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A Comparison of Original and Inverse Motion in Minkowski Plane

[1]Kemal Eren and [2]Soley Ersoy

[1]Fatsa Science High School
Ordu, Turkey

[2]Department of Mathematics
Faculty of Arts and Sciences
Sakarya, Turkey
[1]kemal.eren1@ogr.sakarya.edu.tr, [2]sersoy@sakarya.edu.tr

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Abstract

In this paper, we investigate the inflection circle, circling-point curve, and center-point curve for the original and inverse motion of Minkowski planes, and we also deal with their degenerate cases individually. For this purpose, we consider the trajectory of origin with respect to the instantaneous invariants of Bottema in Minkowski plane by using hyperbolic numbers. Finally, we give the geometric interpretation of the circling-point and center-point curves by comparing the original and inverse motion in Minkowski planes.

Keywords: Circling-point curve; Center-point curve; Inflection circle; Planar motion; Inverse motion; Instantaneous invariants; Minkowski plane

MSC 2010 No.: 53A10, 53B30, 53C42

1. Introduction

The formulation of special locus curves as inflection circle, circling-point curve, or center-point curve for planar or spatial motions has been a frequently discussed problem that is also handled by Bottema (1961), (1963), and Freudenstein and Sandor (1961), (1965), (1967). The application of the concept of instantaneous invariants (introduced by Bottema (1961), (1963), and (1990)) to this theory has been developed by Veldkamp (1963), (1967a), (1967b). However, there is increasing concern over using these invariants to present the local properties of the trajectories of points and lines through the Euclidean planar, spherical or spatial motions (Kirson and Yang (1978), Koetsier (1986), McCarthy and Ravani (1986), McCarthy and Roth (1982), Roth (2015), Roth and Yang (1977), Yang et al. (1994)), referring to this method in another type of subgeometries has been performed in limited number studies.

The classical Burmester theory has been extended to the Cayley–Klein planes with affine base by a unified method in Eren and Ersoy (2018a) and also the necessary and sufficient condition of a Lorentzian (Minkowski) plane to be at Cardan position with respect to instantaneous invariants has been introduced by Eren and Ersoy (2018b). On the other hand, the circling-point curves and the geometric location of Ball points in the Minkowski plane have been studied by Eren and Ersoy (2018c). In fact, a large number of studies have been performed to assess how the basic kinematics forms in the Minkowski plane (Ergüt et al. (1988), Ergin (1991), Tutar et al. (2001), Güngör et al. (2010), and Solouma (2017)). However, much uncertainty still exists about the relations between the original and inverse motion in the Minkowski planes.

In these regards, we investigate the special locus curves for the inverse motion of the Minkowski planes and we interpret the relationship between the circling-point and center-point curves by comparing the original and inverse motion in the Minkowski plane.

2. Preliminaries

Let us consider the motion of a moving Minkowski plane L_m with respect to a fixed Minkowski plane L_f. Then, the planar motion (it is denoted by L_m/L_f to avoid confusing this original motion with inverse motion else) is represented by

$$\begin{aligned} X(\theta) &= x\cosh\theta + y\sinh\theta + a(\theta), \\ Y(\theta) &= x\sinh\theta + y\cosh\theta + b(\theta), \end{aligned} \tag{1}$$

with respect to Cartesian frames of reference xoy and XOY in L_m and L_f, respectively. Here $a,\ b$ and θ are functions depending on time t. The Minkowski plane L_m is chosen to rotate with a constant angular velocity relative to fixed Minkowski plane L_f, that is, $\theta = t$ since we aim to study the geometric properties of Minkowski planes. The successive differentiations of Equation (1) are

$$\begin{aligned} X^{(2n)} &= x\cosh\theta + y\sinh\theta + a^{(2n)} \quad , \quad Y^{(2n)} = x\sinh\theta + y\cosh\theta + b^{(2n)}, \\ X^{(2n+1)} &= x\sinh\theta + y\cosh\theta + a^{(2n+1)} \ , Y^{(2n+1)} = x\cosh\theta + y\sinh\theta + b^{(2n+1)}, \end{aligned} \tag{2}$$

for $n \in \mathbb{N}$.

Throughout this paper, we prefer the notations $X_n = d^nX/d\theta^n$, $Y_n = d^nY/d\theta^n$, $a_n = d^na/d\theta^n$, and $b_n = d^nb/d\theta^n$ such that the subscript $n \in \mathbb{N}$ denotes the n-th derivative of the functions at $\theta = 0$.

Firstly, let the coordinate systems of L_m and L_f be coincident at the reference position by considering $\theta = 0$ and $a = b = 0$. Afterward, if we consider $X' = Y' = 0$ then the instantaneous center of rotation P can be found by

$$x = a_1 \sinh\theta - b_1 \cosh\theta, \quad y = -a_1 \cosh\theta + b_1 \sinh\theta, \tag{3}$$

and

$$X = -b_1 + a, \quad Y = -a_1 + b. \tag{4}$$

The moving and fixed pole curves ρ_m and ρ_f are determined with Equations (3) and (4), respectively. At the reference position, the coordinates of the instantaneous center of rotation become $X = x = -b_1$, $Y = y = -a_1$. We choose the instantaneous center of rotation to be coincident with the origin. So we have $a_1 = b_1 = 0$.

Finally, let the real axis of the moving and fixed Minkowski planes be collinear with the principal tangent of moving pole curve at the first order pole. Then $a_2 = 0$. Also, we arbitrarily choose $b_2 = -1$.

In this way, we obtain the canonical relative system for which

$$a = b = a_1 = b_1 = a_2 = 0 \quad \text{and} \quad b_2 = -1.$$

The derivatives a_n and b_n are known as the instantaneous invariants of Bottema of the motion. By virtue of (2), the instantaneous invariants a_n and b_n completely characterize the infinitesimal properties of motion of Minkowski planes up to the n-th order as

$$\begin{array}{llll} X = x, & X' = y, & X'' = x, & X''' = y + a_3, \ldots, \\ Y = y, & Y' = x, & Y'' = y - 1, & Y''' = x + b_3, \ \ldots, \end{array} \tag{5}$$

at the reference position. The equation of the inflection circle can be obtained from $X'' : Y'' = X' : Y'$, since the curvature function is

$$\kappa = \frac{X'Y'' - X''Y'}{\left|(X')^2 - (Y')^2\right|^{\frac{3}{2}}}, \tag{6}$$

where $(X')^2 - (Y')^2 \neq 0$ in the Minkowski plane (Eren and Ersoy (2018b), (2018c)).

Under the consideration of the equalities of (5), the locus of the points satisfying $\kappa = 0$ gives us the equation of the inflection circle during planar motion of L_m with respect to L_f as follows,

$$x^2 - y^2 + y = 0, \tag{7}$$

where $(x, y) \neq (0, 0)$, $x \neq \mp y$ or $y \neq 0$ (Eren and Ersoy (2018b), (2018c)).

On the other hand, the hyperbolic number representation of the planar motion of L_m/L_f can be given in the form of

$$Z = ze^{j\varphi} + c, \tag{8}$$

such that

$$Z = X + jY, \quad z = x + jy, \quad c = a + jb, \quad \left(j^2 = 1\right).$$

The successive differentiations of Equation (8) are $Z^{(n)} = j^n z + c_n$, $n \in N$. In this way, the canonical relative systems can be given by the relations

$$c_0 = c_1 = 0, \quad c_2 = j, \tag{9}$$

and the hyperbolic numbers c_n can completely characterize the infinitesimal properties of motion of Minkowski planes up to the $n\text{th}$ order that will be called the hyperbolic form of B-invariants.

3. Inverse Motion and Instantaneous Invariants at Minkowski Plane

Definition 3.1.

The motion of the fixed plane L_f with respect to moving plane L_m, that is, the inverse of the original motion, is called the inverse motion of Minkowski planes and it is denoted L_f/L_m.

Accordingly, if we consider Equation (8), we can give the equation of the inverse motion L_f/L_m as

$$z = (Z - c)\, e^{-j\varphi}. \tag{10}$$

Let us obtain the relationship between the equations of the original and inverse motions. From this last equation, one can see that the pole vectors of each motion are coincident. With this point of view, the canonical systems of motion have a common spacelike $X-$axis. Moreover, two successive differentiations of Equation (10) are

$$\begin{aligned} z' &= -j\,(Z - c)\, e^{-j\varphi} - c' e^{-j\varphi}, \\ z'' &= (Z - c)\, e^{-j\varphi} + 2jc' e^{-j\varphi} - c'' e^{-j\varphi}. \end{aligned}$$

If we rearrange these equations with respect to canonical systems of the motion L_f/L_m, by using (9) we get

$$z_1 = -jZ, \quad z_2 = Z - j,$$

or

$$\begin{aligned} x_1 &= -Y, & x_2 &= X, \\ y_1 &= -X, & y_2 &= Y - 1. \end{aligned}$$

From here we obtain the equations of inflection circle of inverse motion as

$$x^2 - y^2 - y = 0.$$

In the Minkowski plane, the equations of the inflection circles $X^2 - Y^2 + Y = 0$, generated by L_m/L_f, and $x^2 - y^2 - y = 0$, generated by L_f/L_m, denote circles with imaginary diameters $|j|$ and center $(0, 1/2)$ and $(0, -1/2)$, respectively (see Figure 1).

In this connection, it is seen that the canonical systems of the inverse motion L_f/L_m and original motion L_m/L_f are symmetrical with respect to pole tangent. The inverse motion, with respect to

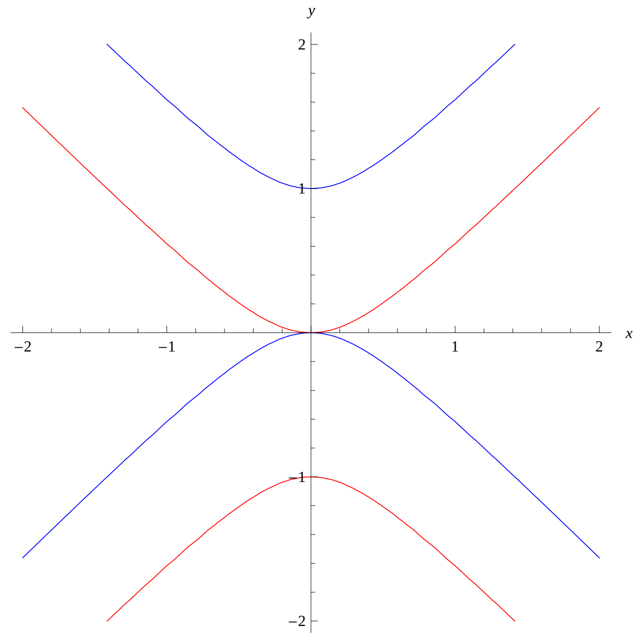

Figure 1. The inflection (blue and red) circles of inverse motion in Minkowski with respect to the canonical system of original and inverse motion, respectively.

its own canonical systems, is given by

$$Z = ze^{j\varphi} + \tilde{c}, \quad \tilde{c} = -\bar{c}e^{j\varphi}, \tag{11}$$

where $\bar{c}$ is the hyperbolic conjugate of c. From the equations of (9), we get

$$\tilde{c}^{(n)} = -je^{j\varphi} \sum_{k=0}^{n} \binom{n}{k} j^k \,\bar{c}^{(n-k)}.$$

Since $\bar{c}_0 = \bar{c}_1 = 0$ at reference position, we get

$$\tilde{c}_n = -\sum_{k=0}^{n-2} \binom{n}{k} j^k \,\bar{c}_{n-k}. \tag{12}$$

This last equation gives us the relationships between the instantaneous invariants of inverse original motions in hyperbolic numbers form as

$$\tilde{c}_0 = \tilde{c}_1 = 0, \quad \tilde{c}_2 = j, \tag{13}$$

and

$$\tilde{c}_3 = 3 - \bar{c}_3, \quad \tilde{c}_4 = 6j - 4j\,\bar{c}_3 - \bar{c}_4, \quad \tilde{c}_5 = 10 - 10\bar{c}_3 - 5j\,\bar{c}_4 - \bar{c}_5.$$

Also, if we pass from hyperbolic numbers representation to Minkowskian coordinates, the relationships can be given by

$$\begin{array}{lll} \tilde{a}_3 = -a_3 + 3, & \tilde{a}_4 = -a_4 + 4b_3, & \tilde{a}_5 = -a_5 - 10a_3 - 5b_4 + 10, \\ \tilde{b}_3 = b_3, & \tilde{b}_4 = -4a_3 + b_4 + 6, & \tilde{b}_5 = -5a_4 + b_5 + 10b_3. \end{array} \tag{14}$$

4. The Trajectory of Origin of Minkowski Plane

The trajectory of the point $(0,0)$ of the Minkowski plane L_m, which is coincident with the pole has been studied in Eren and Ersoy (2018c) by considering

$$X=\sum_{n=3}^{\infty}\frac{a_n}{n!}\varphi^n, \quad Y=\frac{-1}{2}\varphi^2+\sum_{n=3}^{\infty}\frac{b_n}{n!}\varphi^n, \tag{15}$$

for sufficiently small values of $|\varphi|$ at the reference position with respect to canonical relative systems.

Case 1. Let $a_3\neq 0$. If ε is a sufficiently small positive number, then the trajectory described through the time interval $[-\varepsilon,\varepsilon]$ has a cusp at the pole of reference position since $\lim\limits_{\varphi\to 0}|\kappa|=\infty$ and the tangent of the trajectory is pole normal.

Case 2. Let $a_3=0,\ a_4\neq 0$. In this case $a_2b_3-a_3b_2=0$ and $a_2b_4-a_4b_2\neq 0$. So two branches of the trajectory stay at the same side of the tangent. If ε is a sufficiently small positive number, then the trajectory described through the time interval $[-\varepsilon,\varepsilon]$ has a ramphoid cusp at the pole of the reference position. However the trajectory curvature is undefined for $\varphi=0$ at the ramphoid cusp, for $\varphi\to 0$ the curvature is

$$\kappa=\frac{\frac{a_4}{3}+\left(\frac{a_5}{8}-\frac{a_4b_3}{12}\right)\varphi+\left(\frac{a_6}{30}-\frac{a_5b_3}{24}\right)\varphi^2+\left(\frac{a_7}{144}-\frac{a_6b_3}{80}-\frac{a_5b_4}{144}+\frac{a_4b_5}{144}\right)\varphi^3+\ldots}{\left|-1+b_3\varphi+\left(-\frac{b_3^2}{4}+\frac{b_4}{3}\right)\varphi^2+\left(-\frac{b_3b_4}{6}+\frac{b_5}{12}\right)\varphi^3+\left(\frac{a_4^2}{36}-\frac{b_4^2}{36}-\frac{b_3b_5}{24}+\frac{b_6}{60}\right)\varphi^4+\ldots\right|^{\frac{3}{2}}}. \tag{16}$$

Moreover, since $|\varphi|$ has sufficiently small value the curvature can be given as

$$\begin{aligned}\kappa=&\frac{a_4}{3}+\left(\frac{5a_4b_3}{12}+\frac{a_5}{8}\right)\varphi+\left(\frac{-a_4\,{b_3}^2}{8}+\frac{a_4\,b_4}{6}+\frac{7a_5\,b_3}{48}+\frac{a_6}{30}\right)\varphi^2\\&+\left(\frac{7a_4\,b_5}{144}-\frac{a_4\,{b_3}^2}{12}-\frac{a_4\,b_3\,b_4}{24}+\frac{a_5\,b_4}{18}-\frac{a_5\,{b_3}^2}{16}+\frac{3a_6\,b_3}{80}+\frac{a_7}{144}\right)\varphi^3+\ldots\end{aligned} \tag{17}$$

As a consequence the curvature of the trajectory at the pole is

$$\kappa_0=\frac{a_4}{3}. \tag{18}$$

From the successive differentiation of Equation (17), we get successive curvatures at the pole as

$$\kappa_1=\frac{5a_4b_3}{12}+\frac{a_5}{8}, \tag{19}$$

$$\kappa_2=\frac{-a_4{b_3}^2}{4}+\frac{a_4b_4}{3}+\frac{7a_5b_3}{24}+\frac{a_6}{15}, \tag{20}$$

$$\kappa_3=\frac{7a_4b_5}{24}-\frac{a_4{b_3}^2}{2}-\frac{a_4b_3b_4}{4}+\frac{a_5b_4}{3}-\frac{3a_5{b_3}^2}{8}+\frac{9a_6b_3}{40}+\frac{a_7}{24}. \tag{21}$$

Case 3. Let $a_3 = a_4 = 0$. For sufficiently small values of $|\varphi|$, the trajectory described through the time interval $[-\varepsilon, \varepsilon]$ has cusp or ramphoid cusp, provided that the smallest value of, where $a_n \neq 0$, is odd or even, respectively. Therefore, from Equation (17), we get

$$\kappa = 0 + \frac{a_5}{8}\varphi + \left(\frac{7a_5b_3}{48} + \frac{a_6}{30}\right)\varphi^2 + \left(\frac{a_5b_4}{18} - \frac{a_5{b_3}^2}{16} + \frac{3a_6b_3}{80} + \frac{a_7}{144}\right)\varphi^3 + ...,$$

that is, the successive curvatures at the pole are

$$\kappa_0 = 0, \tag{22}$$

$$\kappa_1 = \frac{a_5}{8}, \tag{23}$$

$$\kappa_2 = \frac{7a_5b_3}{24} + \frac{a_6}{15}, \tag{24}$$

$$\kappa_3 = \frac{a_5b_4}{3} - \frac{3a_5{b_3}^2}{8} + \frac{9a_6b_3}{40} + \frac{a_7}{24}, \tag{25}$$

(Eren and Ersoy (2018c)).

5. Circling-Point Curve of Original and Inverse Motions in Minkowski Plane

Definition 5.1.

The locus of the points with constant trajectory curvature at the reference position of Minkowski plane L_m is called circling-point curve or cubic stationary curvature in Minkowski plane and denoted by cp.

This means that the locus of the points satisfying $\kappa' = 0$, where $(X')^2 - (Y')^2 \neq 0$, is the circling-point curve.

In the Minkowski plane the equation of the circling-point curve cp of the original motion L_m/L_f is

$$\left(x^2 - y^2\right)(a_3x - b_3y) + 3x\left(x^2 - y^2 + y\right) = 0, \tag{26}$$

where $(x, y) \neq (0, 0)$ or $x \neq \mp y$ (Eren and Ersoy (2018c)).

Theorem 5.2.

If (ξ, η) denotes the curvature center of the trajectory of the point (x, y) of the moving Minkowski plane L_m which is coincident with the point (X, Y) of the fixed Minkowski plane L_f, then

$$\xi = \frac{yx}{x^2 - y^2 + y}, \quad \eta = \frac{y^2}{x^2 - y^2 + y}, \tag{27}$$

such that (x, y) is not on the inflection circle and $(x, y) \neq (0, 0)$.

Proof:

The coordinates of the curvature center of the trajectory of the point (x, y) of the moving Minkowski plane L_m which is coincident with the point (X, Y) of the fixed Minkowski plane L_f are

$$\xi = X - \frac{Y'\left((X')^2 - (Y')^2\right)}{X'Y'' - X''Y'}, \quad \eta = Y - \frac{X'\left((X')^2 - (Y')^2\right)}{X'Y'' - X''Y'}.$$

Substituting the equalities of Equation (5) into the last equation completes the proof. ∎

Theorem 5.3.

In the Minkowski plane, the equation of the circling-point curve cp of the original motion L_m/L_f is

$$\left(\xi^2 - \eta^2\right)(a_3\xi - b_3\eta) + 3\xi\eta = 0, \tag{28}$$

where $(\xi, \eta) \neq (0, 0)$ is the curvature center and $\xi \neq \mp\eta$.

Proof:

If we consider the curvature centers of the points (x, y) of the moving Minkowski plane L_m with respect to canonical systems given by Equation (27), we get

$$x^2 - y^2 + y = \frac{yx}{\xi} \quad \text{and} \quad x^2 - y^2 + y = \frac{y^2}{\eta}.$$

If we denote $\frac{x^2 - y^2 + y}{y} = \lambda$, it is easy to say that

$$x = \lambda\xi \quad \text{and} \quad y = \lambda\eta.$$

If we rearrange Equation (26) under this consideration and the conditions $(\xi, \eta) \neq (0, 0)$ and $\xi \neq \mp\eta$, we obtained the desired Equation (28). ∎

Definition 5.4.

The locus of the curvature centers of the trajectories of the points of moving plane L_m is called center-point curve of Minkowski plane and denoted by $\tilde{cp}$.

Theorem 5.5.

In the Minkowski plane, the equation of the center-point curve $\tilde{cp}$ of the planar motion L_m/L_f is the cubic curve

$$\left(x^2 - y^2\right)(a_3x - b_3y) + 3xy = 0, \tag{29}$$

where $(x, y) \neq (0,0)$ or $x \neq \mp y$.

Proof:

The curvature centers (ξ, η) are taken as the points (x, y), based on Definition 5.4. In that way, modifying Equation (28) as Equation (29) completes the proof. ■

The center-point curve $\tilde{cp}$ is a rational curve and the special cases of this curve are given in the following corollary.

Corollary 5.6.

i. If $a_3 \neq 0$ and $b_3 = 0$, the equation of the center-point curve $\tilde{cp}$ of the planar motion L_m/L_f is

$$x\left(a_3\left(x^2 - y^2\right) + 3y\right) = 0, \tag{30}$$

that is, $\tilde{cp}$ consists of the pole normal and circle of Minkowski plane with imaginary diameter $3j/a_3$ and center $(0, 3/2a_3)$ (see Figure 2.i).

ii. If $a_3 = 0$ and $b_3 \neq 0$, the equation of the center-point curve $\tilde{cp}$ of the planar motion L_m/L_f is

$$y\left(b_3\left(x^2 - y^2\right) - 3x\right) = 0, \tag{31}$$

and this geometrically means that $\tilde{cp}$ consists of the pole tangent and circle of Minkowski plane with diameter $3/b_3$ and center $(3/2b_3, 0)$ (see Figure 2.ii).

iii. If $a_3 = b_3 = 0$, the equation of the center-point curve $\tilde{cp}$ of the planar motion L_m/L_f is

$$xy = 0, \tag{32}$$

and this geometrically means that $\tilde{cp}$ consists of the pole tangent and pole normal or the line of the Minkowski plane at infinite (see Figure 2.iii).

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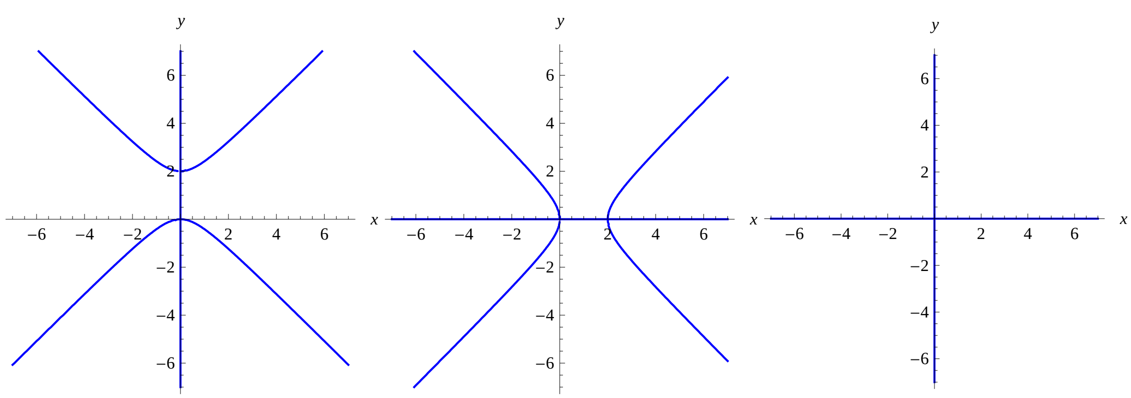

Figure 2. The center-point curves $\tilde{cp}$ in the Minkowski plane for i.$a_3 = \frac{3}{2}$, $b_3 = 0$, ii.$a_3 = 0$, $b_3 = \frac{3}{2}$, and iii. $a_3 = b_3 = 0$, respectively.

Theorem 5.7.

The real asymptotes of the center-point curve $\tilde{cp}$ of the Minkowski plane are

$$\begin{aligned}&\left({a_3}^2-{b_3}^2\right)\left(a_3x-b_3y\right)-3a_3b_3=0,\\&2\left(a_3-b_3\right)\left(x-y\right)+3=0,\\&2\left(a_3+b_3\right)\left(x+y\right)-3=0.\end{aligned} \tag{33}$$

(See Figure 3).

Proof:

Let $y = mx + c$ be the real asymptote of the center-point curve $\tilde{cp}$. If we substitute this equation into equation (29), we find

$$\begin{aligned}&c^3b_3+x(3c-c^2a_3+3c^2mb_3)+x^2(3m-2cma_3-cb_3+3cm^2b_3)\\&\quad+x^3(a_3-m^2a_3-mb_3+m^3b_3)=0.\end{aligned}$$

If the third degree term is solved with respect to m, it is found that $m=-1$, $m=1$ and $m=\frac{a_3}{b_3}$. Also, the solution of the second degree term with respect to c is solved $c=-\frac{3m}{-2ma_3-b_3+3m^2b_3}$. If we write the values of m into c, we obtain

$$c=\frac{3}{2a_3+2b_3},\quad c=-\frac{3}{-2a_3+2b_3},\quad\text{and}\quad c=-\frac{3a_3}{(a_3^2-b_3^2)},$$

for $m=-1$, $m=1$, and $m=\frac{a_3}{b_3}$, respectively. Consequently, by substituting the values c and m into the equation of asymptote, we obtain three real asymptotes of the center-point curve $\tilde{cp}$. ■

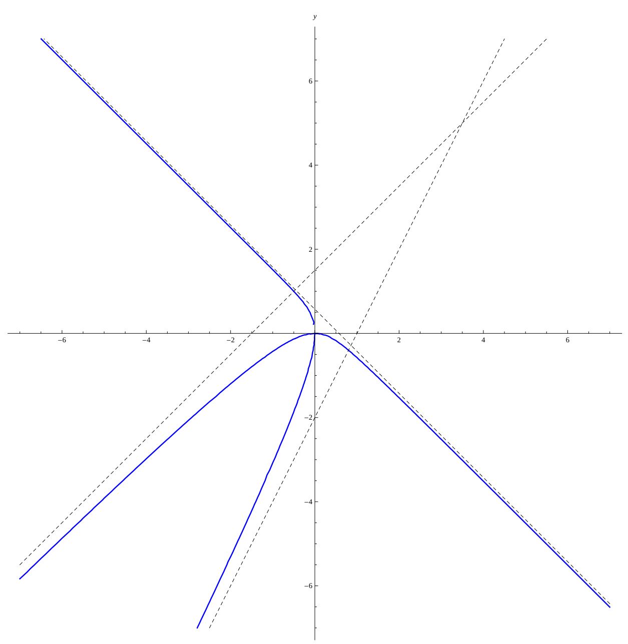

Figure 3. The curve $\tilde{cp}$ and its asymptotes in the Minkowski plane for $a_3 = b_3 = 2$.

Theorem 5.8.

The curves cp and $\tilde{cp}$ degenerate to the same curve at same moment if and only if $b_3 = 0$.

Proof:

If we substitute Equation (14) into Equation (29), we find

$$\left(x^2-y^2\right)\left(\tilde{a}_3x+\tilde{b}_3y\right)+3x\left(x^2-y^2-y\right)=0. \tag{34}$$

■

Moreover, if we put $-y$ instead of y in Equation (33), we get the equation of a center-point curve $c\tilde{p}$ of the inverse motion with respect to canonical systems as

$$\left(x^2-y^2\right)\left(\tilde{a}_3x-\tilde{b}_3y\right)+3x\left(x^2-y^2+y\right)=0.$$

From the previous equation and equation (26) the following corollary is obvious.

Corollary 5.9.

The center-point curve $c\tilde{p}$ is the circling-point curve cp of the inverse motion. Also, the circling-point curve cp is the center-point curve $c\tilde{p}$ of the inverse motion.

6. Conclusion

The present study was designed to compare the original and inverse motion in the Minkowski planes, and the results of this investigation show that the canonical systems of the inverse motion L_f/L_m and original motion L_m/L_f are symmetrical with respect to pole tangent. Moreover, the interconnection between the center-point curve $c\tilde{p}$ and the circling-point curve cp has been executed after a detailed examination of the special locus curves by comparing the original and inverse motions in the Minkowski plane.

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