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On the New Generalized Block Difference Sequence Space

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Abstract

In this current study, the most apparent aspect is to submit a new block sequence space. We investigate its topological properties and inclusion relations. Moreover, we consider the problem of finding the norm of certain matrix operators from ℓ_p into this space and apply our results to Copson and Hilbert matrices.

Keywords: Block sequence space; Semi-norm; Matrix domain; Copson matrix; Hilbert matrix; Generalized difference matrix; Non-absolute type sequence space

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1. Introduction

In studies on the sequence spaces, there are some basic approaches which are determination topologies, inclusion relations and finding the norm of some certain matrix operators. These methods are applied to study the matrix domain of an infinite matrix in a sequence space. Therefore, the matrix domain plays an important role to construct a new sequence space. In literature, there are many studies done by using the matrix domain.

Now, we will give short literature information consisting of recent works about the concept mentioned above as follows. The notion of difference sequence spaces was introduced by Kızmaz (1981). Recently, Kirişçi and Başar (2010) have introduced and studied the generalized difference sequence spaces. In the past, several authors studied matrix transformations on the sequence spaces that are the matrix domains of triangle matrices in classical spaces ℓ_p, ℓ_∞, c and c_0. For instance, some matrix domains of the difference operator and generalized difference operator were studied in Başar and Altay (2003), Candan and Solak (2005), Altay and Başar (2007), and Başar and Altay (2008). In these studies, the matrix domains are obtained by triangle matrices, hence these spaces are normed sequence spaces. For more details on the domain of triangle matrices in some spaces, the reader may refer to chapter 4 of Başar (2012).

The concept of the block sequence space has recently been studied in summability theory. Firstly, Foroutannia (2015) introduced the block sequence space $\ell_p(E)$. After that, the sequence spaces $X(E)$, $\ell_p(\Delta, E)$ and $\ell_p(E, \Delta)$ are introduced and studied by Erfanmanesh and Foroutannia (2015), Roopaei and Foroutannia (2015) and Roopaei and Foroutannia (2016), respectively, where X denotes any of the sequence spaces ℓ_∞, c and c_0.

In this study, the sequence space $\ell_p(E, \Delta)$ is extended to the block sequence space $\ell_p(E, B(r, s))$. The matrix domains given in this paper specify a certain non-triangle matrix, so we should not expect that related spaces are normed sequence spaces.

The plan of the present paper is organized as follows. In Section 2, we give some fundamental concepts which are going to be used in the rest of the article. At the beginning of Section 3, the generalized block difference sequence space $\ell_p(E, B(r, s))$ is introduced. Also, the algebraic and topological properties of this space are examined and some inclusion relations are given. Finally, in the last section, we consider the problem of finding the norm of certain matrix operators from ℓ_p into this space and apply our results to Copson and Hilbert matrices.

2. Preliminaries

By a sequence space, we mean any vector subspace of ω, the space of all real valued sequences $u = (u_n)$. By ℓ_∞, c, c_0 and ℓ_p, we denote the spaces of all bounded, convergent, null sequences and p-absolutely convergent series, respectively.

Let Λ and Ω be two sequence spaces and $D = (d_{nk})$ be an infinite matrix of real numbers where $n, k \in \mathbb{N}$. The matrix D is a matrix transformation from Λ into Ω with the notation $D : \Lambda \to \Omega$, if for every sequence $u = (u_k) \in \Lambda$, the sequence $Du = (Du)_n$, the D-transform of u, is in Ω; here

$$(Du)_n = \sum_k d_{nk} u_k, \quad (n \in \mathbb{N}). \tag{1}$$

Let Λ be a sequence space. Then the matrix domain Λ_D of an infinite matrix D is defined by

$$\Lambda_D = \{u = (u_k) \in \omega : Du \in \Lambda\}, \tag{2}$$

which is also a sequence space.

The new sequence space Λ_D generated by the limitation matrix D from a sequence space Λ can be the expansion or the contraction or the overlap of the original space. A matrix $D = (D_{nk})$ is called triangle if $d_{nk} = 0$ for $k > n$ and $d_{nn} \neq 0$ for all $n \in \mathbb{N}$. If D is triangle, then one can easily observe that the sequence spaces Λ_D and Λ are linearly isomorphic, i.e., $\Lambda_D \cong \Lambda$.

The difference operator $\Delta : \omega \to \omega$ is defined by $\Delta u = (\Delta u_k) = (u_k - u_{k-1})$ or $\Delta u = (\Delta u_k) = (u_{k-1} - u_k)$ for all $u = (u_k) \in \omega$. The matrix domain Λ_Δ is called the difference sequence space whenever Λ is a sequence space. Firstly, the notion of difference sequence spaces was defined by Kizmaz (1981) as $\Lambda(\Delta) = \{u = (u_k) \in \omega : (u_{k-1} - u_k) \in \Lambda\}$ for $\Lambda \in \{\ell_\infty, c, c_0\}$.

The difference sequence space $\ell_p(\Delta)$ is defined by

$$\ell_p(\Delta) = \left\{ u = (u_n) \in \omega : \sum_{n=1}^{\infty} |u_n - u_{n+1}|^p < \infty \right\},$$

with the semi-norm

$$\|u\|_{p,\Delta} = \left(\sum_{n=1}^{\infty} |u_n - u_{n+1}|^p \right)^{1/p}.$$

The space obtained as a domain of the generalized difference matrix $B(r, s)$ in the classical space of absolutely p-summable sequence space was introduced by Kirişçi and Başar (2010). The sequence space $\hat{\ell_p}$ is defined by

$$\hat{\ell_p} = \left\{ u = (u_n) \in \omega : \sum_{n=1}^{\infty} \left| ru_n + su_{n-1} \right|^p < \infty \right\},$$

with the norm

$$\|u\|_{\hat{\ell_p}} = \left(\sum_{n=1}^{\infty} |ru_n + su_{n-1}|^p \right)^{1/p}.$$

Let $E = (E_n)$ be a partition of finite subsets of the positive integers such that

$$\max E_n < \min E_{n+1}, \tag{3}$$

for $n = 1, 2, ...$ Foroutannia defined the sequence space $\ell_p(E)$ by

$$\ell_p(E) = \left\{ u = (u_n) \in \omega : \sum_{n=1}^{\infty} \left| \sum_{j \in E_n} u_j \right|^p < \infty \right\}, \quad (1 \leq p < \infty),$$

with the semi-norm $\|.\|_{p,E}$, which is defined in the following way:

$$\|u\|_{p,E} = \left(\sum_{n=1}^{\infty} \left| \sum_{j \in E_n} u_j \right|^p \right)^{1/p}.$$

It is clear that, in the special case $E_n = \{n\}$ for $n = 1, 2, ...$, we have $\ell_p(E) = \ell_p$ and $\|u\|_{p,E} = \|u\|_p$. For more details about sequence space $\ell_p(E)$, one can see Foroutannia (2015).

Furthermore, Roopaei and Foroutannia (2016) defined the sequence space

$$\ell_p(E, \Delta) = \left\{ u = (u_n) \in \omega : \sum_{n=1}^{\infty} \left| \sum_{j \in E_n} u_j - \sum_{j \in E_{n+1}} u_j \right|^p < \infty \right\},$$

where $E = (E_n)$ is a partition of finite subsets of the positive integers and $p \geq 1$.

In this study, we define the sequence space $\ell_p(E, B(r, s))$ and investigate some topological and algebraical properties of this space and derive inclusion relations concerning with its. Moreover, we shall consider the inequality of the form

$$\|Du\|_{p,E,B(r,s)} \leq M\|u\|_p \ ,$$

for all the sequence $u \in \ell_p$. The constant M is not dependent on u and we seek the smallest possible value of M. We write $\|D\|_{p,E,B(r,s)}$ for the norm of D as an operator from ℓ_p into $\ell_p(E, B(r, s))$. In this research, we examine the problem of finding the upper bound of certain matrix operators from ℓ_p into $\ell_p(E, B(r, s))$ and we consider certain matrix operators such as Copson and Hilbert.

In a similar way, Roopaei and Foroutannia (2016) have introduced the sequence space $\ell_p(E, \Delta)$ and investigated the norm of certain matrix operators on this space.

3. The sequence space $\ell_p(E, B(r, s))$ of non-absolute type

Suppose $E = (E_n)$ is a partition of finite subsets of the positive integers that satisfies the condition (3). We define the sequence space $\ell_p(E, B(r, s))$ by

$$\ell_p(E, B(r, s)) = \left\{ u = (u_n) \in \omega : \sum_{n=1}^{\infty} \left| \sum_{j \in E_n} r u_j + \sum_{j \in E_{n+1}} s u_j \right|^p < \infty \right\},$$

with the semi-norm $\|.\|_{p,E,B(r,s)}$, which is defined in the following way:

$$\|u\|_{p,E,B(r,s)} = \left(\sum_{n=1}^{\infty} \left| \sum_{j \in E_n} r u_j + \sum_{j \in E_{n+1}} s u_j \right|^p \right)^{1/p}. \tag{4}$$

It should be noted that the function $\left\|.\right\|_{p,E,B(r,s)}$ cannot be a norm, since choosing $u = (u_j) = \{(-1)^{j+1}\}_{j=1}^{\infty}$ and $E = \{2n-1, 2n\}$, for all n , then $\left\|u\right\|_{p,E,B(r,s)} = 0$ while $u \neq 0$.

It is also significant that in the special case $E_n = \{n\}$ for $n = 1, 2, ...$, we have $\ell_p(E, B(r, s)) = \ell_p(B(r, s))$, where

$$\ell_p(B(r,s)) = \left\{ u = (u_n) \in \omega : \sum_{n=1}^{\infty} \left| ru_n + su_{n+1} \right|^p < \infty \right\}. \tag{5}$$

Letting $r = 1$ and $s = -1$, we have $\ell_p(E, B(r, s)) = \ell_p(E, \Delta)$ (Roopaei and Foroutannia (2016)), $r = 1$, $s = -1$ and $E_n = \{n\}$, we have $\ell_p(E, B(r, s)) = \ell_p(\Delta)$.

If the infinite matrix $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} r, & \text{if } k \in E_n, \\ s, & \text{if } k \in E_{n+1}, \\ 0, & \text{otherwise,} \end{cases}$$

with the notation (2), we can redefine the space $\ell_p(E, B(r, s))$ as follows:

$$\ell_p(E, B(r, s)) = (\ell_p)_D.$$

Throughout this paper, the cardinal number of the set E_k is denoted by $|E_k|$.

Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of the positive integers that satisfies the condition (3). We can say that the set $\ell_p(E, B(r, s))$ becomes a vector space with coordinatewise addition and scalar multiplication, which is a complete semi-normed space by $\left\|.\right\|_{p,E,B(r,s)}$ defined by (4).

It can be easily checked that the absolute property does not hold on the space $\ell_p(E, B(r, s))$, that is $\|u\|_{p,E,B(r,s)} \neq \||u|\|_{p,E,B(r,s)}$ for at least one sequence in the space $\ell_p(E, B(r, s))$, and this says that $\ell_p(E, B(r, s))$ is a sequence space of nonabsolute type, where $|u| = (|u_k|)$.

Theorem 3.1.

Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of the positive integers that satisfies the condition (3). If

$$M = \left\{ u = (u_n) : \sum_{j \in E_n} ru_j + \sum_{j \in E_{n+1}} su_j = 0, \forall n \right\},$$

then we have $\ell_p(E, B(r, s))/M \simeq \ell_p$.

Proof:

Consider the map $T : \ell_p(E, B(r,s)) \to \ell_p$ defined by

$$(Tu)_n = \sum_{j \in E_n} r u_j + \sum_{j \in E_{n+1}} s u_j,$$

for all $u \in \ell_p(E, B(r,s))$ and for all n. It is trivial that T is well defined, linear and ker $T = M$. Suppose that $v \in \ell_p$. It is clear that by defining

$$u_j = \begin{cases} \dfrac{v_n}{2r|E_n|}, & \text{if} \quad j \in E_n, \\ \\ \dfrac{v_n}{2s|E_{n+1}|}, & \text{if} \quad j \in E_{n+1}, \\ \\ 0, & \text{otherwise}, \end{cases}$$

$u \in \ell_p(E, B(r,s))$ and $Tu = v$, so the map T is surjective. By applying the first isomorphism theorem we have $\ell_p(E, B(r,s))/M \simeq \ell_p$. ■

Note that the map T, defined in Theorem 3.1, is not injective, while $\|Tu\|_p = \|u\|_{p,E,B(r,s)}$, for all $u \in \ell_p(E, B(r,s))$.

Let us derive some inclusion relations concerning with the space $\ell_p(E, B(r,s))$.

Lemma 3.2. (Roopaei and Foroutannia (2016))

Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of positive integers that satisfies the condition (3). If $\sup_n |E_n| < \infty$, then $\ell_p \subset \ell_p(E)$. Moreover if $|E_n| > 1$ for an infinite number of n, then the inclusion is strict.

Theorem 3.3.

Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of positive integers that satisfies the condition (3). Then $\ell_p(E) \subset \ell_p(E, B(r,s))$, furthermore the inclusion strictly holds.

Proof:

To prove the validity of the inclusion $\ell_p(E) \subset \ell_p(E, B(r,s))$, it suffices to show

$$\|u\|_{p,E,B(r,s)} \leq M \|u\|_{p,E},$$

for each $u \in \ell_p(E)$. Suppose that $u \in \ell_p(E)$ is an arbitrary sequence. Since

$$\left(\sum_{n=1}^{\infty} \left| \sum_{j \in E_n} r u_j + \sum_{j \in E_{n+1}} s u_j \right|^p \right)^{1/p} \leq \left(\sum_{n=1}^{\infty} \left(\left| \sum_{j \in E_n} r u_j \right| + \left| \sum_{j \in E_{n+1}} s u_j \right| \right)^p \right)^{1/p},$$

by Minkowski's inequality we obtain

$$
\begin{aligned}
\|u\|_{p,E,B(r,s)} &\leq \left(\sum_{n=1}^{\infty}\left|\sum_{j\in E_n} ru_j\right|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty}\left|\sum_{j\in E_{n+1}} su_j\right|^p\right)^{1/p} \\
&= |r|\left(\sum_{n=1}^{\infty}\left|\sum_{j\in E_n} u_j\right|^p\right)^{1/p} + |s|\left(\sum_{n=1}^{\infty}\left|\sum_{j\in E_{n+1}} u_j\right|^p\right)^{1/p} \\
&\leq |r|\left(\sum_{n=1}^{\infty}\left|\sum_{j\in E_n} u_j\right|^p\right)^{1/p} + |s|\left(\sum_{n=1}^{\infty}\left|\sum_{j\in E_n} u_j\right|^p\right)^{1/p}.
\end{aligned}
$$

By letting $M = |r| + |s|$ and simple calculation we have

$$\|u\|_{p,E,B(r,s)} \leq M\|u\|_{p,E}.$$

If we define the sequence $u = (u_j)$ such that

$$
u_j = \begin{cases} \dfrac{-s}{|E_n|}, & \text{if} \quad j \in E_n, \\[2ex] \dfrac{r}{|E_{n+1}|}, & \text{if} \quad j \in E_{n+1}, \\[2ex] 0, & \text{otherwise,} \end{cases}
$$

it is obvious that $u \in \ell_p(E, B(r,s)) - \ell_p(E)$ and this completes the proof. ∎

Combining Lemma 3.2 and Theorem 3.3, we get the following corollary:

Corollary 3.4.

Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of positive integers that satisfies the condition (3). If $\sup_n |E_n| < \infty$, then $\ell_p \subset \ell_p(E, B(r,s))$. Moreover if $|E_n| > 1$ for an infinite number of n, then the inclusion is strict.

Proof:

From Lemma 3.2, if $\sup_n |E_n| < \infty$ and $|E_n| > 1$, for an infinite number of n, the inclusion $\ell_p \subset \ell_p(E)$ is strict and from Theorem 3.3, the inclusion $\ell_p(E) \subset \ell_p(E, B(r,s))$ strictly holds. So, if $\sup_n |E_n| < \infty$ and $|E_n| > 1$ for an infinite number of n, it is obvious that the inclusion $\ell_p \subset \ell_p(E, B(r,s))$ is strict. ∎

Theorem 3.5.

Neither of the spaces $\ell_p(\Delta)$ and $\ell_p(E, B(r,s))$ includes the other one.

Proof:

Consider the sequences $u = (u_j) = (-1)^j$ and $v = (v_j) = (1, 1, 1, ...)$, for all $j \in \mathbb{N}$. If $E_n = \{2n-1, 2n\}$ for all n, then we conclude that $u \in \ell_p(E, B(r,s))$, but $u \notin \ell_p(\Delta)$. Nevertheless $v \in \ell_p(\Delta)$ but $v \notin \ell_p(E, B(r,s))$. This completes the proof. ■

It is known that the ℓ_p space is not a Hilbert Space with $p \neq 2$. The similar result is valid for the space $\ell_p(E, B(r,s))$ and following theorem gives this result.

Theorem 3.6.

Let $E = (E_n)$ be a partition of finite subsets of positive integers that satisfies the condition (3). The space $\ell_p(E, B(r,s))$ is not a semi-inner product space in the case $p \neq 2$.

Proof:

We must show that the space $\ell_2(E, B(r,s))$ is a semi-inner product space while $\ell_p(E, B(r,s))$ is not a semi-inner product space. If we define

$$\langle u, v \rangle = \sum_{n=1}^{\infty} \sum_{i,j \in E_n} u_i v_j,$$

then it is a semi-inner product on the space $\ell_2(E, B(r,s))$ and

$$\begin{aligned} \|u\|^2_{2,E,B(r,s)} &= \sum_{n=1}^{\infty} \left| \sum_{j \in E_n} r u_j + \sum_{j \in E_{n+1}} s u_j \right|^2 \\ &= \|u_{E,B(r,s)}\|_2^2 \\ &= \langle u_{E,B(r,s)}, u_{E,B(r,s)} \rangle, \end{aligned}$$

where

$$u_{E,B(r,s)} = \left(\sum_{j \in E_1} r u_j + \sum_{j \in E_2} s u_j, \sum_{j \in E_2} r u_j + \sum_{j \in E_3} s u_j, ... \right).$$

Now, by considering the sequences u and v such that

$$\sum_{j \in E_1} u_j = \frac{r+s}{r^2} \ , \ \sum_{j \in E_2} u_j = \frac{-1}{r} \ , \ \sum_{j \in E_3} u_j = \sum_{j \in E_4} u_j = ... = 0,$$

$$\sum_{j \in E_1} v_j = \frac{r-s}{r^2} \ , \ \sum_{j \in E_2} v_j = \frac{1}{r} \ , \ \sum_{j \in E_3} v_j = \sum_{j \in E_4} v_j = ... = 0,$$

we see that

$$\|u+v\|^2_{p,E,B(r,s)} + \|u-v\|^2_{p,E,B(r,s)} \neq 2 \left(\|u\|^2_{p,E,B(r,s)} + \|v\|^2_{p,E,B(r,s)} \right), \quad (p \neq 2).$$

Also, it is easy to see that the equation $2 = 2^{\frac{2}{p}}$ does not hold for $p \neq 2$. This means that the parallelogram equality can not be satisfies by the semi-norm of the space $\ell_p(E, B(r,s))$ for $p \neq 2$. Therefore this semi-norm can not be obtained from a semi-inner product. Hence the space $\ell_p(E, B(r,s))$ with $p \neq 2$ is not semi-inner product space. ■

Let Λ be a semi-normed space with a semi-norm g. A sequence (b_n) of elements of the semi-normed space Λ is called a Schauder basis (or briefly basis) for Λ iff for each $u \in \Lambda$ there exists a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} g\left(u - \sum_{k=1}^{n} \alpha_k b_k\right) = 0.$$

The series $\sum_{k=1}^{n} \alpha_k b_k$ which has the sum u, is then called the expansion of u with respect to (b_n) and written as $u = \sum_{k=1}^{n} \alpha_k b_k$.

Let $p \geq 1$ and $E = (E_n)$ be a partition of finite subsets of the positive integers that satisfies the condition (3). If the sequence $b^{(k)}(r,s) = \{b_j{}^{(k)}(r,s)\}_{j\in\mathbb{N}}$ is defined such that

$$\sum_{j\in E_n} b_j{}^{(k)}(r,s) = \begin{cases} 0 & , \text{ if } n < k, \\ \frac{1}{r}(-\frac{s}{r})^n, & \text{ if } n \geq k, \end{cases}$$

and the remaining elements are zero, for $k = 1, 2, ...$, we can say that the sequence $\left\{b^{(k)}(r,s)\right\}_{k\in\mathbb{N}}$ is a basis for the space $\ell_p(E, B(r,s))$ and any $u \in \ell_p(E, B(r,s))$ has a unique representation of the form

$$u = \sum_k \alpha_k b^{(k)}(r,s),$$

where $\alpha_k = \sum_{j\in E_k} u_j$, for $k = 1, 2, ...$

4. The norm of matrix operators from ℓ_p into $\ell_p(E, B(r,s))$

In this section, the problem of finding the norm of certain matrix operators such as Copson and Hilbert from ℓ_p into $\ell_p(E, B(r,s))$ is considered, where $p \geq 1$. At the beginning, we tend to compute the norm of operators from ℓ_1 into $\ell_1(E, B(r,s))$.

Theorem 4.1.

Let $D = (d_{n,k})$ be a matrix operator and $E = (E_n)$ be a partition that satisfies condition (3). If

$$M = \sup_k \sum_{n=1}^{\infty} \left| \sum_{i\in E_n} r d_{i,k} + \sum_{i\in E_{n+1}} s d_{i,k} \right| < \infty,$$

then D is a bounded operator from ℓ_1 into $\ell_1(E, B(r,s))$ and $\|D\|_{1,E,B(r,s)} = M$. In particular, if

$$\sum_{i\in E_n} r d_{i,k} + \sum_{i\in E_{n+1}} s d_{i,k} \geq 0,$$

and $r + s = 0$, for all n, k, then

$$\|D\|_{1,E,B(r,s)} = \sup_k \sum_{i\in E_1} r d_{i,k}.$$

Proof:

Let (u_n) be in ℓ_1 and

$$t_k = \sum_{n=1}^{\infty} \left| \sum_{i \in E_n} rd_{i,k} + \sum_{i \in E_{n+1}} sd_{i,k} \right|,$$

for all k. We have

$$\begin{aligned}\|Du\|_{1,E,B(r,s)} &\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left| \sum_{i \in E_n} rd_{i,k} + \sum_{i \in E_{n+1}} sd_{i,k} \right| |u_k| \\ &= \sum_{k=1}^{\infty} t_k |u_k| \\ &\leq M \|u\|_1,\end{aligned}$$

which says that $\|D\|_{1,E,B(r,s)} \leq M$. Conversely, we take $u = e_n$ which e_n denotes the sequence having 1 in place n and 0 elsewhere, then $\|u\|_1 = 1$ and $\|Du\|_{1,E,B(r,s)} = t_n$ which proves that $\|D\|_{1,E,B(r,s)} = M$. ∎

In the sequel, we will compute the norms of Copson and Hilbert operators from sequence space ℓ_1 into $\ell_1(E, B(r,s))$. The Copson operator C is defined by $v = Cu$, where

$$v_n = \sum_{k=n}^{\infty} \frac{u_k}{k}, (\forall n).$$

It is given by the Copson matrix:

$$c_{n,k} = \begin{cases} \frac{1}{k}, & \text{if } n \leq k, \\ 0, & \text{if } n > k. \end{cases}$$

Corollary 4.2.

Let C be the Copson operator and $E = (E_n)$ be a partition that satisfies condition (3). If

$$\sum_{i \in E_n} rc_{i,k} + \sum_{i \in E_{n+1}} sc_{i,k} \geq 0,$$

for all n, k and $r + s = 0$, then C is a bounded operator from ℓ_1 into $\ell_1(E, B(r,s))$ and $\|C\|_{1,E,B(r,s)} = r$.

Proof:

Since

$$\begin{aligned}M &= \sup_k \sum_{i \in E_1} rc_{i,k} \\ &= r \sup_k \sum_{i \in E_1} c_{i,k} \\ &= rc_{1,1} = r,\end{aligned}$$

we obtain the desired result from Theorem 4.1. ■

Corollary 4.3.

Suppose that C is the Copson operator, $rc_{n,k} + sc_{n+1,k} \geq 0$, for all n, k, $r + s = 0$ and $E = \{n\}$, for all n. Then, C is a bounded operator from ℓ_1 into $\ell_1(B(r, s))$ and $\|C\|_{1,B(r,s)} = r$.

Proof:

The proof is obvious by letting $E_n = \{n\}$ in Corollary 4.2. ■

Recall the Hilbert operator H defined by the matrix

$$h_{n,k} = \frac{1}{n+k}, \qquad (n, k = 1, 2, ...).$$

Corollary 4.4.

Let H be the Hilbert operator and $E = (E_n)$ be a partition that satisfies the condition (3). If

$$\sum_{i \in E_n} rh_{i,k} + \sum_{i \in E_{n+1}} sh_{i,k} \geq 0,$$

for all n, k and $r + s = 0$, then H is a bounded operator from ℓ_1 into $\ell_1(E, B(r, s))$ and

$$\|H\|_{1,E,B(r,s)} = r\left(\frac{1}{2} + ... + \frac{1}{\max E_1 + 1}\right).$$

Proof:

Since

$$\begin{aligned} M &= \sup_k \sum_{i \in E_1} rh_{i,k} \\ &= r \sup_k \sum_{i \in E_1} h_{i,k} \\ &= r\left(\frac{1}{2} + ... + \frac{1}{\max E_1 + 1}\right), \end{aligned}$$

we obtain the desired result from Theorem 4.1. ■

Corollary 4.5.

If H is the Hilbert operator, $rh_{n,k} + sh_{n+1,k} \geq 0$, for all n, k and $r + s = 0$, then H is a bounded operator from ℓ_1 into $\ell_1(B(r, s))$ and $\|H\|_{1,B(r,s)} = \frac{r}{2}$.

Proof:

The proof is obvious by letting $E_n = \{n\}$ in Corollary 4.4. ■

In the following, the problem of finding the norm of certain matrix operators such as Copson and Hilbert from ℓ_p into $\ell_p(E, B(r,s))$ are investigated for $p > 1$. For this purpose, we give Schur's Theorem and a lemma which are needed to prove our main results.

Theorem 4.6. (Hardy et al. (2001), Theorem 275)

Let $p > 1$ and $B = (b_{n,k})$ be a matrix operator with $b_{n,k} \geq 0$ for all n, k. Suppose that K and R are two strictly positive numbers such that

$$\sum_{n=1}^{\infty} b_{n,k} \leq K, \text{ for all } k, \qquad \sum_{k=1}^{\infty} b_{n,k} \leq R, \text{ for all } n,$$

(bounds for column and row sums respectively). Then,

$$\|B\|_p \leq R^{(p-1)/p} \,.\, K^{1/p}.$$

Lemma 4.7.

If $D = (d_{n,k})$ and $B = (b_{n,k})$ are two matrix operators such that

$$b_{n,k} = \sum_{i \in E_n} r d_{i,k} + \sum_{i \in E_{n+1}} s d_{i,k},$$

then,

$$\|D\|_{p,E,B(r,s)} = \|B\|_p.$$

Hence, if B is a bounded operator on ℓ_p, then D will be a bounded operator from ℓ_p into $\ell_p(E, B(r,s))$.

Proof:

From the definition of operators B and D, the proof is obvious. ■

Now, we are ready to compute the norm of the Copson Matrix operator when $p > 1$.

Theorem 4.8.

Let C is the Copson matrix operator $r > 0$ and $r+s = 0$. If N is a positive integer and $E_n = \{nN - N + 1, nN - N + 2, ..., nN\}$, for all n, then C is a bounded operator from ℓ_p into $\ell_p(E, B(r,s))$ and

$$\|C\|_{p,E,B(r,s)} \leq r \left(N + \frac{N-1}{N+1} + \frac{N-2}{N+2} + ... + \frac{1}{2N-1} \right)^{\frac{(p-1)}{p}}.$$

Proof:

By applying Lemma 4.7 we have

$$\|C\|_{p,E,B(r,s)} = \|B\|_p,$$

where

$$b_{n,k} = \sum_{i \in E_n} r c_{i,k} + \sum_{i \in E_{n+1}} s c_{i,k}.$$

Let K and R defined as in Theorem 4.6. By a simple calculation we deduce that $K_n \leq r$ and $R_n \leq R_1$ for all n. Since

$$b_{1,k} = \begin{cases} r, & \text{if} \quad k \leq N, \\ \dfrac{(2N-k)r}{k}, & \text{if} \quad N < k \leq 2N-1, \\ 0, & \text{if} \quad k \geq 2N, \end{cases}$$

and

$$\begin{aligned} R_1 = \sum_{k=1}^{\infty} b_{1,k} &= \sum_{k=1}^{N} b_{1,k} + \sum_{k=N+1}^{2N-1} b_{1,k} + \sum_{k=2N}^{\infty} b_{1,k} \\ &= r\left(N + \frac{N-1}{N+1} + \frac{N-2}{N+2} + ... + \frac{1}{2N-1}\right), \end{aligned}$$ ■

it can be concluded that

$$\|C\|_{p,E,B(r,s)} \leq r\left(N + \frac{N-1}{N+1} + \frac{N-2}{N+2} + ... + \frac{1}{2N-1}\right)^{\frac{(p-1)}{p}}.$$

Theorem 4.9.

Suppose that $p > 1$, $r > 0$, $r + s = 0$, N is a positive integer and $E_n = \{nN - N + 1, nN - N + 2, ..., nN\}$, for all n. If H is the Hilbert matrix operator, then it is a bounded operator from ℓ_p into $\ell_p(E, B(r, s))$ and

$$\|H\|_{p,E,B(r,s)} \leq r\left(\frac{1}{2} + \frac{2}{3} + ... + \frac{N}{N+1} + ... + \frac{1}{2N}\right)^{\frac{(p-1)}{p}} \left(\frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2N}\right)^{\frac{1}{p}}.$$

Proof:

By applying Lemma 4.7 we have

$$\|H\|_{p,E,B(r,s)} = \|B\|_p,$$

where

$$b_{n,k} = \sum_{i \in E_n} r h_{i,k} + \sum_{i \in E_{n+1}} s h_{i,k}.$$

Let K and R defined as in Theorem 4.6. By a simple calculation we deduce that $K_n \leq K_1$ and

$R_n \leq R_1$ for all n. Since

$$R_1 = \sum_{k=1}^{\infty} b_{1,k} = \sum_{k=1}^{\infty} \left(\sum_{i \in E_1} rh_{i,k} + \sum_{i \in E_2} sh_{i,k} \right) \\ = r\left(\frac{1}{2} + \frac{2}{3} + ... + \frac{N}{N+1} + ... + \frac{1}{2N}\right),$$

and

$$K_1 = \sum_{n=1}^{\infty} b_{n,1} = \sum_{n=1}^{\infty} \left(\sum_{i \in E_n} rh_{i,1} + \sum_{i \in E_{n+1}} sh_{i,1} \right) \\ = \sum_{i \in E_1} rh_{i,1} \\ = r\left(\frac{1}{2} + \frac{1}{3} + ... + \frac{1}{N+1}\right),$$

we have

$$\|H\|_{p,E,B(r,s)} \leq R^{\frac{p-1}{p}} K^{\frac{1}{p}} \\ \leq r\left(\frac{1}{2} + \frac{2}{3} + ... + \frac{N}{N+1} + ... + \frac{1}{2N}\right)^{\frac{p-1}{p}} \left(\frac{1}{2} + \frac{1}{3} + ... + \frac{1}{N+1}\right)^{\frac{1}{p}}. \qquad \blacksquare$$

5. Conclusion

The concept of block sequence space has recently been studied in summability theory. In 2016, Roopaei and Foroutannia studied the semi-normed difference sequence space $\ell_p(E, \Delta)$. In this study, the sequence space $\ell_p(E, \Delta)$ is extended to the block sequence space $\ell_p(E, B(r, s))$. Also, we investigate its algebraic and topological properties and inclusion relations. We think that this article will contribute to the current literature in the field of block sequence spaces. It would be interesting also to determine the α-, β- and γ- duals of the space $\ell_p(E, B(r, s))$, characterize some matrix classes on this space and apply our results to certain matrix operators such as Holder, Euler and Hausdorff.

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