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Analysis of Batch Arrival Single and Bulk Service Queue with Multiple Vacation Closedown and Repair

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Abstract

In this paper, we analyze batch arrival single and bulk service queueing model with multiple vacation, closedown and repair. The single server provides single service if the queue size is '$< a$' and bulk service if the queue size is '$\geq a$'. After completing the service (single or bulk), the server may breakdown with probability ξ and then it will be sent for repair. When the system becomes empty or the server is ready to serve after the repair but no one is waiting, the server resumes closedown and then goes for a multiple vacation of random length. Using supplementary variable technique, the steady-state probability generating function (PGF) of the queue size at an arbitrary time is obtained. The performance measures and cost model are also derived. Numerical illustrations are presented to visualize the effect of system parameters.

Keywords: Batch arrival; Single service; Bulk service; Multiple vacation; Closedown; Repair; Supplementary variable technique

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1. Introduction

Queueing models where the server performs closedown work and resumes vacation if there is no customer waiting for the service are quite common in various practical situations related to manufacturing industries, service systems, etc. Whenever the system becomes empty, the server starts to do some other supplementary job (called vacation). During those jobs, the server is not available for the arriving customers. When the server returns from the vacation, if no one is waiting for service, he starts another vacation. The server will continue this process until he finds at least one customer waiting in the queue for service. This situation is called multiple vacation of server (Arumuganathan and Jeyakumar (2004), Arumuganathan and Jeyakumar (2005), Jeyakumar and Senthilnathan (2012), Krishna Reddy et al. (1998)). Before the commencement of vacation, the server does the closedown work (Arumuganathan and Jeyakumar (2004), Arumuganathan and Jeyakumar (2005), Jeyakumar and Senthilnathan (2012), Ke (2007)).

The server breakdown is an important factor to be analyzed in many practical situations related to communication systems, flexible manufacturing systems, etc. The breakdown of a server occurs at random. The server is then sent for repair without interrupting the customer (or batch of customers) in service (refer to Jeyakumar and Senthilnathan (2012)). Some of the authors analyzed situations related to breakdown and immediate repair (refer to Jain et al. (2015), Ke (2007), Li et al. (1997), Madan et al. (2003), Wang et al. (2005), Wang et al. (2007), Wang et al. (2009)).

Neuts (1967) initiated the concept of bulk queues and analyzed a general class of such models. A literature survey on vacation queueing models can be found in Doshi (1986) and Takagi (1991) which include some applications. Lee (1991) developed a systematic procedure to calculate the system size probabilities for a bulk queueing model. Krishna Reddy et al. (1998) considered an $M^{[X]}/G(a,b)/1$ queueing model with multiple vacations, setup times and N policy. They derived the steady-state system size distribution, cost model, expected length of idle and busy period. Li et al. (1997) considered an $M/G/1$ queueing model with server breakdowns and Bernoulli vacations. They derived the time-dependent system size probabilities and reliability measures using supplementary variable method. Madan et al. (2003) derived PGF of various system characteristics for two $M^{[X]}/M(a,b)/1$ queueing models where the service station undergoes random breakdowns.

Arumuganathan and Jeyakumar (2004) obtained the PGF of queue length distributions at an arbitrary time epoch for the bulk queueing model with multiple vacation and closedown times. Also they have developed a cost model with a numerical study for their queueing model. Arumuganathan and Jeyakumar (2005) obtained the PGF of queue size distribution at an arbitrary time epoch and a cost model for the $M^{[X]}/G(a,b)/1$ queueing model with multiple vacation, closedown, setup times and N-policy. Avi-Itzhak and Naor (1963) analyzed five different single server queueing models and derived the expected queue lengths for those models. They also assumed arbitrary service and repair times. Choudhury and Ke (2012) considered an $M^{[X]}/G/1$ queueing model in which they derived the steady-state system size probabilities. They also have obtained various performance measures and reliability indices of the model.

Jain and Agrawal (2009) analyzed an $M^{[X]}/M/1$ queueing model with multiple types of server

breakdown, unreliable server and N-policy. They obtained the mean queue length and other system characteristics using matrix geometric method. Ke (2007) investigated an $M^{[X]}/G/1$ queueing model with vacation policies, breakdown and startup/closedown times where the vacation, startup, closedown and repair times are generally distributed. Jeyakumar and Arumuganathan (2008) obtained the PGF of queue size at an arbitrary time epoch in the steady state case for the $M^{[X]}/G/1$ queueing model with two service modes and multiple vacation. Jeyakumar and Senthilnathan (2012) analyzed an $M^{[X]}/G(a,b)/1$ queueing model with multiple vacation and closedown in which the server breakdown without interrupting the batch in service. They also obtained the PGF of queue size at an arbitrary time epoch and some performance measures with cost model.

Wang et al. (2005) derived the approximate results for the steady-state probability distributions of the queue length for a single unreliable server $M/G/1$ queueing model using maximum entropy principle and performed a comparative analysis of these approximate results with the available exact results. Wang et al. (2007) considered an unreliable $M/G/1$ queueing model with general service, repair and startup times. They obtained the cost function to determine the optimum value of N at a minimum cost and various performance measures. Wang et al. (2009) investigated an $M/G/1$ queueing model with server breakdown, general startup times and T policy where the server is turned on after a fixed length of time T repeatedly until an arrival occurs. A machine repair problem with standby server, two modes of failure, discouragement and switching failure was analyzed by Jain and Preeti (2014). They have derived the transient system size probabilities and also obtained various system performance measures as well as cost model. Jain et al. (2015) carried out the transient analysis of a machine repair problem with warm spares and two failure modes. They also obtained performance measures, cost model and sensitivity analysis of the model.

The main contribution of this paper lies in applying both single and bulk service pattern. That is, an interesting service pattern is assumed in this model such that the service of customers is carried out one by one when the queue size is less than the minimum threshold value 'a' whereas it is in bulk when the size of the queue is at least 'a'. The other significant parameters like multiple vacation, closedown and repair are also included in this model. Jeyakumar and Arumuganathan (2008) considered a batch arrival queueing model in which the service is single (one by one). But Arumuganathan and Jeyakumar (2004), Arumuganathan and Jeyakumar (2005), and Jeyakumar and Senthilnathan (2012) considered batch arrival and bulk service queueing model. In the proposed research work, both single and bulk service patterns are assumed in the same model itself. Also, this is the main difference of this paper with the existing literatures.

The rest of the paper is organized as follows. In Section 2, batch arrival single and bulk service queueing model with multiple vacation, closedown and repair is described and the steady-state system size equations are obtained. In Section 3, using supplementary variable technique, the PGF of the queue size are derived and a particular case is obtained. In Section 4, performance measures like expected length of busy and idle periods, expected queue length and expected waiting time are obtained. In Section 5, the cost model is presented. In Section 6, the analytical expressions of performance measures are verified numerically. In Section 7, this research work is concluded with the proposed future work.

2. Model Description

In this paper, we analyze batch arrival single and bulk service queueing model with multiple vacation, closedown and repair. The arrival of batch of customers follows compound Poisson process. The service time follows general distribution for both single and bulk service. The server provides bulk service only if the queue length is at least 'a' and the maximum bulk service capacity is 'b'. If the queue length is less than 'a', the server provides single service. After completing the service (single or bulk) if the server is breakdown with probability ξ, then the server will be sent for repair. When the system becomes empty or the server is ready to serve after the repair but no one is waiting, the server resumes closedown and then goes for a vacation of random length. Otherwise, the server starts the busy period. After the completion of vacation period, still there is no customer waiting in the queue, the server goes for another vacation, continuing this procedure until he finds at least one customer waiting in the queue. Otherwise, the server resumes service to the waiting customers.

A *real time application* relevant to this model is as follows: In *manufacturing systems*, a semi automatic milling machine performs cutting operation with the help of an end mill cutter. The work pieces arrive in batches. If the number of work pieces are less than a minimum threshold value, the cutting operation is done one by one. If the queue size reaches the minimum threshold value, then the cutting operation is done in bulk. The cutter is moved over the work piece(s) and the cutting operation is performed. If any failure occurs, the cutter will be sent for repair. When there are no work pieces, the machine will be shut down and then start the maintenance works such as cleaning, sharpening the cutter, etc.

2.1. Notations

The following notations are used in this paper:

λ - Arrival rate,
Y - Group size random variable,
$h_k - Pr\{Y = k\}$,
$Y(z)$ - PGF of Y.

The supplementary variables $B^0(t), G^0(t), R^0(t), V^0(t)$ and $L^0(t)$ are introduced in order to obtain the bivariate Markov process $\{\Omega(t), O(t)\}$, where $\Omega(t) = \{\Omega_q(t) \cup \Omega_s(t)\}$ and

$$\begin{aligned} O(t) &= (0)[1]\{2\}\langle 3\rangle(4), \text{if the server is on(single service)[bulk service]} \\ &\quad \{\text{repair}\}\langle\text{vacation}\rangle(closedown), \\ Z(t) &= j, \text{if the server is on } j^{th} \text{ vacation}, \\ \Omega_s(t) &= \text{Number of customers in the service at time } t, \\ \Omega_q(t) &= \text{Number of customers in the queue at time } t. \end{aligned}$$

Here $B(.), G(.), R(.), V(.)$ and $L(.)$ represent the cumulative distribution function (CDF) of service time for single service, service time for bulk service, repair time, vacation time and closedown

time and their corresponding probability density functions are $b(w), g(w), r(w), v(w)$ and $l(w)$ respectively. $B^0(t), G^0(t), R^0(t), V^0(t)$ and $L^0(t)$ represent the remaining single service time, remaining bulk service time, repair time, vacation time and closedown time at time t respectively. $\tilde{B}(\phi), \tilde{G}(\phi), \tilde{R}(\phi), \tilde{V}(\phi)$, and $\tilde{L}(\phi)$ represent the Laplace-Stieltjes transform of B, G, R, V and L respectively. The supplementary variables $B(t), G(t), R(t), V(t)$ and $L(t)$ are introduced in order to obtain the bivariate Markov process $\{\Omega(t), Y(t)\}$, where $\Omega(t) = \{\Omega_q(t) \cup \Omega_s(t)\}$.

Define the probabilities as follows:

$$\begin{aligned}
D_{1,j}(w,t)dt =& P\left\{\Omega_q(t) = j, w \le B^0(t) \le w + dw, O(t) = 0\right\},\ j \ge 0,\\
P_{i,j}(w,t)dt =& P\left\{\Omega_s(t) = i, \Omega_q(t) = j, w \le G^0(t) \le w + dw, O(t) = 1\right\},\ a \le i \le b,\ j \ge 0,\\
R_n(w,t)dt =& P\left\{\Omega_q(t) = n, w \le R^0(t) \le w + dw, O(t) = 2\right\},\ a \le i \le b,\ j \ge 0,\\
Q_{j,n}(w,t)dt =& P\left\{\Omega_q(t) = n, w \le V^0(t) \le w + dw, O(t) = 3, Z(t) = j\right\},\ n \ge 0,\ j \ge 1,\\
L_n(w,t)dt =& P\left\{\Omega_q(t) = n, w \le L^0(t) \le w + dw, O(t) = 4\right\},\ n \ge 0.
\end{aligned}$$

Using the supplementary variable technique which was introduced by Cox (1965), the following equations are obtained for the queueing system using supplementary variable technique. The equations are obtained at time $t + \Delta t$ considering all possibilities. Note that when time t is increased by Δt, the remaining service time for single service, service time for bulk service, repair time, closedown time and vacation time will be reduced by $w - \Delta t$.

$$\begin{aligned}
D_{1,0}(w - \Delta t, t + \Delta t) = D_{1,0}(w,t)(1 - \lambda\Delta t) &+ (1 - \xi)D_{1,1}(0,t)b(w)\Delta t\\
&+ (1 - \xi)\sum_{m=a}^{b} P_{m,1}(0,t)b(w)\Delta t + R_1(0,t)b(w)\Delta t\\
&+ \sum_{l=1}^{\infty} Q_l(0,t)b(w)\Delta t.
\end{aligned}$$

The above equation explains possible cases for the probability that there is one customer under service and 0 customers in the queue when the remaining service time is $w - \Delta t$ at time $t + \Delta t$. In a similar way, the following equations are obtained.

$$\begin{aligned}
D_{1,n}(w - \Delta t, t + \Delta t) =& D_{1,n}(w,t)(1 - \lambda\Delta t)\\
&+ \sum_{k=1}^{n} D_{1,n-k}(w,t)\lambda h_k\Delta t + R_{n+1}(0,t)b(w)\Delta t\\
&+ (1 - \xi)B_{1,n+1}(0,t)b(w)\Delta t + (1 - \xi)\sum_{m=a}^{b} P_{m,n+1}(0,t)b(w)\Delta t\\
&+ \sum_{l=1}^{\infty} Q_{l,n+1}(0,t)b(w)\Delta t, 1 \le n \le a - 2,\\
D_{1,n}(w - \Delta t, t + \Delta t) =& D_{1,n}(w,t)(1 - \lambda\Delta t) + \sum_{k=1}^{n} D_{1,n-k}(w,t)\lambda h_k\Delta t, \quad n \ge a - 1,
\end{aligned}$$

$$
\begin{aligned}
P_{i,0}(w-\Delta t, t+\Delta t) =& P_{i,0}(w,t)(1-\lambda\Delta t) \\
&+ (1-\xi)C_{1,i}(0,t)g(w)\Delta t + R_i(0,t)g(w)\Delta t \\
&+ \sum_{l=1}^{\infty} Q_{l,i}(0,t)g(w)\Delta t \\
&+ (1-\xi)\sum_{m=a}^{b} P_{m,i}(0,t)g(w)\Delta t, \quad a \le i \le b, \\
P_{i,j}(w-\Delta t, t+\Delta t) =& P_{i,j}(w,t)(1-\lambda\Delta t) \\
&+ \sum_{k=1}^{j} P_{i,j-k}(w,t)\lambda h_k \Delta t, \quad j \ge 1,\ a \le i \le b-1, \\
P_{b,j}(w-\Delta t, t+\Delta t) =& P_{b,j}(w,t)(1-\lambda\Delta t) + (1-\xi)C_{1,b+j}(0,t)g(w)\Delta t \\
&+ R_{b+j}(0,t)g(w)\Delta t + \sum_{k=1}^{j} P_{b,j-k}(w,t)\lambda h_k \Delta t \\
&+ \sum_{l=1}^{\infty} Q_{l,b+j}(0,t)g(w)\Delta t \\
&+ (1-\xi)\sum_{m=a}^{b} P_{m,b+j}(0,t)g(w)\Delta t, \quad j \ge 1, \\
L_0(w-\Delta t, t+\Delta t) =& L_0(w,t)(1-\lambda\Delta t) + (1-\xi)\sum_{m=a}^{b} P_{m,0}(0,t)l(w)\Delta t \\
&+ (1-\xi)D_{1,0}(0,t)l(w)\Delta t + R_0(0,t)l(w)\Delta t, \\
L_n(w-\Delta t, t+\Delta t) =& L_n(w,t)(1-\lambda\Delta t) + \sum_{k=1}^{n} L_{n-k}(w,t)\lambda h_k \Delta t, \quad n \ge 1, \\
R_0(w-\Delta t, t+\Delta t) =& R_0(w,t)(1-\lambda\Delta t) \\
&+ \xi\sum_{m=a}^{b} P_{m,0}(0,t)r(w)\Delta t + \xi D_{1,0}(0,t)r(w)\Delta t, \\
R_n(w-\Delta t, t+\Delta t) =& R_n(w,t)(1-\lambda\Delta t) + \sum_{k=1}^{n} R_{n-k}(w,t)\lambda h_k \Delta t \\
&+ \xi\sum_{m=a}^{b} P_{m,n}(0,t)r(w)\Delta t + \xi D_{1,n}(0,t)r(w)\Delta t,\ n \ge 1, \\
Q_{1,0}(w-\Delta t, t+\Delta t) =& Q_{1,0}(w,t)(1-\lambda\Delta t) + L_0(0,t)v(w)\Delta t, \\
Q_{1,n}(w-\Delta t, t+\Delta t) =& Q_{1,n}(w,t)(1-\lambda\Delta t) + L_n(0,t)v(w)\Delta t \\
&+ \sum_{k=1}^{n} Q_{1,n-k}(x)\lambda h_k \Delta t,\ n \ge 1,
\end{aligned}
$$

$$\begin{aligned}
Q_{j,0}(w-\Delta t, t+\Delta t) =& Q_{j,0}(w,t)(1-\lambda\Delta t) \\
&+ Q_{j-1,0}(0,t)v(w)\Delta t, \quad j \geq 2, \\
Q_{j,n}(w-\Delta t, t+\Delta t) =& Q_{j,n}(w,t)(1-\lambda\Delta t) \\
&+ \sum_{k=1}^{n} Q_{j,n-k}(w,t)\lambda h_k \Delta t,\ j \geq 2, \quad n \geq 1.
\end{aligned}$$

The steady-state system size equations are obtained as follows:

$$\begin{aligned}
-D'_{1,0}(w) =& -\lambda D_{1,0}(w) + (1-\xi)D_{11}(0)b(w) \\
&+ (1-\xi)\sum_{m=a}^{b} P_{m,1}(0)b(w) + R_1(0)b(w) + \sum_{l=1}^{\infty} Q_l(0)b(w),
\end{aligned} \tag{1}$$

$$\begin{aligned}
-D'_{1,n}(w) =& -\lambda D_{1,n}(w) + \sum_{k=1}^{n} D_{1,n-k}(w)\lambda h_k + (1-\xi)D_{1,n+1}(0)b(w) \\
&+ R_{n+1}(0)b(w) + (1-\xi)\sum_{m=a}^{b} P_{m,n+1}(0)b(w) \\
&+ \sum_{l=1}^{\infty} Q_{l,n+1}(0)b(w), 1 \leq n \leq a-2,
\end{aligned} \tag{2}$$

$$-D'_{1,n}(w) = -\lambda D1, n(w) + \sum_{k=1}^{n} D_{1,n-k}(w)\lambda h_k, \quad n \geq a-1, \tag{3}$$

$$\begin{aligned}
-P'_{i,0}(w) =& -\lambda P_{i,0}(w) + (1-\xi)D_{1,i}(0)g(w) + (1-\xi)\sum_{m=a}^{b} P_{m,i}(0)g(w) \\
&+ \sum_{l=1}^{\infty} Q_{l,i}(0)g(w) + R_i(0)g(w), \quad a \leq i \leq b,
\end{aligned} \tag{4}$$

$$-P'_{i,j}(w) = -\lambda P_{i,j}(w) + \sum_{k=1}^{j} P_{i,j-k}(w)\lambda h_k, \quad j \geq 1,\ a \leq i \leq b-1, \tag{5}$$

$$\begin{aligned}
-P'_{b,j}(w) =& -\lambda P_{b,j}(w) + (1-\xi)D_{1,b+j}(0)g(w) + (1-\xi)\sum_{m=a}^{b} P_{m,b+j}(0)g(w) \\
&+ \sum_{k=1}^{j} P_{b,j-k}(w)\lambda h_k + \sum_{l=1}^{\infty} Q_{l,b+j}(0)g(w) + R_{b+j}(0)g(w), \quad j \geq 1,
\end{aligned} \tag{6}$$

$$\begin{aligned}
-L'_0(w) =& -\lambda L_0(w) + (1-\xi)\sum_{m=a}^{b} P_{m,0}(0)l(w) + (1-\xi)D_{10}(0)l(w) \\
&+ R_0(0)l(w),
\end{aligned} \tag{7}$$

$$-L_n'(w) = -\lambda L_n(w) + \sum_{k=1}^{n} L_{n-k}(w)\lambda h_k, \quad n \geq 1, \tag{8}$$

$$-R_0'(w) = -\lambda R_0(w) + \xi \sum_{m=a}^{b} P_{m,0}(0)r(w) + \xi D_{10}(0)r(w), \tag{9}$$

$$\begin{aligned}-R_n'(w) = &-\lambda R_n(w) + \sum_{k=1}^{n} R_{n-k}(w)\lambda h_k + \xi \sum_{m=a}^{b} P_{m,n}(0)r(w) \\ &+ \xi D_{1n}(0)r(w),\ n \geq 1,\end{aligned} \tag{10}$$

$$-Q_{1,0}'(w) = -\lambda Q_{1,0}(w) + L_0(0)v(w), \tag{11}$$

$$-Q_{1,n}'(w) = -\lambda Q_{1,n}(w) + L_n(0)v(w) + \sum_{k=1}^{n} Q_{1,n-k}(x)\lambda h_k, \quad n \geq 1, \tag{12}$$

$$-Q_{j,0}'(w) = -\lambda Q_{j,0}(w) + Q_{j-1,0}(0)v(w), \quad j \geq 2, \tag{13}$$

$$-Q_{j,n}'(w) = -\lambda Q_{j,n}(w) + \sum_{k=1}^{n} Q_{j,n-k}(w)\lambda h_k,\ j \geq 2, \quad n \geq 1. \tag{14}$$

The Laplace-Stieltjes transform of $P_{i,j}(w), D_{1,j}(w), L_n(w), Q_{j,n}(w), R_n(w)$ are defined as follows:

$$\tilde{P}_{i,j}(\phi) = \int_0^\infty e^{-\phi w} P_{i,j}(w)dw,\ \tilde{D}_{1,j}(\phi) = \int_0^\infty e^{-\phi w} D_{1,j}(w)dw,$$

$$\tilde{L}_n(\phi) = \int_0^\infty e^{-\phi w} L_n(w)dw, \quad \tilde{Q}_{j,n}(\phi) = \int_0^\infty e^{-\phi w} Q_{j,n}(w)dw,\ \tilde{R}_n(\phi) = \int_0^\infty e^{-\phi w} R_n(w)dw.$$

Taking Laplace-Stieltjes transform from (1) to (14), we get

$$\begin{aligned}\phi\tilde{D}_{1,0}(\phi) - D_{1,0}(0) = &\lambda\tilde{D}_{1,0}(\phi) - \tilde{B}_1(\phi)\Big[(1-\xi)D_{11}(0) \\ &+ (1-\xi)\sum_{m=a}^{b} P_{m,1}(0) + \sum_{l=1}^{\infty} Q_{l,1}(0) + R_1(0)\Big],\end{aligned} \tag{15}$$

$$\begin{aligned}\phi\tilde{D}_{1,n}(\phi) - D_{1,n}(0) = &\lambda\tilde{D}_{1,n}(\phi) - \sum_{k=1}^{n} \tilde{D}_{1n-k}(\phi)\lambda h_k \\ &- \tilde{B}_1(\phi)\Big[(1-\xi)D_{1,n+1}(0) + (1-\xi)\sum_{m=a}^{b} P_{m,n+1}(0) \\ &+ \sum_{l=1}^{\infty} Q_{l,n+1}(0) + R_{n+1}(0)\Big], 1 \leq n \leq a-2,\end{aligned} \tag{16}$$

$$\phi\tilde{D}_{1,n}(\phi) - D_{1,n}(0) = \lambda\tilde{D}_{1,n}(\phi) - \sum_{k=1}^{n}\tilde{D}_{1n-k}(\phi)\lambda h_k, \quad n \geq a-1, \tag{17}$$

$$\begin{aligned}\phi\tilde{P}_{i,0}(\phi) - P_{i,0}(0) =& \lambda\tilde{P}_{i,0}(\phi) - \tilde{G}(\phi)\Bigg[(1-\xi)\sum_{m=a}^{b}P_{m,i}(0) + (1-\xi)C_{1,i}(0) \\ &+ \sum_{l=1}^{\infty}Q_{l,i}(0) + R_i(0)\Bigg],\ a \leq i \leq b,\end{aligned} \tag{18}$$

$$\phi\tilde{P}_{i,j}(\phi) - P_{i,j}(0) = \lambda\tilde{P}_{i,j}(\phi) - \sum_{k=1}^{j}\tilde{P}_{i,j-k}(\phi)\lambda h_k, \tag{19}$$

$$\begin{aligned}\phi\tilde{P}_{b,j}(\phi) - P_{b,j}(0) =& \lambda\tilde{P}_{b,j}(\phi) - \sum_{k=1}^{j}\tilde{P}_{b,j-k}(\phi)\lambda h_k \\ &- \tilde{G}(\phi)\Bigg[(1-\xi)\sum_{m=a}^{b}P_{m,b+j}(0) + (1-\xi)D_{1,b+j}(0) \\ &+ \sum_{l=1}^{\infty}Q_{l,b+j}(0) + R_{b+j}(0)\Bigg],\ j \geq 1,\end{aligned} \tag{20}$$

$$\begin{aligned}\phi\tilde{L}_0(\phi) - L_0(0) =& \lambda\tilde{L}_0(\phi) - \tilde{L}(\phi)\Bigg[(1-\xi)D_{10}(0) \\ &+ (1-\xi)\sum_{m=a}^{b}P_{m0}(0) + R_0(0)\Bigg],\end{aligned} \tag{21}$$

$$\phi\tilde{L}_n(\phi) - L_n(0) = \lambda\tilde{L}_n(\phi) - \sum_{k=1}^{n}\tilde{L}_{n-k}(\phi)\lambda h_k,\ n \geq 1, \tag{22}$$

$$\phi\tilde{R}_0(\phi) - R_0(0) = \lambda\tilde{R}_0(\phi) - \tilde{R}(\phi)\Bigg[\xi B_{10}(0) + \xi\sum_{m=a}^{b}P_{m0}(0)\Bigg], \tag{23}$$

$$\begin{aligned}\phi\tilde{R}_n(\phi) - R_n(0) =& \lambda\tilde{R}_n(\phi) - \sum_{k=1}^{n}\tilde{R}_{n-k}(\phi)\lambda h_k \\ &- \tilde{R}(\phi)\Bigg[\xi B_{1n}(0) + \xi\sum_{m=a}^{b}P_{mn}(0)\Bigg],\ n \geq 1,\end{aligned} \tag{24}$$

$$\phi\tilde{Q}_{1,0}(\phi) - Q_{1,0}(0) = \lambda\tilde{Q}_{1,0}(\phi) - \tilde{V}(\phi)L_0(0), \tag{25}$$

$$\phi\tilde{Q}_{1,n}(\phi) - Q_{1,n}(0) = \lambda\tilde{Q}_{1,n}(\phi) - \tilde{V}(\phi)L_n(0) - \sum_{k=1}^{n}\tilde{Q}_{1,n-k}(\phi)\lambda h_k,\ n \geq 1, \tag{26}$$

$$\phi\tilde{Q}_{j,0}(\phi) - Q_{j,0}(0) = \lambda\tilde{Q}_{j,0}(\phi) - \tilde{V}(\phi)Q_{j-1,0}(0), \tag{27}$$

$$\phi\tilde{Q}_{j,n}(\phi) - Q_{j,n}(0) = \lambda\tilde{Q}_{j,n}(\phi) - \sum_{k=1}^{n}\tilde{Q}_{j,n-k}(\phi)\lambda h_k. \tag{28}$$

To find the PGF of queue size, we define the following PGFs:

$$\begin{aligned}
\tilde{P}(z,\phi) &= \sum_{j=0}^{\infty} \tilde{P}_{i,j}(\phi) z^j, \quad P_i(z,0) = \sum_{j=0}^{\infty} P_{i,j}(0) z^j, \quad a \le i \le b, \\
\tilde{D}(z,\phi) &= \sum_{j=0}^{\infty} \tilde{D}_{1,j}(\phi) z^j, \quad D(z,0) = \sum_{j=0}^{\infty} D_{1,j}(0) z^j, \\
\tilde{R}(z,\phi) &= \sum_{n=0}^{\infty} \tilde{R}_n(\phi) z^n, \quad R(z,0) = \sum_{n=0}^{\infty} R_n(0) z^n, \\
\tilde{L}(z,\phi) &= \sum_{n=0}^{\infty} \tilde{L}_n(\phi) z^n, \quad L(z,0) = \sum_{n=0}^{\infty} L_n(0) z^n, \\
\tilde{Q}_j(z,\phi) &= \sum_{n=0}^{\infty} \tilde{Q}_{j,n}(\phi) z^n, \quad Q_j(z,0) = \sum_{n=0}^{\infty} Q_{j,n}(0) z^j, \quad j \ge 1.
\end{aligned} \tag{29}$$

By multiplying the equations from (15) to (28) by suitable power of z^n and summing over n, ($n = 0$ to ∞) and using (29), we get

$$(\phi - \lambda + \lambda Y(z))\tilde{Q}_1(z,\phi) = Q_1(z,0) - C(z,0)\tilde{V}(\phi), \tag{30}$$

$$(\phi - \lambda + \lambda Y(z))\tilde{Q}_j(z,\phi) = Q_j(z,0) - \tilde{V}(\phi) Q_{j-1,0}(0), \; j \ge 2, \tag{31}$$

$$\begin{aligned}
(\phi - \lambda + \lambda Y(z))\tilde{L}(z,\phi) =& L(z,0) - \tilde{L}(\phi)\Bigg[(1-\xi)D_{10}(0) \\
&+ (1-\xi)\sum_{m=a}^{b} P_{m,0}(0) + R_0(0)\Bigg],
\end{aligned} \tag{32}$$

$$\begin{aligned}
(\phi - \lambda + \lambda Y(z))\tilde{D}(z,\phi) =& D(z,0) - \tilde{B}(\phi)\Bigg[(1-\xi)\sum_{n=1}^{a-1} D_{1n}(0) \\
&+ (1-\xi)\sum_{m=a}^{b} P_{m,n}(0) + \sum_{l=1}^{\infty} Q_{l,n}(0) + R_n(0)]\Bigg] z^{n-1},
\end{aligned} \tag{33}$$

$$\begin{aligned}
(\phi - \lambda + \lambda Y(z))\tilde{P}_i(z,\theta) =& P_i(z,0) - \tilde{G}(\phi)\Bigg[(1-\xi)D_{1,i}(0) \\
&+ (1-\xi)\sum_{m=a}^{b} P_{m,i}(0) + \sum_{l=1}^{\infty} Q_{l,i}(0) \\
&+ R_i(0)\Bigg], \; a \le i \le b-1,
\end{aligned} \tag{34}$$

$$(\phi-\lambda+\lambda Y(z))\tilde{P}_b^{(1)}(z,\phi)=P_b^{(1)}(z,0)-\frac{\tilde{G}(\phi)}{z^b}\Big[(1-\xi)(D(z,0)-\sum_{j=0}^{b-1}D_{1,j}(0)z^j)+(1-\xi)(\sum_{m=a}^{b}P_m(z,0)-\sum_{m=a}^{b}\sum_{j=0}^{b-1}P_{m,j}(0)z^j)+\sum_{l=1}^{\infty}Q_l(z,0)-\sum_{l=1}^{\infty}\sum_{j=0}^{b-1}Q_{l,j}(0)z^j+R(z,0)-\sum_{j=0}^{b-1}R_j(0)z^j\Big], \tag{35}$$

$$(\phi-\lambda+\lambda Y(z))\tilde{R}(z,\phi)=\tilde{R}(z,0)-\tilde{R}(\phi)\xi(D(z,0)+\sum_{m=a}^{b}P_m(z,0)). \tag{36}$$

By substituting $\phi=\lambda-\lambda Y(z)$ in (30) to (36), we get

$$Q_1(z,0)=\tilde{V}(\lambda-\lambda Y(z))L(z,0), \tag{37}$$

$$Q_j(z,0)=\tilde{V}(\lambda-\lambda Y(z))\sum_{n=0}^{a-1}Q_{j-1,0}(0),\ j\geq 2, \tag{38}$$

$$L(z,0)=\tilde{L}(\lambda-\lambda Y(z)))\Big[(1-\xi)(D_{10}(0)+\sum_{m=a}^{b}P_{m,0}(0))+R_0(0)]\Big], \tag{39}$$

$$R(z,0)=\tilde{R}(\lambda-\lambda Y(z)))\xi)\Big[D(z,0)+\sum_{m=a}^{b}P_m(z,0)]\Big], \tag{40}$$

$$P_i(z,0)=\tilde{G}(\lambda-\lambda Y(z))\Big[(1-\xi)D_{1i}(0)+\sum_{l=1}^{\infty}Q_{li}(0)+(1-\xi)\sum_{m=a}^{b}P_{m,i}(0)+R_i(0)\Big],\qquad a\leq i\leq b-1, \tag{41}$$

$$z^bP_b(z,0)=\tilde{G}(\lambda-\lambda Y(z))\Big[(1-\xi)\Big[D(z,0)-\sum_{j=0}^{b-1}D_{1,j}(0)z^j\Big]+(1-\xi)\Big[\sum_{m=a}^{b}P_m(z,0)-\sum_{j=0}^{b-1}\sum_{m=a}^{b}P_{m,j}(0)z^j\Big]+\sum_{l=1}^{\infty}Q_l(z,0)-\sum_{j=0}^{b-1}\sum_{l=1}^{\infty}Q_{l,j}(0)z^j+R(z,0)-\sum_{j=0}^{b-1}R_j(0)z^j\Big]. \tag{42}$$

Solving for $P_b(z,0)$,

$$P_b(z,0)=\frac{\tilde{G}(\lambda-\lambda Y(z))f(z)}{z^b-g(z)}, \tag{43}$$

where

$$
\begin{aligned}
f(z) =& \Big[(1-\xi)+\xi\tilde{R}(\lambda-\lambda Y(z))\Big]D(z,0)-(1-\xi)\sum_{n=0}^{b-1}D_{1n}(0)z^n \\
&+\Big[(1-\xi)+\xi\tilde{R}(\lambda-\lambda X(z))\Big]\sum_{m=a}^{b-1}P_m^{(1)}(z,0)-(1-\xi)\sum_{j=0}^{b-1}P_{mj}z^j \\
&+\sum_{l=1}^{\infty}(Q_l(z,0)-\sum_{j=0}^{b-1}Q_{lj}z^j)-\sum_{j=0}^{b-1}R_j(0)z^j, \\
g(z) =& (1-\xi)\tilde{G}(\lambda-\lambda Y(z))+\xi\tilde{G}(\lambda-\lambda Y(z))\tilde{R}(\lambda-\lambda Y(z)),
\end{aligned}
$$

$$
\begin{aligned}
D(z,0) =& \tilde{B}(\lambda-\lambda Y(z))\sum_{n=1}^{a-1}\Big[(1-\xi)(D_{1,n}(0)+\sum_{m=a}^{b}P_{m,n}^{(2)}(0)) \\
&+\sum_{l=1}^{\infty}Q_{l,n}(0)+R_n(0)\Big]z^{n-1}. \qquad (44)
\end{aligned}
$$

Substitute (37) to (44) in (30) to (36), we get

$$
\tilde{Q}_1(z,\phi) = \frac{\left[\tilde{V}(\lambda-\lambda Y(z))-\tilde{V}(\phi)\right]L(z,0)}{(\phi-\lambda+\lambda Y(z))}, \qquad (45)
$$

$$
\tilde{Q}_j(z,\phi) = \frac{\left[\tilde{V}(\lambda-\lambda Y(z))-\tilde{V}(\phi)\right]\sum_{l=1}^{\infty}Q_{l0}(0)}{(\phi-\lambda+\lambda Y(z))},\ j\geq 2, \qquad (46)
$$

$$
\tilde{L}(z,\phi) = \frac{\left(\tilde{L}(\lambda-\lambda Y(z))-\tilde{L}(\phi)\right)\left[(1-\xi)(D_{10}(0)+\sum_{m=a}^{b}P_{m,0}(0))+R_0(0)]\right]}{(\theta-\lambda+\lambda Y(z))}, \qquad (47)
$$

$$
\tilde{R}(z,\phi) = \frac{\left(\tilde{R}(\lambda-\lambda Y(z))-\tilde{R}(\phi)\right)\xi(D(z,0)+\sum_{m=a}^{b}P_m(z,0))}{(\theta-\lambda+\lambda Y(z))}, \qquad (48)
$$

$$
\tilde{P}_i(z,\phi) = \frac{\left(\tilde{G}(\lambda-\lambda Y(z))-\tilde{G}(\phi)\right)\begin{bmatrix}(1-\xi)D_{1i}(0)+\sum_{l=1}^{\infty}Q_{li}(0) \\ +(1-\xi)\sum_{m=a}^{b}P_{m,i}(0)+R_i(0)\end{bmatrix}}{(\phi-\lambda+\lambda Y(z))}, \qquad (49)
$$

$$
\tilde{P}_b(z,\phi) = \frac{\tilde{G}(\lambda-\lambda Y(z))-\tilde{G}(\phi)f(z)}{z^b-g(z)}, \qquad (50)
$$

$$\tilde{D}(z,\phi) = \frac{\left[\begin{array}{l}\left(\tilde{B}(\lambda-\lambda Y(z)) - \tilde{B}(\phi)\right)\sum\limits_{n=1}^{a-1}\Big[(1-\xi)(D_{1,n}(0) \\ \qquad + \sum\limits_{m=a}^{b} P_{m,n}(0)) + \sum\limits_{l=1}^{\infty} Q_{l,n}(0) + R_n(0)\Big] z^{n-1}\end{array}\right]}{(\phi-\lambda+\lambda Y(z))}. \tag{51}$$

Let

$$\begin{aligned} p_i =& \sum_{m=a}^{b} P_{m,i}(0),\ r_i = R_i(0),\ q_i = \sum_{l=1}^{\infty} Q_{l,i}(0), \\ b_i =& D_{1,i}(0),\ d_i = (1-\xi)p_i + (1-\xi)b_i + r_i,\ g_i = d_i + q_i. \end{aligned} \tag{52}$$

3. Probability generating function of queue size

In this section, the PGF of the queue size at an arbitrary time epoch is derived.

3.1. PGF of queue size at an arbitrary time epoch

If $P(z)$ be the PGF of the queue size at an arbitrary time epoch, then

$$P(z) = \sum_{m=a}^{b-1} \tilde{P}_m(z,0) + \tilde{P}_b(z,0) + \tilde{D}(z,0) + \tilde{L}(z,0) + \sum_{l=1}^{\infty} \tilde{Q}_l(z,0) + \tilde{R}(z,0). \tag{53}$$

By substituting $\theta = 0$ into Equations (45) to (51), then Equation (53) becomes

$$P(z) = \frac{\left[\begin{array}{r}\Big[(z^b-1)\Big[(1-\xi)\tilde{B}(\lambda-\lambda Y(z)) + \xi\tilde{B}(\lambda-\lambda Y(z))\tilde{R}(\lambda-\lambda Y(z)) - 1\Big] \\ -(z-1)(g(z)-1)\Big]\sum\limits_{n=1}^{a-1} g_n z^n + z\Big[g(z)-1\Big]\sum\limits_{n=a}^{b-1}(z^b - z^n)g_n \\ +z\Big[\tilde{V}(\lambda-\lambda Y(z))\tilde{L}(\lambda-\lambda Y(z)) - 1\Big](z^b-1)d_0 \\ + z(\tilde{V}(\lambda-\lambda Y(z)) - 1)(z^b-1)q_0\end{array}\right]}{(-\lambda+\lambda Y(z))z(z^b - g(z))}. \tag{54}$$

Equation (54) has $b+1$ unknowns $g_1, g_2, \ldots, g_{b-1}, d_0, q_0$. Using the following result, we express q_0 in terms of d_0 in such a way that numerator has only b constants. Now Equation (54) gives the PGF of the number of customers involving only 'b' unknowns. By Rouche's theorem of complex variables, it can be proved that $(z^b - (1-\xi)\tilde{G}(\lambda-\lambda Y(z)) - \xi\tilde{G}(\lambda-\lambda Y(z))\tilde{R}(\lambda-\lambda Y(z)))$ has $b-1$ zeros inside and one on the unit circle $|z| = 1$. Since $P(z)$ is analytic within and on the unit circle, the numerator must vanish at these points, which gives b equations in b unknowns. These equations can be solved by any suitable numerical technique.

3.2. Steady-state condition

Using $P(1) = 1$, the steady state condition is derived as $\rho = \lambda E(Y)\,[E(G)]\,/b$.

Theorem 3.1.

Let q_0 can be expressed in terms of d_0 as

$$q_0 = \frac{\gamma_0 \tau_0 d_0}{1-\gamma_0}. \tag{55}$$

Proof:

From Equations (37) and (38), we have

$$\begin{aligned}\sum_{n=0}^{\infty} q_n z^n &= \tilde{V}(\lambda - \lambda Y(z))\left[\tilde{L}(\lambda - \lambda Y(z))d_0 + q_0\right] \\ &= \sum_{n=0}^{\infty} \gamma_n z^n \left[\sum_{i=0}^{\infty} \tau_i z^i [d_0 + q_0]\right].\end{aligned} \tag{56}$$

Equating constant term, we get

$$q_0 = \frac{\gamma_0 \tau_0 d_0}{1-\gamma_0}. \qquad \blacksquare$$

3.3. Particular case

When there is no breakdown, repair and closedown, then

$$P(z) = \frac{\left[\begin{aligned}&(z^b-1)\left(\tilde{B}(\lambda-\lambda Y(z))-1\right) - (z-1)\left(\tilde{G}(\lambda-\lambda Y(z))-1\right)\sum_{n=1}^{a-1} k_n z^n \\ &\qquad + z\left(\tilde{G}(\lambda-\lambda Y(z))-1\right)\sum_{n=a}^{b-1}(z^b - z^n)k_n \\ &\qquad + z\left(\tilde{V}(\lambda-\lambda Y(z))-1\right)(z^b-1)k_0\end{aligned}\right]}{(-\lambda+\lambda Y(z))z(z-\tilde{G}(\lambda-\lambda Y(z)))}, \tag{57}$$

which coincides with the PGF of Jayakumar and Arumuganathan (2008).

4. Performance measures

In this section, various performance measures like the average queue length, average waiting time, expected length of busy and idle periods are derived.

4.1. Expected queue length

The expected queue length $E(A)$ at an arbitrary epoch is obtained by differentiating $P(z)$ at $z = 1$ and is given by

$$E(A) = \frac{\left[\begin{array}{l} f_1(Y,B,G,R)\left[\sum_{n=1}^{a-1} g_n\right] + f_2(Y,B,G,R)\sum_{n=1}^{a-1} ng_n \\ \qquad\qquad + f_3(Y,G,R)\sum_{n=a}^{b-1}(b-n)g_n \\ \quad +f_4(Y,G,R)\left[\sum_{n=a}^{b-1}\left[b(b-1)-n(n-1)\right]g_n\right] \\ \qquad + f_5(Y,V,G,R)(d_0+q_0) + f_6(Y,V,G,R,L)d_0 \end{array}\right]}{2.\left[(\lambda.Y_1).(b-G^{(1)}-\xi R^{(1)})\right]^2}, \tag{58}$$

where

$$\begin{aligned} f_1(Y,B,G,R) =& \Big[b(b-1).(B^{(1)}+\xi R^{(1)}) + b(B^{(2)}+\xi R^{(2)}+2.\xi.B^{(1)}R^{(1)}) \\ & - (G^{(2)}+\xi R^{(2)}+2.\xi.G^{(1)}.R^{(1)})\Big].X_1 \\ & - \Big[b.(B^{(1)}+\xi R^{(1)}) - G^{(1)} - \xi R^{(1)}\Big].X_2, \\ f_2(Y,B,G,R) =& 2.\Big[b.(B^{(1)}+\xi R^{(1)}) - G^{(1)} - \xi R^{(1)}\Big].X_1, \\ f_3(Y,G,R) =& \Big[2.(G^{(1)}+\xi R^{(1)}) + G^{(2)} + \xi R^{(2)} + 2.\xi.G^{(1)}.R^{(1)}\Big].X_1 \\ & - (G^{(1)}+\xi R^{(1)}).X_2, \end{aligned}$$

$$\begin{aligned} f_4(Y,G,R) =& (G^{(1)}+\xi R^{(1)}).X_1, \\ f_5(Y,G,R,V) =& \Big[2.b.V^{(1)} + b.V^{(2)} + b(b-1).V^{(1)}\Big].X_1 - b.V^{(1)}.X_2, \\ f_6(Y,G,R,V,L) =& \Big[2.b.L^{(1)} + b.(L^{(2)} + 2.L^{(1)}.V^{(1)}) + b(b-1).L^{(1)}\Big].X_1 - b.L^{(1)}.X_2. \end{aligned}$$

Further, we have

$$\begin{aligned} X_1 =& (\lambda.Y_1).\left(b - G^{(1)} - \xi R^{(1)}\right), \\ X_2 =& (\lambda.Y_2).\left(b - G^{(1)} - \xi R^{(1)}\right) + (\lambda.Y_1).\left(b.(b-1) - G^{(2)} - \xi R^{(2)} - 2.\xi.G^{(1)}.R^{(1)}\right) \\ & + 2(\lambda.Y_1).\left(b - G^{(1)} - \xi R^{(1)}\right), \end{aligned}$$

and

$$\begin{aligned}
Y_1 =& E(Y), \qquad B^{(1)} = \lambda.Y_1.E(B), \quad R^{(1)} = \lambda Y_1.E(R),\\
V^{(1)} =& \lambda.Y_1.E(V), \quad L^{(1)} = \lambda Y_1.E(L),\\
G^{(1)} =& \lambda.Y_1.E(G),\\
B^{(2)} =& \lambda.Y_2.E(B) + \lambda^2.(E(Y))^2.E(B^2),\\
R^{(2)} =& \lambda.Y_2.E(R) + \lambda^2.(E(Y))^2.E(R^2),\\
G^{(2)} =& \lambda.Y_2.E(G) + \lambda^2.(E(Y))^2.E(G^2),\\
V^{(2)} =& \lambda.Y_2.E(V) + \lambda^2(E(Y))^2.E(V^2),\\
L^{(2)} =& \lambda.Y_2.E(L) + \lambda^2.(E(Y))^2.E(L^2).
\end{aligned}$$

4.2. Expected waiting time

The expected waiting time is obtained by using Little's formula as

$$E(W) = \frac{E(A)}{\lambda E(Y)},$$

where $E(Q)$ is given in (58).

4.3. Expected length of busy period

Theorem 4.1.

Let M be the busy period random variable. Then, the expected length of busy period is

$$E(M) = \frac{E(H)}{d_0}. \tag{59}$$

Proof:

Let H be the residence time that the server is rendering single service or bulk service or under repair.

$$E(H) = E(B) + E(G) + \xi E(R).$$

Define a random variable J_1 as

$$J_1 = \begin{cases} 0, & \text{if the server finds no customer after the residence time,}\\ 1, & \text{if the server finds at least one customer after the residence time.}\end{cases}$$

Now the expected length of the busy period is given by

$$\begin{aligned}
E(M) &= E(M/J_1 = 0)P(J_1 = 0) + E(M/J_1 = 1)P(J_1 = 1)\\
&= E(H)P(J_1 = 0) + [E(H) + E(M)]\, P(J_1 = 1).
\end{aligned}$$

Solving for $E(M)$, we get

$$E(M) = \frac{E(H)}{P(J_1 = 0)} = \frac{E(H)}{d_0}. \qquad \blacksquare$$

4.4. Expected length of idle period

Theorem 4.2.

Let F be the idle period random variable. Then, the expected length of idle period is given by

$$E(F) = E(L) + E(F_1), \tag{60}$$

where

$$E(F_1) = \frac{E(V)}{1 - \gamma_0 \tau_0 d_0}, \tag{61}$$

F_1 is the idle period due to multiple vacation process, $E(L)$ is the expected closedown time.

Proof:

Define a random variable J_2 as

$$J_2 = \begin{cases} 0, & \text{if the server finds atleast one customer after the first vacation,} \\ 1, & \text{if the server finds no customer after the first vacation.} \end{cases}$$

The expected length of idle period due to multiple vacations $E(I_1)$ is given by

$$\begin{aligned} E(F_1) &= E(F_1/J_2 = 0)P(J_2 = 0) + E(F_1/J_2 = 1)P(J_2 = 1) \\ &= E(V)P(J_2 = 0) + [E(V) + E(F_1)]\, P(J_2 = 1). \end{aligned}$$

On solving, we get

$$E(F_1) = \frac{E(V)}{P(J_2 = 0)} = \frac{E(V)}{1 - P(J_2 = 1)} = \frac{E(V)}{1 - Q_{10}(0)}. \tag{62}$$

From Equation (37), we get

$$\begin{aligned} Q_{10}(0) &= \text{coefficient of } z^n \text{ in } Q_1(z, 0), \\ Q_1(z, 0) &= \tilde{V}(\lambda - \lambda Y(z))\tilde{C}(\lambda - \lambda Y(z))d_0 \\ &= \sum_{n=0}^{\infty} \gamma_n z^n \sum_{i=0}^{\infty} \tau_i z^i d_0. \end{aligned}$$

Equating the coefficient of z^0 on both sides, we get

$$Q_{10}(0) = \gamma_0 \tau_0 d_0. \tag{63}$$

Substitute (63) in (62), we get (61). $\blacksquare$

5. Cost Model

We derive the expression for finding the total average cost with the following assumptions:

C_s - Start up cost,

C_v - Reward per unit time due to vacation,

C_h - Holding cost per customer,

C_o - Operating cost per unit time,

C_r - Repair cost per unit time,

C_u - Closedown cost per unit time.

The length of cycle is the sum of the idle period and busy period. Now, the expected length of the cycle $E(T_c)$ is obtained as

$$E(T_c) = E(F) + E(M) = \frac{E(V)}{P(J_2 = 0)} + E(L) + \frac{E(H)}{d_0},$$

$$TAC = \left[C_s + \xi.C_r.E(R) + C_u.E(L) - C_v.\frac{E(V)}{P(J_2 = 0)}\right].\frac{1}{E(T_c)} + C_h.E(A) + C_O.\rho,$$

where

$$\rho = \lambda E(Y)\left[E(G) + \xi E(R)\right]/b.$$

6. Numerical illustration

In this section, various performance measures which are computed in earlier sections are verified numerically. Numerical example is analyzed using MATLAB, the zeros of the function $(z^b - g(z))$ are obtained and the zeroes are substituted in $P(z)$, we get the simultaneous equations. Solving these equations using Gauss elimination method, we get the probabilities and substituting these probabilities into the equations (58), (59) and (60) to get $E(A), E(M), E(F)$. The values of $E(W)$ are obtained from $E(A)$. The values of TAC are obtained from $E(A), E(M), E(F), E(W)$. A numerical example is analyzed with the following assumptions:

1. Batch size distribution of the arrival is geometric with mean two,

2. Single service time distribution is exponential and service rate is μ,

3. Bulk service time distribution is Erlang - k with $k = 2$ and service rate is μ^*,

4. Vacation time and closedown time are exponential with parameter $\gamma = 9$ and $\tau = 7$, respectively,

5. ξ = 0.2,

6. Repair cost: Rs. 1,

7. Start-up cost: Rs. 3,

8. Holding cost per customer: Rs. 0.50,

9. Operating cost per unit time: Rs. 2,

10. Reward per unit time due to vacation: Rs. 3,

11. Closedown cost per unit time: Rs. 0.25.

Table 1. Arrival rate vs performance measures for $\mu = 6,\ \mu^* = 8, \xi = 0.1,\ a = 2,\ b = 5$

λ	$E(A)$	$E(W)$	$E(M)$	$E(F)$	TAC
1.0	1.2157	0.6079	0.2871	0.0953	10.3281
1.5	2.4568	0.8189	0.2895	0.0940	10.5503
2.0	3.4589	0.8647	0.2933	0.0912	11.2389
2.5	4.5596	0.9119	0.2952	0.0875	11.7288
3.0	5.6239	0.9373	0.2974	0.0843	12.3720

Tables 1 and 2 show the performance of various measures like $E(A), E(M), E(F)$, $E(W)$ and TAC with the increment of arrival rate λ for the values of $\mu = 6$, $\mu^* = 8$ and $\mu = 9$, $\mu^* = 10$, respectively. It is also evident that the average queue length, expected busy period, average waiting time and total average cost increase as the increase of arrival rate. However, average queue length decreases as the increase of service rate.

Table 2. Arrival rate vs performance measures for $\mu = 9,\ \mu^* = 10, \xi = 0.1,\ a = 2,\ b = 5$

λ	$E(A)$	$E(W)$	$E(M)$	$E(F)$	TAC
1.0	1.1081	0.5541	0.2825	0.0982	10.0073
1.5	2.2156	0.7385	0.2863	0.0961	10.3328
2.0	2.9985	0.7496	0.2910	0.0943	10.8574
2.5	3.7526	0.7505	0.2937	0.0915	11.2098
3.0	4.5269	0.7545	0.2953	0.0882	11.7322

Table 3 shows that the performance of various measures like $E(A), E(M), E(F), E(W)$ and TAC with the increment of probability of breakdown ξ for the values of $\mu = 9$, $\mu^* = 10$. When the probability of breakdown increases, $E(A), E(M), E(W)$ and TAC increase where as $E(F)$ decreases.

Figures 1, 2 and 3 depict that average queue length, the expected waiting time and expected length of busy period increase with the increment of probability of breakdown. In Figure 4, it is evident that the expected length of idle period decreases as the increase of probability of breakdown. The tables and graphs are useful for manufacturing industries to take precise managerial decisions. The management can predict from the tables and graphs whenever the end mill cutter breaks down, then the average number of work pieces in the queue, the average waiting time of work pieces and the average length of busy period increase whereas the average length of idle period decreases.

Table 3. Probability of breakdown vs performance measures for $\mu = 9,\ \mu^* = 10,\ a = 2,\ b = 5$

ξ	$E(A)$	$E(W)$	$E(M)$	$E(F)$	TAC
0.1	1.1276	0.2819	0.2110	0.1093	8.7176
0.2	1.5345	0.3836	0.2185	0.1075	8.9594
0.3	1.9581	0.4895	0.2223	0.1042	9.1195
0.4	2.3875	0.5969	0.2251	0.1011	9.5075
0.5	2.7432	0.6858	0.2274	0.0990	9.8128
0.6	3.2063	0.8016	0.2296	0.0975	10.3383
0.7	3.5274	0.8819	0.2318	0.0954	10.7196
0.8	4.1027	0.9907	0.2337	0.0923	11.0181
0.9	4.8189	1.1797	0.2352	0.0910	11.4263
1.0	5.5538	1.3885	0.2379	0.0885	11.9091

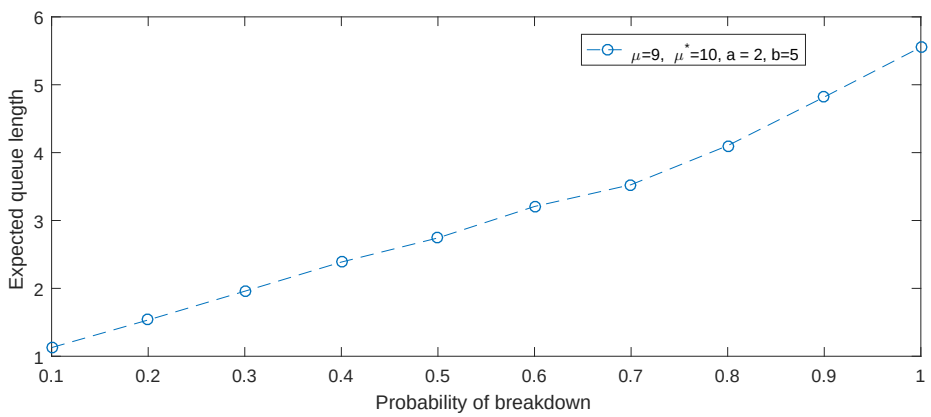

Figure 1. Expected queue length varies with probability of breakdown

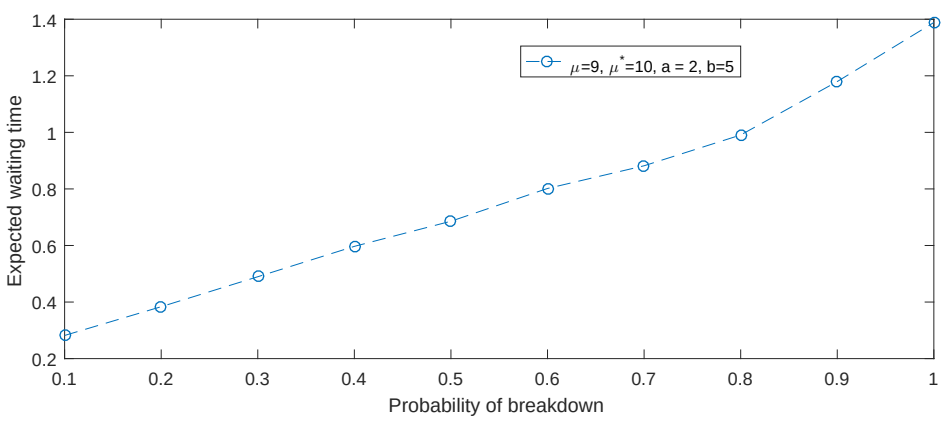

Figure 2. Expected waiting time varies with probability of breakdown

7. Conclusion and future work

In this paper, we have derived the PGF of the queue size for batch arrival single and bulk service queue with multiple vacation, closedown and repair under the steady-state case. Various perfor-

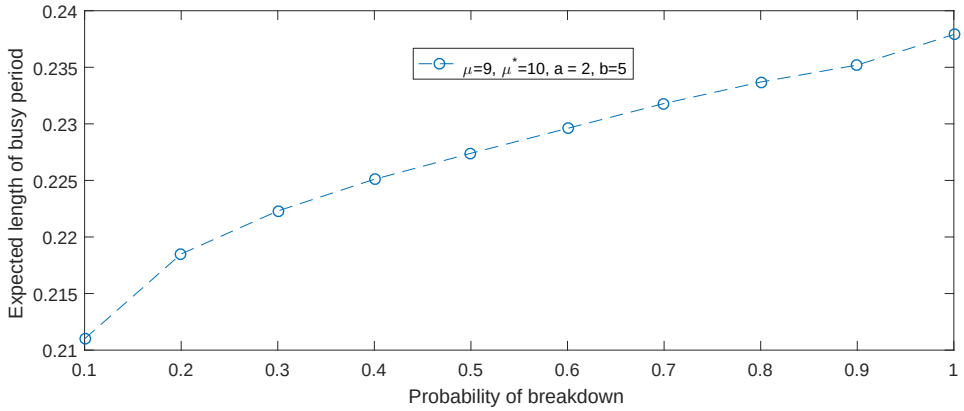

Figure 3. Expected length of busy period varies with probability of breakdown

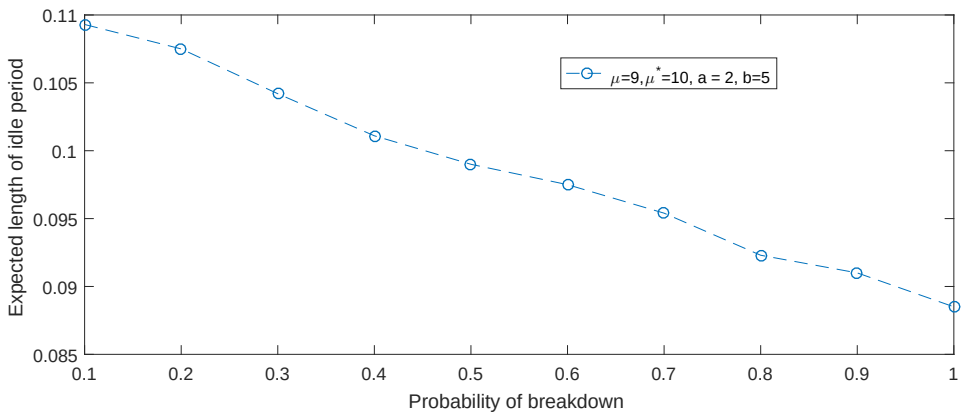

Figure 4. Expected length of idle period varies with probability of breakdown

mance measures are also obtained and verified them numerically. In the future, this work may be extended into a queueing model with two stages of service and modified vacation.

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