psi_4(s,z))]+\alpha[\frac{1-\overline{B}_1(\psi_1(s,z))}{\psi_1(s,z)}]\overline{W}(\psi_4(s,z)) \\
&\qquad \times \overline{R}_1(\psi_4(s,z))\Big\} = \lambda_1 C(g(z_2))\overline{I}_0(0,s,z_2)[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}] \\
&\qquad + \overline{P}_0^{(2)}(0,s,z_2)\Big\{\lambda_1 r C(z_1) z_2[\frac{1-\overline{B}_2(\psi_2(s,z))}{\psi_2(s,z)}] + \overline{B}_2(\psi_2(s,z)) \\
&\qquad \times [\overline{B}_3(\sigma_3(s,z)) - \overline{B}_3(\psi_3(s,z))]\Big\}. \qquad (97)
\end{aligned}$$

We substitute the above in (94), and we get

$$\begin{aligned}
&\overline{I}_0(0,s,z_2)\Big\{1-\lambda_1 C(g(z_2))[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}]\Big\} = 1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s) \\
&\qquad + \overline{P}_0^{(2)}(0,s,z_2)\Big\{\lambda_1 r C(g(z_2)) z_2[\frac{1-\overline{B}_2(\psi_2(s,z))}{\psi_2(s,z)}] + \overline{B}_2(\psi_2(s,z_2)) \\
&\qquad \times \overline{B}_3(\sigma_2(s,z)) + \lambda^-(1-q(1-z_2))\big\{[\frac{1-\overline{B}_2(\psi_2(s,z))}{\psi_2(s,z)}] + \overline{B}_2(\psi_2(s,z)) \\
&\qquad \times [\frac{1-\overline{B}_3(\psi_3(s,z))}{\psi_3(s,z)}]\big\}\Big\}. \qquad (98)
\end{aligned}$$

We substitute Equation (98) in (96), and we get

$$\overline{P}_0^{(2)}(0,s,z_2) = \frac{\left\{\begin{aligned} &\lambda_2 C(z_2)\overline{I}_{0,0}(s)\big\{1-\lambda_1 C(g(z_2))[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}]\big\} \\ &\quad + \big\{1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s)\big\}\big\{\overline{M}(s+\lambda_1+\lambda_2) \\ &\quad + \lambda_2 C(z_2)[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}]\big\} \end{aligned}\right\}}{\left\{\begin{aligned} &\big\{z_2-\lambda_2 bpC(z_2)[\frac{1-\overline{B}_2(\psi_2(s,z))}{\psi_2(s,z)}]\big\}\big\{1-\lambda_1 C(g(z_2)) \\ &\times [\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}]\big\} - T(s,z)\big\{\overline{M}(s+\lambda_1+\lambda_2) \\ &+ \lambda_2 C(z_2)[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}]\big\} \end{aligned}\right\}} \qquad (99)$$

We substitute Equation (98) into (97), and we get

$$\overline{I}_0(0,s,z_2)=\frac{\left\{\begin{array}{l}\{1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s)\}\{z_2-\lambda_2 bpC(z_2)[\frac{1-\overline{B}_2(\psi_2(s,z))}{\psi_2(s,z)}]\}\\ \quad +\lambda_2 C(z_2)\overline{I}_{0,0}(s)T(s,z)\end{array}\right\}}{\left\{\begin{array}{l}\{z_2-\lambda_2 bpC(z_2)[\frac{1-\overline{B}_2(\psi_2(s,z))}{\psi_2(s,z)}]\}\{1-\lambda_1 C(g(z_2))\\ \times[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}]\}-T(s,z)\{\overline{M}(s+\lambda_1+\lambda_2)\\ +\lambda_2 C(z_2)[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}]\}\end{array}\right\}}. \tag{100}$$

Finally substituting Equations (97), (99) and (100) in (95), we get

$$\overline{P}^{(1)}(0,s,z_1,z_2)=\frac{\left\{\begin{array}{l}\lambda_1(C(z_1)-C(g(z_2)))\overline{I}_0(0,s,z_2)[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}]\\ +\overline{P}_0^{(2)}(0,s,z_2)\{\lambda_1 r[C(z_1)-C(g(z_2))]z_2[\frac{1-\overline{B}_2(\psi_2(s,z))}{\psi_2(s,z)}]\\ +\overline{B}_2(\psi_2(s,z))[\overline{B}_3(\phi_3(s,z))-\overline{B}_3(\sigma_3(s,z))]\}\end{array}\right\}}{\left\{\begin{array}{l}z_1-[1-\theta+\theta\overline{V}(\phi_4(s,z))]\overline{B}_1(\phi_1(s,z))\\ -\alpha[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}]\overline{W}(\phi(s,z))\overline{R}_1(\phi_4(s,z))\end{array}\right\}}, \tag{101}$$

where

$$\begin{aligned}T(s,z)&=\lambda_1 rC(g(z_2))z_2[\frac{1-\overline{B}_2(\psi_2(s,z))}{\psi_2(s,z)}]\overline{B}_2(\psi_2(s,z))\overline{B}_3(\sigma_3(s,z))\\ &\qquad+\lambda^-(1-q(1-z_2))\{[\frac{1-\overline{B}_2(\psi_2(s,z))}{\psi_2(s,z)}]+\overline{B}_2(\psi_2(s,z))[\frac{1-\overline{B}_3(\psi_3(s,z))}{\psi_3(s,z)}]\}.\end{aligned}$$

The other boundary conditions are

$$\overline{P}_0^{(3)}(0,s,z_2)=\overline{P}_0^{(2)}(0,s,z_2)\overline{B}_2(\psi_2(s,z)), \tag{102}$$

$$\overline{Q}^{(1)}(0,s,z_1,z_2)=\alpha\overline{P}^{(1)}(0,s,z_1,z_2)[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}], \tag{103}$$

$$\overline{R}_1(0,s,z_1,z_2)=\alpha\overline{P}^{(1)}(0,s,z_1,z_2)[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}]\overline{W}(\phi(s,z)), \tag{104}$$

$$\overline{R}_2(0,s,z_1,z_2)=\alpha\overline{P}_0^{(2)}(0,s,z_1,z_2)[\frac{1-\overline{B}_2(\phi_2(s,z))}{\phi_2(s,z)}], \tag{105}$$

$$\overline{R}_3(0,s,z_1,z_2)=\alpha\overline{P}_0^{(2)}(0,s,z_1,z_2)\overline{B}_2(\psi_2(s,z))[\frac{1-\overline{B}_3(\phi_3(s,z))}{\phi_3(s,z)}], \tag{106}$$

$$\overline{V}(0,s,z_1,z_2)=\theta\overline{P}^{(1)}(0,s,z_1,z_2)\overline{B}_1(\phi_1(s,z)).$$

Theorem 3.1.

The probability generating function of the Laplace transforms of the number of units in the respective queue while the system was in regular service, working breakdown service, repair and vacation are given by

$$\overline{I}_0(s,z_2)=\overline{I}_0(0,s,z_2)\big[\frac{1-\overline{M}(s+\lambda_1+\lambda_2)}{s+\lambda_1+\lambda_2}\big], \tag{107}$$

$$\overline{P}^{(1)}(s,z_1,z_2)=\overline{P}^{(1)}(0,s,z_1,z_2)\big[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}\big], \tag{108}$$

$$\overline{P}^{(2)}(s,z_1,z_2)=\overline{P}_0^{(2)}(0,s,z_2)\big[\frac{1-\overline{B}_2(\phi_2(s,z))}{\phi_2(s,z)}\big], \tag{109}$$

$$\overline{P}^{(3)}(s,z_1,z_2)=\overline{P}_0^{(2)}(0,s,z_2)\overline{B}_2(\psi_2(s,z))\big[\frac{1-\overline{B}_3(\phi_3(s,z))}{\phi_3(s,z)}\big], \tag{110}$$

$$\overline{Q}^{(1)}(s,z_1,z_2)=\alpha\overline{P}^{(1)}(0,s,z_1,z_2)\big[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}\big]\big[\frac{1-\overline{W}(\phi(s,z))}{\phi(s,z)}\big], \tag{111}$$

$$\overline{R}_1(s,z_1,z_2)=\alpha\overline{P}^{(1)}(0,s,z_1,z_2)\big[\frac{1-\overline{B}_1(\phi_1(s,z))}{\phi_1(s,z)}\big]\overline{W}(\phi(s,z))\big[\frac{1-\overline{R}_1(\phi_4(s,z))}{\phi_4(s,z)}\big], \tag{112}$$

$$\overline{R}_2(s,z_1,z_2)=\alpha\overline{P}_0^{(2)}(0,s,z_2)\big[\frac{1-\overline{B}_2(\phi_2(s,z))}{\phi_2(s,z)}\big]\big[\frac{1-\overline{R}_2(\phi_4(s,z))}{\phi_4(s,z)}\big], \tag{113}$$

$$\overline{R}_3(s,z_1,z_2)=\alpha\overline{P}_0^{(2)}(0,s,z_2)\overline{B}_2(\psi_2(s,z))\big[\frac{1-\overline{B}_3(\phi_3(s,z))}{\phi_3(s,z)}\big]\big[\frac{1-\overline{R}_3(\phi_4(s,z))}{\phi_4(s,z)}\big], \tag{114}$$

$$\overline{V}(s,z_1,z_2)=\theta\overline{P}^{(1)}(0,s,z_1,z_2)\overline{B}_1(\phi_1(s,z))\big[\frac{1-\overline{V}(\phi_4(s,z))}{\phi_4(s,z)}\big]. \tag{115}$$

Proof:

Integrating Equations (79) to (85), (88) and (89) with respect to ℓ and using the well known result of renewal theory

$$\int_0^\infty \big[1-H(\ell)\big]e^{-s\ell}d\ell=\frac{1-\overline{h(s)}}{s}, \tag{116}$$

where $\overline{h(s)}$ is the LST of the distribution function of a random variable $H(\ell)$, we get the results (107) to (115) respectively. Thus, we obtain the complete solution for the probability generating functions for the following states $\overline{I}_{(0)}(z_2,s)$, $\overline{P}^{(i)}(z_1,z_2,s)$, $\overline{Q}^{(1)}(z_1,z_2,s)$, $\overline{R}_i(z_1,z_2,s)$, and $\overline{V}(z_1,z_2,s)$. ∎

4. Steady State Analysis

By applying the well-known Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s) = \lim_{t\to\infty} f(t),$$

the normalizing condition of this model is

$$\sum_{i=1}^{3}\big\{P^{(i)}(1,1)+R_i(1,1)\big\}+Q^{(1)}(1,1)+V(1,1)+I_0(1)+I_{0,0}=1.$$

The probability generating function of the queue size irrespective of the state of the system

$$W_q(z_1,z_2)=\sum_{i=1}^{3}\big\{P^{(i)}(z_1,z_2)+R_i(z_1,z_2)\big\}+Q^{(1)}(z_1,z_2)+V(z_1,z_2)+I_0(z_2), \tag{117}$$

$$W_q(z_1,z_2)=I_{0,0}\frac{Nr(z_1,z_2)}{Dr(z_1,z_2)}, \tag{118}$$

where

$$\begin{aligned}Nr(z_1,z_2)&=N_1(z)\psi_2(z)\phi_2(z)\phi_3(z)[\frac{1-\overline{M}(\lambda_1+\lambda_2)}{\lambda_1+\lambda_2}]\big\{\lambda_1\big(C(z_1)-C(g(z_2))\big)F(z)\\&\quad+\phi(z)\phi_4(z)S_1(z)\big\}+N_2(z)\big\{\phi_2(z)\phi_3(z)S_2(z)F(z)+\phi(z)S_3(z)S_1(z)\big\},\end{aligned}$$

$$Dr(z_1,z_2)=\psi_2(z)\phi_2(z)\phi_3(z)\phi_4(z)\phi(z)S_1(z),$$

$$\begin{aligned}F(z)&=[1-\overline{B}_1(\phi_1(z))]\Big\{\phi(z)\phi_4(z)+\alpha\phi_4(z)[1-\overline{W}(\phi(z))]+\alpha\phi(z)\overline{W}(\phi(z))\\&\quad\times[1-\overline{R}_1(\phi_4(z))]\Big\}+\theta\phi(z)\phi_1(z)\overline{B}_1(\phi_1(z))[1-\overline{V}(\phi_4(z))],\end{aligned}$$

$$\begin{aligned}S_1(z)&=z_1\phi_1(z)-\phi_1(z)\overline{B}_1(\phi_1(z))[1-\theta+\theta\overline{V}(\phi_4(z))]-\alpha[1-\overline{B}_1(\phi_1(z))]\overline{W}(\phi(z))\\&\quad\times\overline{R}_1(\phi_4(z)),\end{aligned}$$

$$\begin{aligned}S_2(z)&=\lambda_1 r\big(C(z_1)-C(g(z_2))\big)z_2[1-\overline{B}_2(\psi_2(z))]+\psi_2(z)\overline{B}_2(\psi_2(z))[\overline{B}_3(\phi_3(z))\\&\quad-\overline{B}_3(\sigma_3(z))],\end{aligned}$$

$$\begin{aligned}S_3(z)&=\psi_2(z)\phi_3(z)\phi_4(z)[1-\overline{B}_2(\phi_2(z))]+\psi_2(z)\phi_2(z)\phi_4(z)\overline{B}_2(\psi_2(z))[1-\overline{B}_3(\phi_3(z))]\\&\quad+\alpha\phi_2(z)\phi_3(z)[1-\overline{B}_2(\psi_2(z))][1-\overline{R}_2(\phi_4(z))]+\alpha\psi_2(z)\phi_2(z)\overline{B}_2(\psi_2(z))\\&\quad\times[1-\overline{B}_3(\psi_3(z))][1-\overline{R}_3(\phi_4(z))].\end{aligned}$$

Now using the normalizing condition, we get

$$I_{0,0} = \frac{\left\{\begin{array}{l} \psi_2(1)(1)\lambda^-(\lambda_1 r + \lambda_2 bp + \lambda^-)D(1)\Big\{\alpha\big(1+\theta\phi_4^-(1)\overline{B}_1(\alpha)E(V)\big) \\ +[1-\overline{B}_1(\alpha)]\big(\phi_1'(1)+\alpha(\phi'(1)E(W)+\phi_4'(1)E(R_1))\big)\Big\} \end{array}\right\}}{\left\{\begin{array}{l} \psi_2(1)\lambda^-(\lambda_1 r + \lambda_2 bp + \lambda^-)\big\{D(1)+N_1(1)[\frac{\overline{M}(\lambda_1+\lambda_2)}{\lambda_1+\lambda_2}]\big\}\Big\{\alpha\big(1+\theta\phi_4'(1) \\ \times\overline{B}_1(\alpha)E(V)\big)+[1-\overline{B}_1(\alpha)]\big(\phi_1'(1)+\alpha(\phi'(1)E(W)+\phi_4'(1)E(R_1))\big)\Big\} \\ +\lambda^-(\lambda_1 r+\lambda_2 bp+\lambda^-)\Big\{N_1(1)\psi_2(1)\lambda_1 r(1-E(X_1))E(X)[\frac{\overline{M}(\lambda_1+\lambda_2)}{\lambda_1+\lambda_2}] \\ +N_2(1)\big\{\lambda_1 r(1-E(X_1))E(X)[1-\overline{B}_2(\psi_2(1))]+\psi_2(1)\overline{B}_2(\psi_2(1))\overline{B}_3'(\lambda^-) \\ \times(\phi_3'(1)-\sigma_3'(1))\big\}\Big\}\Big\{[1-\overline{B}_1(\alpha)]\big(1+\alpha(E(W)+E(R_1))\big) \\ +\theta\alpha\overline{B}_1(\alpha)E(V)\Big\}+N_2(1)\Big\{\psi_2(1)\lambda^-[1-\overline{B}_2(\lambda_1 r+\lambda_2 bp+\lambda^-)] \\ +(\lambda_1 r+\lambda_2 bp+\lambda^-)\big\{\psi_2(1)\overline{B}_2(\psi_2(1))[1-\overline{B}_3(\lambda^-)](1+\alpha E(R_3)) \\ +\alpha\lambda^-[1-\overline{B}_2(\psi_2(1))]E(R_2)\big\}\Big\} \end{array}\right\}}, \quad (119)$$

and the utilization factor is given by

$$\rho = \frac{\left\{\begin{array}{l} \lambda^-(\lambda_h r+\lambda_l bp+\lambda^-)\Big\{N_1(1)\psi_2(1)\lambda_h r(1-E(X_1))E(X)[\frac{\overline{M}(\lambda_h+\lambda_l)}{\lambda_h+\lambda_l}] \\ +N_2(1)\big\{\lambda_h r(1-E(X_1))E(X)[1-\overline{B}_2(\psi_2)]+\psi_2(1)\overline{B}_2(\psi_2(1))\overline{B}_3'(\lambda^-) \\ \times(\phi_3'(1)-\sigma_3'(1))\big\}\Big\}\Big\{[1-\overline{B}_1(\alpha)]\big(1+\alpha(E(W)+E(R_1))\big)+\theta\alpha\overline{B}_1(\alpha) \\ \times E(V)\Big\}+N_2(1)\Big\{\psi_2(1)\lambda^-[1-\overline{B}_2(\lambda_h r+\lambda_l bp+\lambda^-)]+(\lambda_h r+\lambda_l bp \\ +\lambda^-)\big\{\psi_2(1)\overline{B}_2(\psi_2(1))[1-\overline{B}_3(\lambda^-)](1+\alpha E(R_3))+\alpha\lambda^-[1-\overline{B}_2(\psi_2(1))] \\ \times E(R_2)\big\}\Big\} \end{array}\right\}}{\left\{\begin{array}{l} \psi_2(1)\lambda^-(\lambda_1 r + \lambda_2 bp + \lambda^-)\big\{D(1)+N_1(1)[\frac{\overline{M}(\lambda_1+\lambda_2)}{\lambda_1+\lambda_2}]\big\}\Big\{\alpha\big(1+\theta\phi_4'(1) \\ \times\overline{B}_1(\alpha)E(V)\big)+[1-\overline{B}_1(\alpha)]\big(\phi_1'(1)+\alpha(\phi'(1)E(W)+\phi_4'(1)E(R_1))\big)\Big\} \\ +\lambda^-(\lambda_1 r+\lambda_2 bp+\lambda^-)\Big\{N_1(1)\psi_2(1)\lambda_1 r(1-E(X_1))E(X)[\frac{\overline{M}(\lambda_1+\lambda_2)}{\lambda_1+\lambda_2}] \\ +N_2(1)\big\{\lambda_1 r(1-E(X_1))E(X)[1-\overline{B}_2(\psi_2(1))]+\psi_2(1)\overline{B}_2(\psi_2(1))\overline{B}_3'(\lambda^-) \\ \times(\phi_3'(1)-\sigma_3'(1))\big\}\Big\}\Big\{[1-\overline{B}_1(\alpha)]\big(1+\alpha(E(W)+E(R_1))\big)+\theta\alpha\overline{B}_1(\alpha) \\ \times E(V)\Big\}+N_2(1)\Big\{\psi_2(1)\lambda^-[1-\overline{B}_2(\lambda_1 r+\lambda_2 bp+\lambda^-)]+(\lambda_1 r+\lambda_2 bp \\ +\lambda^-)\big\{\psi_2(1)\overline{B}_2(\psi_2(1))[1-\overline{B}_3(\lambda^-)](1+\alpha E(R_3))+\alpha\lambda^-[1-\overline{B}_2(\psi_2(1))] \\ \times E(R_2)\big\}\Big\} \end{array}\right\}}, \quad (120)$$

where $\rho < 1$ is the stability condition under which steady state exist, for the model.

5. The Expected Queue Lengths

The expected priority queue size

$$L_{q_1} = \frac{d}{dz_1} W_q(z_1, 1)|_{z_1=1} \ , \tag{121}$$

and the expected ordinary orbit size

$$L_{q_2} = \frac{d}{dz_2} W_q(1, z_2)|_{z_2=1} \ , \tag{122}$$

then,

$$L_{q_1} = \frac{DR'''(1)NR^{(iv)}(1) - DR^{(iv)}(1)NR'''(1)}{4(DR'''(1))^2}, \tag{123}$$

$$L_{q_2} = \frac{dr'''(1)nr^{(iv)}(1) - dr^{(iv)}(1)nr'''(1)}{4(dr'''(1))^2}, \tag{124}$$

where

$$\begin{aligned} NR'''(1) = & N_1(1)\psi_2(1)[\frac{1-\overline{M}(\lambda_1+\lambda_2)}{\lambda_1+\lambda_2}]\Big\{\lambda^-(\lambda_1 r + \lambda_2 bp + \lambda^-)\big\{3\lambda_1 r E(X_1) F^{''}(1) \\ & + 6\overline{\phi}^{'}(1)\overline{\phi}_4^{'}(1)S_1^{'}(1)\big\}\Big\} + N_2(1)\Big\{3\lambda^-(\lambda_1 r + \lambda_2 bp + \lambda^-)F^{''}(1)S_2^{'}(1) \\ & + 6\overline{\phi}^{'}(1)S_1^{'}(1)S_3^{'}(1)\Big\}, \end{aligned}$$

$$\begin{aligned} NR^{(iv)}(1) = & N_1(1)\psi_2(1)[\frac{1-\overline{M}(\lambda_1+\lambda_2)}{\lambda_1+\lambda_2}]\Big\{4\big((\lambda_1 r + \lambda_2 bp + \lambda^-)\overline{\phi}_3^{'}(1) + \lambda^-\overline{\phi}_2^{'}(1)\big) \\ & \times\big\{3\lambda_1 r E(X_1)F^{''}(1) + 6\overline{\phi}^{'}(1)\overline{\phi}_4^{'}(1)S_1^{'}(1)\big\} + \lambda^-(\lambda_1 r + \lambda_2 bp + \lambda^-)\big\{2\lambda_1 r \\ & \times\big(3E(X^2)F^{''}(1) + 2E(X)F^{'''}(1)\big) + 12\big(\overline{\phi}^{''}(1)\overline{\phi}_4^{'}(1) + \overline{\phi}^{'}(1)\overline{\phi}_4^{''}(1)\big)S_1^{'}(1) \\ & + 12\overline{\phi}^{'}(1)\overline{\phi}_4^{'}(1)S_1^{''}(1)\big\}\Big\} + N_2(1)\Big\{12\big((\lambda_1 r + \lambda_2 bp + \lambda^-)\overline{\phi}_3^{'}(1) + \lambda^-\overline{\phi}_2^{'}(1)\big) \\ & \times S_2^{'}(1)F^{''}(1) + 2\lambda^-(\lambda_1 r + \lambda_2 bp + \lambda^-)\big\{3S_2^{''}(1)F^{''}(1) + 2S_2^{'}(1)F^{'''}(1)\big\} \\ & + 12\overline{\phi}^{''}S_1^{'}(1)S_3^{'}(1) + 12\overline{\phi}^{'}(1)\big(S_1^{''}(1)S_3^{'} + S_1^{'}(1)S_3^{''}(1)\big)\Big\}, \end{aligned}$$

$$DR'''(1) = 6\lambda^-(\lambda_1 r + \lambda_2 bp + \lambda^-)\psi_2(1)\overline{\phi}^{'}(1)\overline{\phi}_4^{'}(1)D(1)S_1^{'}(1),$$

$$\begin{aligned} DR^{(iv)}(1) = & \psi_2(1)D(1)\Big\{24\big((\lambda_1 r + \lambda_2 bp + \lambda^-)\overline{\phi}_3^{'}(1) + \lambda^-\overline{\phi}_2^{'}(1)\big)\overline{\phi}^{'}(1)\overline{\phi}_4^{'}(1)S_1^{'}(1) \\ & + 12\lambda^-(\lambda_1 r + \lambda_2 bp + \lambda^-)\big\{\big(\overline{\phi}^{''}(1)\overline{\phi}_4^{'}(1) + \overline{\phi}^{'}(1)\overline{\phi}_4^{''}(1)\big)S_1^{'}(1) + \overline{\phi}^{'}(1)\overline{\phi}^{'}(1) \\ & \times S_1^{''}(1)\big\}\Big\}, \end{aligned}$$

$$F^{''}(1) = 2\overline{\phi}^{'}(1)\overline{\phi}_4^{'}(1)\Big\{[1-\overline{B}_1(\alpha)]\big(1+\alpha(E(W)+E(R_1))\big)+\theta\alpha\overline{B}_1(\alpha)E(V)\Big\},$$

$$\begin{aligned} F^{'''}(1) = {} & 6\overline{\phi}^{'}(1)\overline{\phi}_1^{'}(1)\overline{\phi}_4^{'}(1)\big\{-\overline{B}_1^{'}(1)\big(1+\alpha(E(W)+E(R_1))\big)+\theta[\overline{B}_1(\alpha)+\alpha\overline{B}_1^{'}(\alpha)] \\ & \times E(V)\big\}+3\big(\overline{\phi}^{''}(1)\overline{\phi}_4^{'}(1)+\overline{\phi}^{'}(1)\overline{\phi}_4^{''}(1)\big)\Big\{[1-\overline{B}_1(\alpha)]\big(1+\alpha(E(W)+E(R_1))\big) \\ & +\theta\alpha\overline{B}_1(\alpha)E(V)\Big\}-3\alpha\Big\{[1-\overline{B}_1(\alpha)]\big\{(\overline{\phi}^{'}(1))^2\overline{\phi}_4^{'}(1)\big(E(W^2)+2E(W)E(R_1)\big) \\ & +\overline{\phi}^{'}(1)(\overline{\phi}_4^{'}(1))^2E(R_1^2)\big\}+\theta\overline{\phi}^{'}(1)(\overline{\phi}_4^{'}(1))^2\overline{B}_1(\alpha)E(V^2)\Big\}, \end{aligned}$$

$$S_1^{'}(1) = \alpha+[1-\overline{B}_1(\alpha)]\big\{\overline{\phi}_1^{'}(1)+\alpha\big(\overline{\phi}^{'}(1)E(W)+\overline{\phi}_4^{'}(1)E(R_1)\big)\big\}+\alpha\theta\overline{\phi}_4^{'}(1)\overline{B}_1(\alpha)E(V),$$

$$\begin{aligned} S_1^{''}(1) = {} & 2\overline{\phi}_1^{'}(1)[1-\overline{\phi}_1^{'}(1)\overline{B}_1^{'}(\alpha)]+\overline{\phi}_1^{''}(1)[1-\overline{B}_1(\alpha)]+\alpha\big\{\theta\overline{B}_1(\alpha)\overline{\phi}_4^{''}(1)E(V) \\ & +[1-\overline{B}_1(\alpha)]\big(\overline{\phi}^{''}(1)E(W)+\overline{\phi}_4^{''}(1)E(R_1)\big)\big\}-\alpha\theta\overline{B}_1(\alpha)(\overline{\phi}_4^{'}(1))^2E(V^2) \\ & -\alpha[1-\overline{B}_1(\alpha)]\big\{(\overline{\phi}^{'}(1))^2E(W^2)+(\overline{\phi}_4^{'}(1))^2E(R_1^2)+2\overline{\phi}^{'}(1)\overline{\phi}_4^{'}(1)E(W)E(R_1)\big\} \\ & +2\theta\overline{\phi}_1^{'}(1)\overline{\phi}_4^{'}(1)E(V)[\overline{B}_1(\alpha)+\alpha\overline{B}_1^{'}(\alpha)]-2\alpha\overline{\phi}_1^{'}(1)\overline{B}_1^{'}(\alpha)\big(\overline{\phi}^{'}(1)E(W) \\ & +\overline{\phi}_4^{'}(1)E(R_1)\big), \end{aligned}$$

$$S_2^{'}(1) = \lambda_1r[1-\overline{B}_2(\psi_2(1))]E(X)+\psi_2(1)\overline{\phi}_3^{'}(1)\overline{B}_2(\psi_2(1))\overline{B}_3^{'}(\lambda^-),$$

$$S_2^{''}(1) = \lambda_1r[1-\overline{B}_2(\psi_2(1))]E(X^2)+\psi_2(1)\overline{B}_2(\psi_2(1))\big\{\overline{\phi}_3^{''}(1)\overline{B}_3^{'}(\lambda^-)+(\overline{\phi}_3^{'}(1))^2\overline{B}_3^{''}(\lambda^-)\big\},$$

$$\begin{aligned} S_3^{'}(1) = {} & \lambda^-\psi_2(1)\phi_4^{'}(1)[1-\overline{B}_2(\lambda_1r+\lambda_2bp+\lambda^-)]+\alpha\lambda^-(\lambda_1r+\lambda_2bp+\lambda^-)\overline{\phi}_4^{'}(1)E(R_2) \\ & \times[1-\overline{B}_2(\psi_2(1))]+(\lambda_1r+\lambda_2bp+\lambda^-)\psi_2(1)\overline{\phi}_4^{'}(1)\overline{B}_2(\psi_2(1))[1-\overline{B}(\lambda^-)] \\ & \times(1+\alpha E(R_3)), \end{aligned}$$

$$\begin{aligned} S_3^{''}(1) = {} & \psi_2(1)[1-\overline{B}_2(\lambda_1r+\lambda_2bp+\lambda^-)]\big(2\overline{\phi}_3^{'}(1)\overline{\phi}_4^{'}(1)+\lambda^-\overline{\phi}_4^{''}(1)\big)-2\lambda^-\psi_2(1)\overline{\phi}_2^{'}(1) \\ & \times\overline{\phi}_4^{'}(1)\overline{B}_2^{'}(\lambda_1r+\lambda_2bp+\lambda^-)+\psi_2(1)\overline{B}_2(\psi_2(1))(1+\alpha E(R_3))\big\{2[1-\overline{B}_3(\lambda^-)] \\ & \times\overline{\phi}_2^{'}(1)\overline{\phi}_4^{'}(1)+(\lambda_1r+\lambda_2bp+\lambda^-)\big(\overline{\phi}_4^{''}(1)[1-\overline{B}_3(\lambda^-)]-2\overline{\phi}_3^{'}(1)\overline{\phi}_4^{'}(1)\overline{B}_3^{'}(\lambda^-)\big)\big\} \\ & -\alpha[1-\overline{B}_2(\psi_2(1))]\big\{(\overline{\phi}_4^{'}(1))^2E(R_2^2)+2\lambda^-\overline{\phi}_2^{'}(1)\overline{\phi}_4^{'}(1)E(R_2)+(\lambda_1r+\lambda_2bp \\ & +\lambda^-)\big\{2\overline{\phi}_3^{'}(1)\overline{\phi}_4^{'}(1)E(R_2)+\lambda^-\big(\overline{\phi}_4^{''}(1)E(R_2)-(\overline{\phi}_4^{'}(1))^2E(R_2^2)\big)\big\}\big\}, \end{aligned}$$

$$\begin{aligned} nr'''(1) = {} & 3N_1(1)\psi_2(1)\lambda^-(\lambda_1r+\lambda_2bp+\lambda^-)[\frac{1-\overline{M}(\lambda_1+\lambda_2)}{\lambda_1+\lambda_2}]\big\{-\lambda_1rE(X_1)E(X) \\ & \times f^{''}(1)+2\underline{\phi}^{'}(1)\underline{\phi}_4^{'}(1)s_1^{'}(1)\big\}+3N_2(1)\big\{\lambda^-(\lambda_1r+\lambda_2bp+\lambda^-)s_2^{'}(1)f^{''}(1) \\ & +2\underline{\phi}^{'}(1)s_1^{'}(1)s_3^{'}(1)\big\}, \end{aligned}$$

$$nr^{(iv)}(1) = 12[\frac{1-\overline{M}(\lambda_1+\lambda_2)}{\lambda_1+\lambda_2}]\Big\{(\lambda_1 r+\lambda_2 bp+\lambda^-)\big\{N_1^{'}(1)\lambda^-\psi_2(1)+N_1(1)\lambda^-\psi_2^{'}(1) + N_1(1)\psi_2(1)\underline{\phi}_3^{'}(1)\big\} + N_1(1)\lambda^-\psi_2(1)\underline{\phi}_2^{'}(1)\Big\}\big\{-\lambda_1 rE(X_1)E(X)f^{''}(1) + 2\underline{\phi}^{'}(1)\underline{\phi}_4^{'}(1)s_1^{'}(1)\big\} + 6N_1(1)\psi_2(1)\lambda^-(\lambda_1 r+\lambda_2 bp+\lambda^-)[\frac{1-\overline{M}(\lambda_1+\lambda_2)}{\lambda_1+\lambda_2}] \times\Big\{-\lambda_1 r\big[E(X_2)E(X)+(E(X_1))^2E(X^2)\big]f^{''}(1)-\lambda_1 rE(X_1)E(X)f^{'''}(1) + 2\big(\underline{\phi}^{''}(1)\underline{\phi}_4^{'}(1)+\underline{\phi}^{'}(1)\underline{\phi}_4^{''}(1)\big)s_1^{'}(1)+2\underline{\phi}^{'}(1)\underline{\phi}_4^{'}(1)s_1^{''}(1)\Big\} + 3N_2^{'}(1)\big\{\lambda^-(\lambda_1 r + \lambda_2 bp+\lambda^-)s_2^{'}(1)f^{''}(1)+2\underline{\phi}^{'}(1)s_1^{'}(1)s_3^{'}(1)\big\} + N_2(1)\Big\{(\lambda_1 r+\lambda_2 bp+\lambda^-) \times\big\{12\underline{\phi}_3^{'}(1)s_2^{'}(1)f^{''}(1)+6\lambda^-s_2^{''}(1)f^{''}(1)+4\lambda^-s_2^{'}(1)f^{'''}(1)\big\}+12\lambda^-\underline{\phi}_2^{'}(1)s_2^{'}(1) \times f^{''}(1)+12\underline{\phi}^{''}(1)s_1^{'}(1)s_3^{'}(1)+12\underline{\phi}^{'}(1)\big(s_1^{''}(1)s_3^{'}(1)+s_1^{'}(1)s_3^{''}(1)\big)\Big\},$$

$$dr'''(1) = 6\lambda^-(\lambda_1 r+\lambda_2 bp+\lambda^-)D(1)\psi_2(1)\underline{\phi}^{'}(1)\underline{\phi}_4^{'}(1)s_1^{'}(1),$$

$$dr^{(iv)}(1) = 24\Big\{(\lambda_1 r+\lambda_2 bp+\lambda^-)\big\{D^{'}(1)\lambda^-\psi_2(1)+D(1)\lambda^-\psi_2^{'}(1)+D(1)\psi_2(1)\underline{\phi}_3^{'}(1)\big\} + D(1)\lambda^-\psi_2(1)\underline{\phi}_2^{'}(1)\Big\}\underline{\phi}^{'}(1)\underline{\phi}_4^{'}(1)s_1^{'}(1)+12\lambda^-(\lambda_1 r+\lambda_2 bp+\lambda^-)D(1)\psi_2(1) \times\Big\{\big(\underline{\phi}^{''}(1)\underline{\phi}_4^{'}(1)+\underline{\phi}^{'}(1)\underline{\phi}_4^{''}(1)\big)s_1^{'}+\underline{\phi}^{'}(1)\underline{\phi}_4^{'}(1)s_1^{''}(1)\Big\},$$

$$f^{''}(1) = 2\underline{\phi}^{'}(1)\underline{\phi}_4^{'}(1)\big\{[1-\overline{B}_1(\alpha)]\big(1+\alpha(E(W)+E(R_1))\big)+\theta\alpha\overline{B}_1(\alpha)E(V)\big\},$$

$$f^{'''}(1) = 6\underline{\phi}^{'}(1)\underline{\phi}_4^{'}(1)\Big\{\underline{\phi}_1^{'}(1)\overline{B}_1^{'}(\alpha)\big(1+\alpha(E(W)+E(R_1))\big)+\theta\big(\overline{B}_1(\alpha)+\alpha\overline{B}_1^{'}(\alpha)\big) \times\underline{\phi}_1^{'}(1)E(V)\Big\}+3\big(\underline{\phi}^{''}(1)\underline{\phi}_4^{'}(1)+\underline{\phi}^{'}(1)\underline{\phi}_4^{''}(1)\big)\big(1+\alpha(E(W)+E(R_1))\big) \times\Big\{[1-\overline{B}_1(\alpha)]+\theta\alpha\overline{B}_1(\alpha)E(V)\Big\}-3\alpha[1-\overline{B}_1(\alpha)](\underline{\phi}^{'}(1))^2\underline{\phi}_4^{'}(1)\big(E(W^2) + 2E(W)E(R_1)\big)-3\alpha\underline{\phi}^{'}(1)(\underline{\phi}_4^{'}(1))^2\big\{[1-\overline{B}_1(\alpha)]E(R_1^2)+\theta\overline{B}_1(\alpha)E(V^2)\big\},$$

$$s_1^{'}(1) = \theta\alpha\underline{\phi}_4^{'}(1)\overline{B}_1(\alpha)E(V)+[1-\overline{B}_1(\alpha)]\big\{\underline{\phi}_1^{'}(1)+\alpha\big(\underline{\phi}^{'}(1)E(W)+\underline{\phi}_4^{'}(1)E(R_1)\big)\big\},$$

$$s_1^{''}(1) = [1-\overline{B}_1(\alpha)]\big\{\underline{\phi}_1^{''}(1)+\alpha\big(\underline{\phi}^{''}(1)E(W)+\underline{\phi}_4^{''}(1)E(R_1)\big)\big\}-2\big\{(\underline{\phi}_1^{'}(1))^2+\alpha\underline{\phi}_1^{'}(1) \times\big(\underline{\phi}^{'}(1)E(W)+\underline{\phi}_4^{'}(1)E(R_1)\big)\big\}\overline{B}_1^{'}(\alpha)-\alpha[1-\overline{B}_1(\alpha)]\big\{(\underline{\phi}^{'}(1))^2E(W^2) + (\underline{\phi}_4^{'}(1))^2E(R_1^2)+2\underline{\phi}^{'}(1)\underline{\phi}_4^{'}(1)E(W)E(R_1)\big\}+2\theta\underline{\phi}_1^{'}(1)\underline{\phi}_4^{'}(1)\big[\overline{B}_1(\alpha) + \underline{\phi}_4^{'}(1)\overline{B}_1^{'}(\alpha)\big]E(V)+\alpha\theta\underline{\phi}_4^{''}(1)\overline{B}_1(\alpha)E(V)-\alpha\theta(\underline{\phi}_4^{'}(1))^2\overline{B}_1(\alpha)E(V^2),$$

$$s_2^{'}(1) = -\lambda_1 rE(X_1)E(X)[1-\overline{B}_2(\psi_2(1))]+\psi_2(1)\overline{B}_2(\psi_2(1))\overline{B}_3^{'}(\lambda^-)\big(\underline{\phi}_3^{'}(1)-\sigma_3^{'}(1)\big),$$

$$s_2^{''}(1) = \lambda_1 r\{E(X_1^2)E(X) + (E(X_1))^2E(X^2)\}[1-\overline{B}_2(\psi_2(1))] - 2\lambda_1 rE(X_1)E(X) \times \{1-\overline{B}_2(\psi_2(1)) - \psi_2^{'}(1)\overline{B}_2^{'}(\psi_2(1))\} + 2\psi_2^{'}(1)\overline{B}_3^{'}(\lambda^-)\big(\underline{\phi}_3^{'}(1) - \sigma_3^{'}(1)\big) \times \{\overline{B}_2(\psi_2(1)) + \psi_2^{'}(1)\overline{B}_2^{'}(\psi_2(1))\} + \psi_2(1)\overline{B}_2(\psi_2(1))\{\big(\underline{\phi}_3^{''}(1) - \sigma_3^{''}(1)\big) \times \overline{B}_3^{'}(\lambda^-) + \big((\underline{\phi}_3^{'}(1))^2 - (\sigma_3^{'}(1))^2\big)\overline{B}_3^{''}(\lambda^-)\},$$

$$s_3^{'}(1) = \lambda^-\psi_2(1)\underline{\phi}_4^{'}(1)[1-\overline{B}_2(\lambda_1 r + \lambda_2 bp + \lambda^-)] + (\lambda_1 r + \lambda_2 bp + \lambda^-)\underline{\phi}_4^{'}(1)\{\alpha\lambda^- \times [1-\overline{B}_2(\psi_2(1))]E(R_2) + \psi_2(1)\overline{B}_2(\psi_2(1))[1-\overline{B}_3(\lambda^-)](1+\alpha E(R_3))\},$$

$$s_3^{''}(1) = 2\underline{\phi}_4^{'}(1)\Big\{[1-\overline{B}_2(\lambda_1 r + \lambda_2 bp + \lambda^-)]\big(\lambda^-\psi_2^{'}(1) + \psi_2(1)\underline{\phi}_3^{'}(1)\big) - \lambda^-\psi_2(1)\psi_2^{'}(1) \times \overline{B}_2^{'}(\lambda_1 r + \lambda_2 bp + \lambda^-)\Big\} + \lambda^{'}\psi_2(1)\underline{\phi}_4^{''}(1)[1-\overline{B}_2(\lambda_1 r + \lambda_2 bp + \lambda^-)].$$

6. The Expected Waiting Time

The expected waiting time of priority and ordinary units are

$$W_{q_1} = \frac{L_{q_1}}{\lambda_1}, \tag{125}$$

and

$$W_{q_2} = \frac{L_{q_2}}{\lambda_2}. \tag{126}$$

7. Particular Cases

Case 1: If there are no priority queue, no breakdown, no preferred unit and single service i.e $\lambda_1 = 0$, $\alpha = 0$, $p = 0$, $\overline{B}_3(.) = 1$. Then, this model becomes to $M^X/G/1$ queueing system with balking

$$W_q(z) = \Big[\frac{I_{0,0}\lambda_2(C(z_2)-1)}{z_2 - \overline{B}_2(\lambda_2 b(1-C(z_2)))}\Big]\Big[\frac{1-\overline{B}_2(\lambda_2 b(1-C(z_2)))}{\lambda_2 b(1-C(z_2))}\Big].$$

The above result coincides with the result of Sing et al. (2014).

Case 2: If there are no priority queue, no breakdown, no preferred unit, no balking and single arrival for ordinary unit i.e $\lambda_1 = 0$, $\alpha = 0$, $p = 0$, $b = 1$ and $C(z_2) = z_2$. Then, this model becomes to $M/G/1$ queue with two stage of service

$$W_q(z) = \frac{I_{0,0}\big[1-\overline{B}_2(\lambda_2(1-z_2))\overline{B}_3(\lambda_2(1-z_2))\big]}{z_2 - \overline{B}_2(\lambda_2(1-z_2))\overline{B}_3(\lambda_2(1-z_2))}.$$

The above result coincides with the result of Zadeh and Shahkar (2008).

8. Numerical Results

Here, we present some numerical examples with the following assumptions. We consider that the regular service time, working breakdown service time, repair time and vacation time are to be exponentially and Erlangianly distributed. Also, we assume that the units will arrive one by one, so $E(X) = 1$ and $E(X(X-1)) = 0$.

Table 1 shows that when an increase in the vacation probability (θ), then decreases the idle probability and increase the expected queue size and waiting time for the arbitrary values to the parameters are chosen as $\lambda_1 = 0.2$, $\lambda_2 = 1.5$, $\lambda^- = 3$, $\mu = 3$, $\omega = 0.9$, $\alpha = 7$, $\eta = 9$, $\gamma = 0.6$, $p = 0.2$, $q = 0.4$, $b = 0.5$, $r = 0.4$, $\xi = 0.8$, $\beta = 15$, while θ, varies from 0.5 to 0.9, such that the stability condition is satisfied.

Table 2 shows that when an increase in the retrial rate (β), then increases the idle probability and decrease the expected queue size and waiting time for the arbitrary values to the parameters are chosen as $\lambda_1 = 0.2$, $\lambda_2 = 1.5$, $\lambda^- = 3$, $\mu = 3$, $\omega = 0.9$, $\alpha = 7$, $\eta = 9$, $\gamma = 0.6$, $p = 0.2$, $q = 0.4$, $b = 0.5$, $r = 0.4$, $\xi = 0.8$, $\theta = 0.8$, while β, varies from 15 to 19 such that the stability condition is satisfied.

Table 3 shows that when an increase in the breakdown rate (α), then increase the expected queue size and waiting time for the arbitrary values to the parameters are chosen as $\lambda_1 = 3$, $\lambda_2 = 0.5$, $\lambda^- = 3$, $\mu = 14$, $\omega = 0.9$, $\eta = 0.1$, $\gamma = 12$, $\beta = 22$, $p = 0.2$, $q = 0.4$, $b = 0.5$, $\xi = 0.8$, $\theta = 0.1$, while α, varies from 3.1 to 3.5 such that the stability condition is satisfied.

Table 4 shows that when an increase in the priority arrival rate (λ_1), then increase the expected queue size and the expected waiting time for the arbitrary values to the parameters are chosen as $\lambda_2 = 0.8$, $\lambda^- = 1.5$, $\alpha = 0.1$, $\mu = 12$, $\omega = 10$, $\eta = 5$, $\gamma = 0.6$, $p = 0.2$, $q = 0.2$, $b = 0.6$, $\xi = 0.1$, $\theta = 0.6$ while λ_1, varies from 0.3 to 0.7 such that the stability condition is satisfied.

Two dimensional graphs are illustrated in Figure 2 and 3. Figure 2, presents the expected priority queue size (L_{q_1}) which increases for increase in priority arrival rate (λ_1) as compared to the first stage of ordinary service disciplines. The expected priority queue size (L_{q_1}) increases for increasing breakdown rate (α) as compared to the first stage of ordinary service disciplines as shown in Figure 3.

Three-dimensional figures are shown:

- Figure 4, shows the behaviour of the expected queue size (L_{q_1}) which increases in order to increase the value of the priority arrival rate (λ_1) and ordinary arrival rate (λ_2).
- Figure 5, shows the behaviour of the expected orbit size (L_{q_2}) which increases in order to increase the value of the priority arrival rate (λ_1) and ordinary arrival rate (λ_2).
- Figure 6, shows the behaviour of the expected queue size (L_{q_1}) which decreases in order to increase the value of the reneging rate (ξ) and slow service rate (ω).
- Figure 7, shows the behaviour of the expected orbit size (L_{q_2}) which decreases in order to increase the value of the reneging rate (ξ) and slow service rate (ω).

Table 1. Effect of vacation probability (θ)

	Exponential					Erlang-2				
θ	I_0	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}	I_0	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}
0.5	0.4915	0.7427	3.7137	0.3016	0.2011	0.4931	0.7559	3.7797	0.4330	0.2887
0.6	0.4905	0.7778	3.8891	0.3017	0.2011	0.4921	0.7917	3.9585	0.4340	0.2893
0.7	0.4895	0.8121	4.0604	0.3017	0.2012	0.4911	0.8268	4.1340	0.4349	0.2900
0.8	0.4887	0.8455	4.2277	0.3018	0.2012	0.4903	0.8612	4.3059	0.4358	0.2908
0.9	0.4879	0.8781	4.3907	0.3019	0.2013	0.4895	0.8948	4.4740	0.4366	0.2911

Table 2. Effect of retrial rate (β)

	Exponential					Erlang-2				
β	I_0	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}	I_0	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}
15	0.4887	0.8455	4.2277	0.3018	0.2012	0.4903	0.8612	4.3059	0.4358	0.2905
16	0.4890	0.8447	4.2236	0.2607	0.1738	0.4906	0.8603	4.3016	0.3980	0.2653
17	0.4893	0.8440	4.2199	0.2253	0.1502	0.4909	0.8595	4.2977	0.3656	0.2437
18	0.4895	0.8433	4.2166	0.1947	0.1298	0.4911	0.8588	4.2942	0.3376	0.2251
19	0.4898	0.8727	4.2136	0.1679	0.1119	0.4814	0.8582	4.2911	0.3133	0.2089

Table 3. Effect of breakdown rate (α)

	Pre-emptive				Non-pre-emptive			
α	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}
3.1	3.4932	1.1644	4.9772	9.9545	24.5168	8.1723	4.2712	8.5424
3.2	6.9730	2.3243	5.1688	10.3375	28.4909	9.4970	4.3369	8.6738
3.3	10.3045	3.4348	5.3467	10.6934	32.3247	10.7749	4.3976	8.7952
3.4	13.5059	4.5020	5.5123	11.0247	36.0349	12.0116	4.4536	8.9073
3.5	16.5922	5.5307	5.6668	11.3335	39.6358	13.2119	4.5052	9.0105

Table 4. Effect of priority arrival rate (λ_1)

	Pre-emptive				Non-pre-emptive			
λ_1	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}
0.3	0.1019	0.3398	37.7016	47.1270	0.1643	0.5475	2.9548	3.6935
0.4	0.4996	1.2490	61.8236	77.2795	0.8165	2.0262	5.3196	6.6495
0.5	1.5831	3.1662	74.5742	93.2177	2.5832	5.1664	6.7939	8.4923
0.6	4.3783	7.2972	81.8400	102.3000	7.1934	11.9890	7.7243	9.6554
0.7	12.0315	17.1878	86.1267	107.6583	19.9327	28.4753	8.3059	10.3824

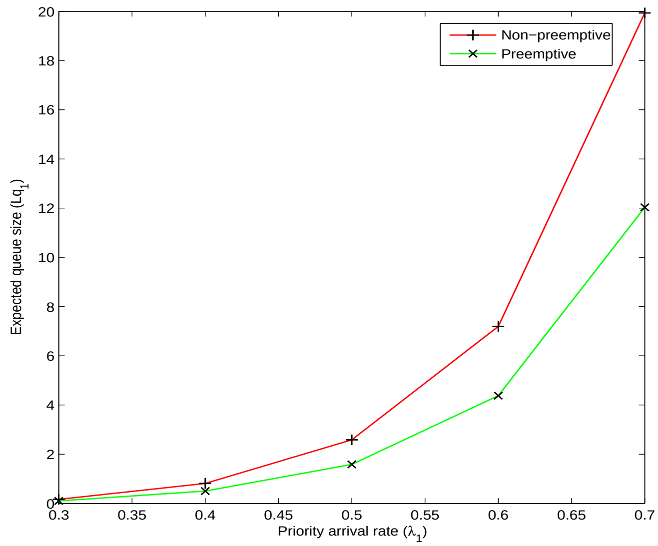

Figure 2. Lq_1 versus λ_1

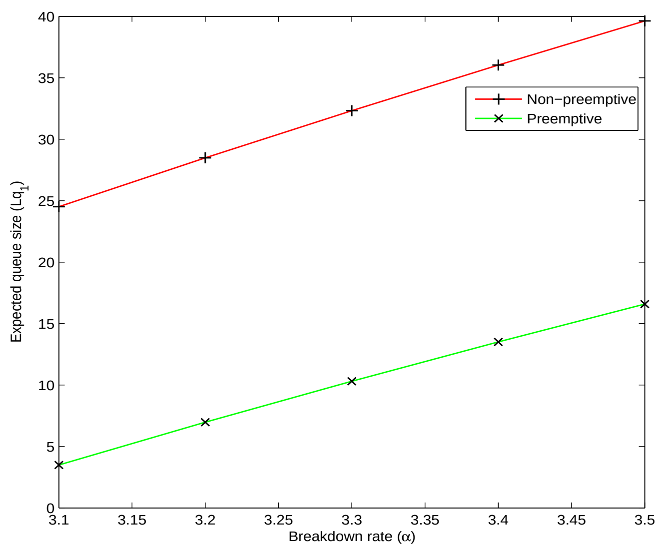

Figure 3. Lq_1 versus α

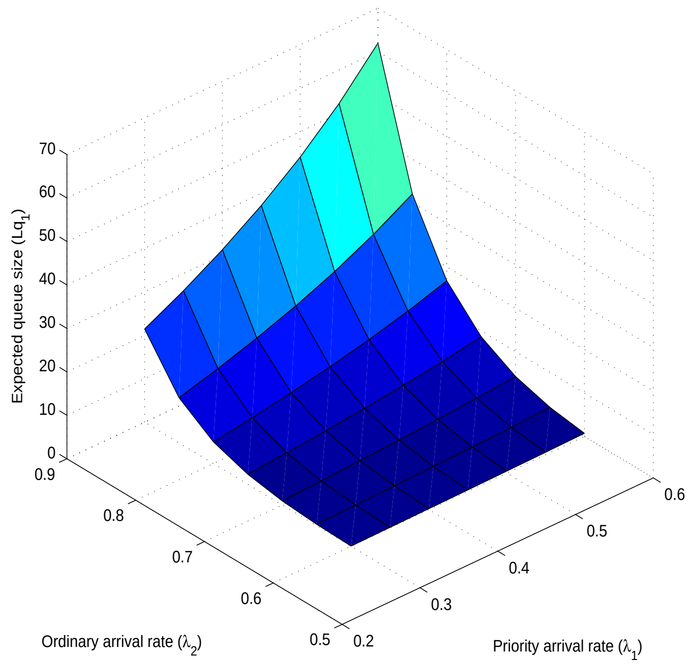

Figure 4. Lq_1 versus λ_1 and λ_2

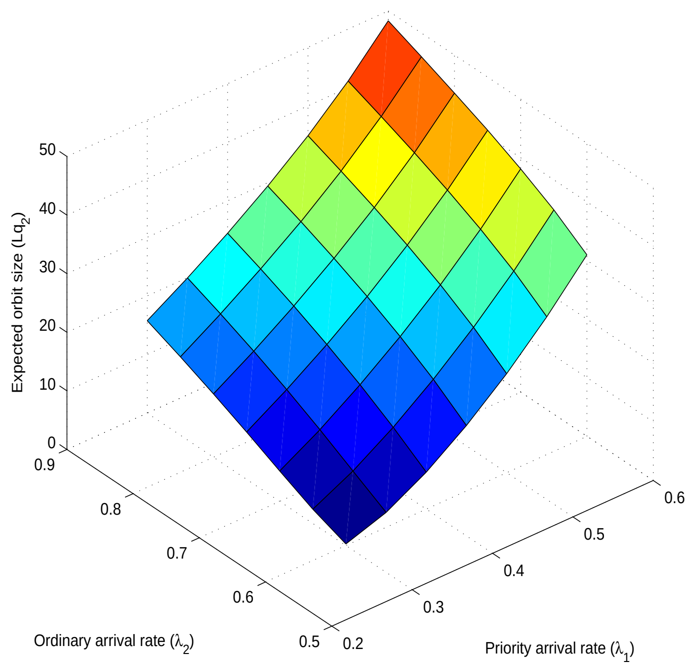

Figure 5. Lq_2 versus λ_1 and λ_2

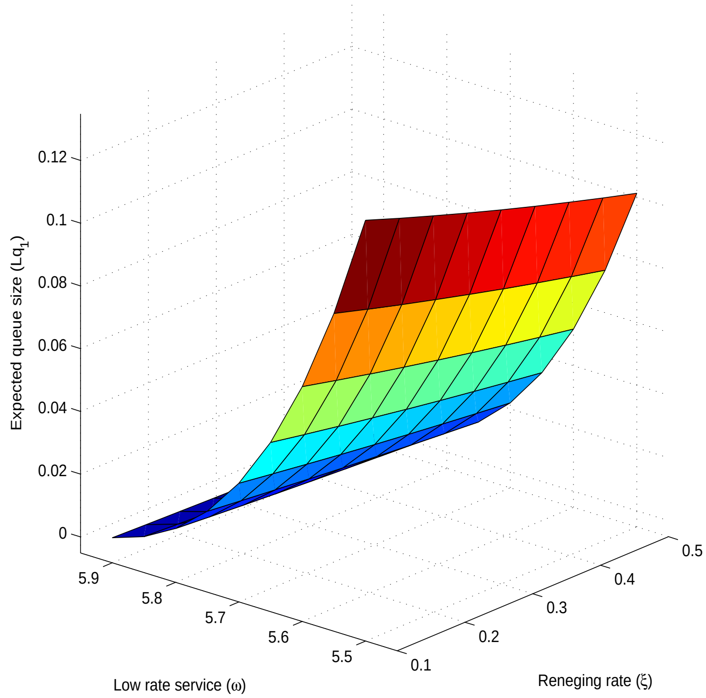

Figure 6. Lq_1 versus ξ and ω

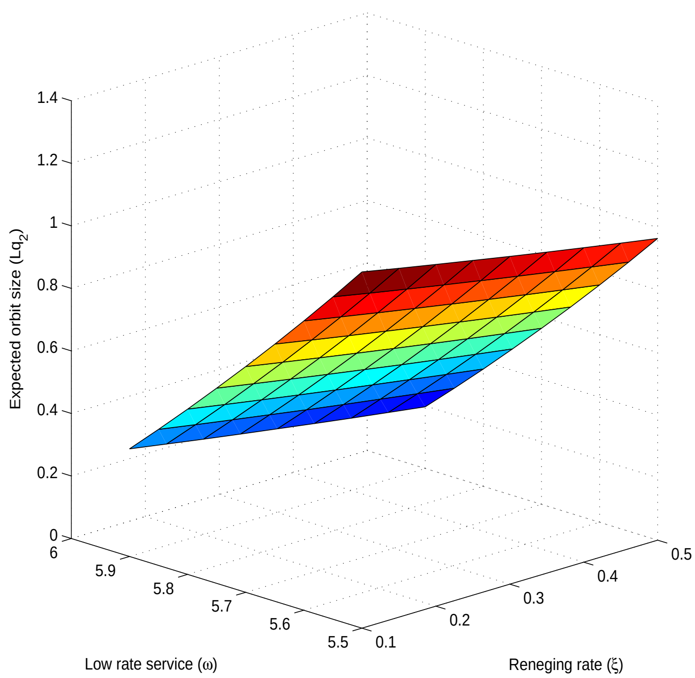

Figure 7. Lq_2 versus ξ and ω

9. Conclusion

In this paper, we have studied two types of batch arrivals, priority and ordinary units with discretionary priority services, working breakdown, negative arrival, Bernoulli vacation, preferred and impatient units. The corresponding steady state results for time-dependent probability generating functions are obtained explicitly. Performance measures like, the mean queue size and mean waiting times are obtained. Finally, some numerical results are computed along with graphical representations are given.

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